ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.10, 8685–8702
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.10.94

TRIGONOMETRIC RATIOS USING GEOMETRIC METHODS

SAMEEN AHMED KHAN

ABSTRACT. Obtaining exact expressions for the trigonometric ratios is as old as the subject of trigonometry itself. In this article, we shall state the results from geometric methods. Tables of exact trigonometric ratios are presented and the patterns in them are illustrated. The power and limitations of the geometric methods for deriving the exact values of the trigonometric ratios, based on certain theorems is discussed in detail. The irrational sets of trigonometric ratios of rational angles are also discussed. Results from number theory are presented wherever required.

1. INTRODUCTION

Exact values of trigonometric ratios of selected angles have been known since the beginning of the subject and the search for additional exact solutions still continues [1]- [7]. Earlier attempts were based on geometric constructions and later on there was a switchover to equations. In Section 2, we shall state the geometric methods used for obtaining the exact ratios of certain angles. Section 3 has a general description on the geometric constructions along with the related theorems. In Section 4, we shall derive the results for a set of angles using quadratic equations. Section 5 covers the irrationality of trigonometric numbers. Section 6, our final Section has our concluding remarks.

2020 *Mathematics Subject Classification.* 33B10, 97G60.
Key words and phrases. Trigonometric Ratios, Geometric Methods, Ailles Rectangle, Constructible Regular Polygons, Fermat Primes, Niven's Theorem, Paolillo-Vincenzi Theorem.

2. Geometric methods

Exact values of trigonometric functions of certain angles such as 30°, 45° and 60° can be found directly using simple geometric constructions, without resorting to the trigonometric identities or any equations. For the aforementioned angles, it suffices to use the Pythagorean theorem $a^2 + b^2 = c^2$, where c is the length of the hypotenuse and a and b are the lengths of the right-angled triangle's other two sides. The trigonometric ratios of these angles are in Table (1).

Table 1: Exact Ratios: 30°, 45° and 60°

	Angle, θ			
S. No.	**Deg.**	**Rad.**	$\sin\theta$	$\cos\theta$
1	30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
2	45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
3	60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

In this Section, we shall make use of the geometric constructions using the Pythagorean theorem and obtain exactly the trigonometric ratios of the two sets of angles, $\{15^\circ, 75^\circ\}$ and $\{18^\circ, 36^\circ, 54^\circ, 72^\circ\}$.

2.1. Exact Ratios: 15° and 75°.

In this Section, we shall derive some trigonometric ratios using geometric methods involving simple constructions. In the

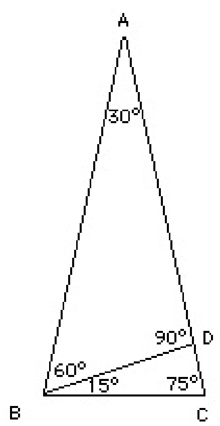

Figure 1. Trigonometric Ratios for 15° and 75°

Figure (1), $AB = AC = 1$, $BD = AB\sin 30^\circ = \frac{1}{2}$, $AD = AB\sin 60^\circ = \frac{\sqrt{3}}{2}$ and $CD = AC - AD = 1 - \frac{\sqrt{3}}{2}$. Applying the Pythagorean theorem to the triangles, we obtain $BC^2 = BD^2 + DC^2 = 2 - \sqrt{3}$, leading to $BC = \sqrt{2-\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{2}}$. This gives the trigonometric ratios

$$\begin{aligned} \sin 15^\circ &= \cos 75^\circ = \frac{CD}{BC} = \frac{\sqrt{3}-1}{2\sqrt{2}} \\ \sin 75^\circ &= \cos 15^\circ = \frac{BD}{BC} = \frac{\sqrt{3}+1}{2\sqrt{2}}\,. \end{aligned}$$

2.2. **Exact Ratios:** 18°, 36°, 54° **and** 72°**.** In Figure (2), the trigonometric ratios for 18°, 36°, 54° and 72° are functions of x (see [8] for details). Applying the

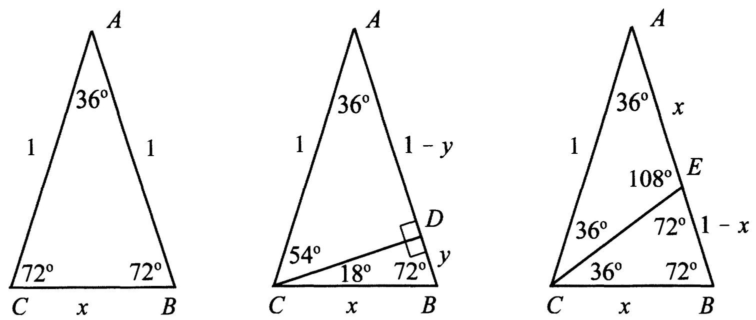

FIGURE 2. Trigonometric Ratios for 18°, 36°, 54° and 72°.

Pythagorean theorem to the two triangles, we obtain $CD^2 = x^2 - y^2$ and $CD^2 = 1 - (1-y)^2 = 2y - y^2$, which leads to the relation $y = x^2/2$. This enables us to express all the sides of the various triangles as

$$AD = 1 - \frac{1}{2}x^2\,, \quad BD = \frac{1}{2}x^2\,, \quad CD = x\sqrt{1 - \frac{1}{4}x^2}\,.$$

The similarity of the triangles, ΔABC and ΔCEB leads to

$$\frac{AB}{CB} = \frac{CB}{BE}\,, \qquad \frac{1}{x} = \frac{x}{1-x}\,.$$

This gives $x = \frac{-1+\sqrt{5}}{2}$. Finally, the trigonometric ratios are

$$\begin{aligned}
\sin 18^\circ &= \cos 72^\circ = \frac{\sqrt{5}-1}{4} \\
\sin 36^\circ &= \cos 54^\circ = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}} \\
\sin 54^\circ &= \cos 36^\circ = \frac{\sqrt{5}+1}{4} \\
\sin 72^\circ &= \cos 18^\circ = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}.
\end{aligned}$$

2.3. **Ailles Rectangle for 15° and 75°.** An elegant geometric method for deriving the trigonometric ratios of 15° and 75° is by using the Ailles Rectangle [9]-[11]. The Ailles rectangle also gives the trigonometric ratios of 30°, 45° and

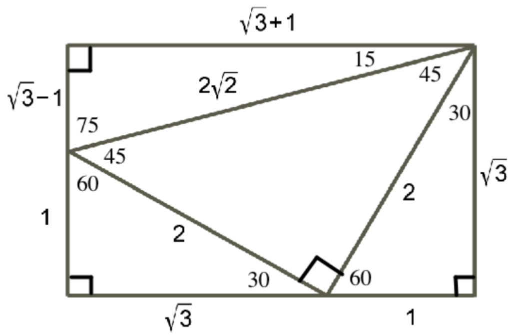

FIGURE 3. Ailles Rectangle and the Trigonometric Ratios for 15° and 75°

60°. Douglas S. Ailles, a high school teacher at Etobicoke Collegiate Institute in Etobicoke, Ontario, Canada published his work in the year 1971 [9]. A similar rectangle for other angles such as 18° and 72° has remained elusive!

Geometric methods are not possible for certain angles, which is discussed in Section 3. Even, when possible, the geometric methods are not straightforward for certain angles. In such cases, it is better to use the trigonometric equations. This approach of identities and equations is done in detail in Section 4.

3. A NOTE ON GEOMETRIC CONSTRUCTIONS

In Section 2, we used geometric constructions to find the exact trigonometric ratios of the two sets of angles namely, $\{15^\circ, 75^\circ\}$ and $\{18^\circ, 36^\circ, 54^\circ, 72^\circ\}$. In this Section, we shall discuss the constructability of regular polygons.

A figure is said to be constructible, if it can be constructed using only a compass and a straightedge. The ancient Greek mathematicians knew how to construct a regular polygon with 3, 4, 5 or 6 sides, using a compass and a straightedge. They also knew that, it is possible to construct a regular polygon with double the number of sides of a given regular polygon. But they could not figure out which of the n-gons (polygons with n sides/edges) are constructible and which are non-constructible. This question was finally settled by Carl Friedrich Gauss in the eighteenth century [12]- [14].

The derivation of the exact values of the trigonometric ratios is based on the ability to construct the required right-angled triangles. This translates to constructing regular polygons. Let us consider a regular n-gon inscribed on a circle. The n-gon has $2n$ right-angled triangles with angles $\frac{180^\circ}{n}$ at the centre of the circle/n-gon and $(90^\circ - \frac{180^\circ}{n})$ on the circle. The right-angle is on the chord.

The results of Gauss make use of the *Fermat primes*. A *Fermat number* has the form $F_n = 2^{2^n}+1$ for $n \geq 0$. The infinite sequence of Fermat numbers is 3, 5, 17, 257, 65537, 4294967297, $\cdots$. If a Fermat number happens to be a prime, it is known as the Fermat prime. To date, there are only five known Fermat primes, which are $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$ and $F_4 = 65537$. The Fermat numbers for $5 \leq n \leq 32$ are all composite. Extending the range beyond $n = 32$ looks very difficult using current computational methods and hardware. In *The On-Line Encyclopedia of Integer Sequences* (OEIS), created and maintained by Neil Sloane [15], the Fermat numbers are designated by the Sequence A000215 [16]. The Fermat primes form the sequence 3, 5, 17, 257, 65537 (only five known terms) and are designated by the Sequence A019434 [17].

Theorem 3.1. *Gauss-Wantzel Theorem: A regular n-gon is constructible with straightedge and compass if and only if $n = 2^k p_1 p_2 p_3 \ldots$, where p_i are distinct Fermat primes and $k \geq 0$.*

Using the Gauss-Wantzel theorem and the five known Fermat primes, a regular n-gon is *constructible* if and only if $n = 3, 4, 5, 6, 8, 10, 12, \ldots$ which is the Sequence A003401 in the OEIS [18]. Likewise, a regular n-gon is *non-constructible* if $n = 7, 9, 11, 13, 14, 18, 19, \ldots$ which is the Sequence A004169 [19].

To summarize, it is possible to find the trigonometric ratios for $\sin(180°/n)$ and $\cos(180°/n)$ exactly, using geometric constructions, where n has the form given in the Gauss-Wantzel theorem. The resulting values can be expressed using the radicals. The same results can be obtained using equations. The use of equations enables us to get results for those angles, which are not possible due to the non-constructible polygons. A prime example is that of $20°$.

4. Trigonometric ratios from identities and equations

Exact values of certain trigonometric ratios can be obtained by employing trigonometric identities and equations. The Pythagorean theorem leads to the basic identity, $\sin^2 A + \cos^2 A = 1$. The identities for the sines and cosines of sums of two angles are

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B\,. \end{aligned} \tag{4.1}$$

Theese identities can be compactly written using a matrix

$$\begin{bmatrix} \cos A & \sin A \\ -\sin A & \cos A \end{bmatrix} \begin{bmatrix} \cos B & \sin B \\ -\sin B & \cos B \end{bmatrix} = \begin{bmatrix} \cos(A+B) & \sin(A+B) \\ -\sin(A+B) & \cos(A+B) \end{bmatrix}.$$

A special case is

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{bmatrix}.$$

The same formulae can also be derived using complex exponentials and the Euler's formula or the de Moivre's formula as follows

$$\begin{aligned}\cos(A+B)+\ \mathrm{i}\sin(A+B) &= e^{\mathrm{i}(A+B)}\\ &= e^{\mathrm{i}A}e^{\mathrm{i}B}\\ &= (\cos A+\ \mathrm{i}\sin A)(\cos B+\ \mathrm{i}\sin B)\\ &= (\cos A\cos B-\sin A\sin B)\\ &\quad +\mathrm{i}(\sin A\cos B+\cos A\sin B)\,.\end{aligned}$$

Equating the real and imaginary parts along with the substitution $-B$ for B gives both the identities in (4.1).

In (4.1), if we substitute $A = 30°$ and $B = 45°$, then we obtain

$$\begin{aligned}\sin 75° &= \sin 30°\cos 45° + \cos 30°\sin 45°\\ &= \frac{1}{2}\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}=\frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

and

$$\begin{aligned}\cos 75° &= \cos 30°\cos 45° - \sin 30°\sin 45°\\ &= \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}-\frac{1}{2}\frac{1}{\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}\,.\end{aligned}$$

For the difference of two angles, we have

$$\begin{aligned}\sin(A-B) &= \sin A\cos B-\cos A\sin B\\ \cos(A-B) &= \cos A\cos B+\sin A\sin B\,.\end{aligned}\tag{4.2}$$

In (4.2), if we substitute $A = 45°$ and $B = 30°$, then we obtain

$$\begin{aligned}\sin 15° &= \sin 45°\cos 30° - \cos 45°\sin 30°\\ &= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

and

$$\begin{aligned}\cos 15° &= \cos 45°\cos 30° + \sin 45°\sin 30°\\ &= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}\,.\end{aligned}$$

Some basic identities involving the multiples and submultiples of angles are

$$\sin 2A = 2\sin A\cos A$$

and

$$\cos 2A = \cos^2 A - \sin^2 A \;=\; 2\cos^2 A - 1 = 1 - 2\sin^2 A\,. \tag{4.3}$$

It is useful to write the identities in (4.3) as

$$\begin{aligned} \sin\left(\frac{A}{2}\right) &= \frac{1}{2}\sqrt{2\,(1-\cos A)}\,, \\ \cos\left(\frac{A}{2}\right) &= \frac{1}{2}\sqrt{2\,(1+\cos A)}\,. \end{aligned} \tag{4.4}$$

If we choose $A = 45^\circ$, then

$$\sin\frac{45^\circ}{2} = \sqrt{\frac{1-\cos 45^\circ}{2}} = \frac{1}{2}\sqrt{2-\sqrt{2}} \tag{4.5}$$

and

$$\cos\frac{45^\circ}{2} = \sqrt{\frac{1+\cos 45^\circ}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}} \tag{4.6}$$

If we start with an exact value of $\cos A$, we can repeatedly use the identities in (4.4) and express the ratios of the sub-multiples of A using square-roots [20]. After k steps (k square-roots), we obtain the values of $\sin(A/2^k)$ in terms of the k-th root

$$\begin{aligned} 2\sin A &= \sqrt{2-2\cos(2A)} \\ &= \sqrt{2-\sqrt{2+2\cos(2^2A)}} \\ &= \sqrt{2-\sqrt{2+\sqrt{2+2\cos(2^3A)}}} \\ &= \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+2\cos(2^4A)}}}} \\ &= \cdots\,. \end{aligned} \tag{4.7}$$

The similar expression for $\cos(A/2^k)$ is

$$
\begin{aligned}
2\cos A &= \sqrt{2+2\cos(2A)} \\
&= \sqrt{2+\sqrt{2+2\cos(2^2A)}} \\
&= \sqrt{2+\sqrt{2+\sqrt{2+2\cos(2^3A)}}} \\
&= \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+2\cos(2^4A)}}}} \\
&= \cdots .
\end{aligned} \tag{4.8}
$$

Using the identities in (4.8)-(4.9), we obtain the chains for the ratios of the multiples of 3° $(\pi/60)$ in Table (2), the multiples of 9° $(\pi/20)$ in Table (3) and the multiples of 0.5625° $(\pi/32)$ in Table (4) respectively. The three Tables exhibit interesting patterns [21]. Table (3) has the additional feature, that the sines and cosines of multiples of 9° are expressed using the golden ratio, $\phi = (\sqrt{5}+1)/2$, which appears in diverse areas of mathematics and sciences [22]. A combination of identities involving sums/differences along with the identities for multiples/submultiples enable us to obtain a chain of exact values.

The same trigonometric ratio may look very different depending upon the identities used to derive it, but they are definitely equivalent. For instance, the representations of $\sin 9^\circ$ include

$$
\begin{aligned}
\sin 9^\circ &= \frac{1}{4}\left(\sqrt{8-2\sqrt{10+2\sqrt{5}}}\right), \\
&= \frac{1}{4}\left(\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}\right), \\
&= \frac{1}{4}\left(\frac{1}{2}\left(\sqrt{10}+\sqrt{2}\right)-\sqrt{5-\sqrt{5}}\right), \\
&= \frac{1}{2}\sqrt{2-2\sqrt{2+\phi}},
\end{aligned}
$$

where $\phi = (\sqrt{5}+1)/2$ is the golden ratio [22].

Repeated use of the identities in (4.1) leads to

$$
\begin{aligned}
\sin 3A &= 3\sin A - 4\sin^3 A \\
\cos 3A &= 4\cos^3 A - 3\cos A .
\end{aligned}
$$

We shall use the identities for the double and triple angles to find the trigonometric ratios of 18°, 36°, 54° and 72°. We choose, $\theta = 18^\circ = 90^\circ/5$ and write

$$\begin{aligned} 2\theta &= 90^\circ - 3\theta \\ \sin(2\theta) &= \sin(90^\circ - 3\theta) = \cos(3\theta) \\ 2\sin\theta\cos\theta &= 4\cos^3\theta - 3\cos\theta = \cos\theta(1 - 4\sin^2\theta)\,. \end{aligned}$$

This leads to the equation

$$4\sin^2\theta + 2\sin\theta - 1 = 0\,,$$

whose solution is

$$\sin 18^\circ = \cos 72^\circ = \frac{\sqrt{5}-1}{4}\,.$$

The value of $\cos 18^\circ = \sin 72^\circ$ can be calculated as follows

$$\cos 18^\circ = \sin 72^\circ = \sqrt{1 - \sin^2 18^\circ} = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}\,.$$

The trigonometric ratios of 36° are found using the double angle identities in (4.3)

$$\cos 36^\circ = \sin 54^\circ = 1 - 2\sin^2 18^\circ = \frac{\sqrt{5}+1}{4}\,.$$

The value of $\sin 36^\circ = \cos 54^\circ$ can be calculated as follows

$$\sin 36^\circ = \cos 54^\circ = \sqrt{1 - \cos^2 36^\circ} = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}\,.$$

In this Section, we made use of the basic trigonometric identities and quadratic equations. To proceed further, we need to use cubic and higher order equations.

5. Irrationality of trigonometric ratios

The irrationality of the trigonometric ratios is of keen interest. The first major result in this topic is the theorem due to Ivan Morton Niven in the year 1956.

Theorem 5.1. *Niven's Theorem: The only rational values of α in the interval $0^\circ \le \alpha \le 90^\circ$ for which the sine of α degrees is also a rational number are $\sin 0^\circ = 0$, $\sin 30^\circ = \frac{1}{2}$ and $\sin 90^\circ = 1$.*

The theorem appears in the Niven's books on irrational numbers [23, 24]. The theorem implies that for rational angles in degrees the rational values of the trigonometric ratios are $\cos\alpha\,,\sin\alpha \in \{0\,,\pm\frac{1}{2}\,,\pm 1\}$, $\sec\alpha\,,\csc\alpha \in \{\pm 1,\pm 2\}$ and $\tan\alpha\,,\cot\alpha \in \{0\,,\pm 1\}$. A straightforward proof of Niven's theorem with additional results is in the following theorem based on the tangent function [25].

Theorem 5.2. *Paolillo-Vincenzi Theorem: If α is rational in degrees, say $\alpha = (m/n)180^\circ$ for some rational number m/n, and $\tan^2(\alpha)$ is rational, then $\tan^2(\alpha) \in \{0, 1, \frac{1}{3}, 3\}$.*

Using Theorem (5.2) along with the trigonometric identities (such as $\cos^2\alpha = 1/(1+\tan^2\alpha)$, $\cos(2\alpha) = (1-\tan^2\alpha)/(1+\tan^2\alpha)$, $\sin(2\alpha) = 2\tan\alpha/(1+\tan^2\alpha)$ and $\sin^2\alpha = 1-\cos^2\alpha$), we conclude that $\cos^2(\alpha)\,,\sin^2(\alpha) \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. From this set, we conclude that $\cos\alpha\,,\sin\alpha \in \{0, \pm\frac{1}{2}, \pm 1\}$ and $\sec\alpha\,,\csc\alpha \in \{\pm 1, \pm 2\}$. A related study deals with the set of angles, whose squared trigonometric functions are rational [26].

The Gregory numbers are defined as

$$G_x \;=\; \tan^{-1}\left(\frac{1}{x}\right) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k+1)x^{2k+1}}\,,$$

where x is an integer or a rational number. For $x = 1$, $G_1 = \tan^{-1}(1) = 45^\circ(\pi/4)$ is a Gregory number. All Gregory numbers in degrees are irrational with the exception of the pair $G_{-1} = -45^\circ(-\pi/4)$ and $G_1 = 45^\circ(\pi/4)$. The $\tan^{-1}$ identities have been extensively used for calculating the value of π [27].

A trigonometric number is an irrational number produced by taking the sine or cosine of a rational number of degrees (if in radians, it is a rational multiple of π), with the exception of $\cos\alpha\,,\sin\alpha \in \{0, \pm\frac{1}{2}, \pm 1\}$. Any real number different from these exceptions is a trigonometric number if and only if it is the real part of a root of unity [28]. A stronger statement on the *rational linear independence* of trigonometric numbers is in the following theorem due to Arno Berger [29]

Theorem 5.3. *Berger Theorem: Let r_1 and r_2 to be two rational numbers such that either $r_1 - r_2$ and $r_1 + r_2$ is not an integer, then the three numbers 1, $\cos(r_1\pi)$ and $\cos(r_2\pi)$ are rationally independent.*

6. Concluding remarks

The subject of trigonometry is of immense importance within mathematics and across the sciences. Derivation of exact values of the trigonometric ratios is of keen interest. This is evident by the numerous recent publications. The derivation of the exact values is intimately tied to *number theory* and *algebraic geometry*. In this article, we saw the results from geometric methods along with some results using identities and quadratic equations. Tables of exact trigonometric ratios were presented and the patterns in them were illustrated. The power and limitations of the geometric methods for deriving the exact values of the trigonometric ratios, based on certain theorems were discussed in detail. The rational sets of trigonometric ratios of rational angles were also discussed. Results from number theory were presented wherever required.

Some of the numerical data can also be obtained using the *Microsoft Excel* [30]- [35]. One can alternately use the powerful symbolic packages, for instance the *Mathematica* [36,37]. MS Excel is useful in different areas of physics and mathematics [31]- [35]. It has also found applications in specific problems such as the study of quadratic surfaces in the laboratory [38]- [41]; resistor networks [42]- [45]; chemical physics [46]; and number theory [47,48].

Table 2: Exact Ratios: Multiples of 3° $(\pi/60)$

	Angle, θ			
S. No.	**Deg.**	**Rad.**	$\sin\theta$	$\cos\theta$
1	0°	0	0	1
2	3°	$\frac{\pi}{60}$	$\frac{1}{4}\left(\sqrt{8-\sqrt{3}-\sqrt{15}-\sqrt{10-2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8+\sqrt{3}+\sqrt{15}+\sqrt{10-2\sqrt{5}}}\right)$
3	6°	$\frac{\pi}{30}$	$\frac{1}{4}\left(\sqrt{9-\sqrt{5}-\sqrt{30+6\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{7+\sqrt{5}+\sqrt{30+6\sqrt{5}}}\right)$
4	9°	$\frac{\pi}{20}$	$\frac{1}{4}\left(\sqrt{8-2\sqrt{10+2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8+2\sqrt{10+2\sqrt{5}}}\right)$
5	12°	$\frac{\pi}{15}$	$\frac{1}{4}\left(\sqrt{7-\sqrt{5}-\sqrt{30-6\sqrt{5}}}\right)$	$\frac{1}{8}\left(-1+\sqrt{5}+\sqrt{30+6\sqrt{5}}\right)$
6	15°	$\frac{\pi}{12}$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$
7	18°	$\frac{\pi}{10}$	$\frac{1}{4}(\sqrt{5}-1)$	$\frac{1}{4}(\sqrt{10+2\sqrt{5}})$
8	21°	$\frac{7\pi}{60}$	$\frac{1}{4}\left(\sqrt{8+\sqrt{3}-\sqrt{15}-\sqrt{10+2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8-\sqrt{3}+\sqrt{15}+\sqrt{10+2\sqrt{5}}}\right)$
9	24°	$\frac{2\pi}{15}$	$\frac{1}{4}\left(\sqrt{7+\sqrt{5}-\sqrt{30+6\sqrt{5}}}\right)$	$\frac{1}{8}\left(1+\sqrt{5}+\sqrt{30-6\sqrt{5}}\right)$
10	27°	$\frac{3\pi}{20}$	$\frac{1}{4}\left(\sqrt{8-2\sqrt{10-2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8+2\sqrt{10-2\sqrt{5}}}\right)$

Table 2: Exact Ratios: Multiples of 3° $(\pi/60)$, *continued*

S. No.	**Deg.**	**Rad.**	$\sin\theta$	$\cos\theta$
11	30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$
12	33°	$\frac{11\pi}{60}$	$\frac{1}{4}\left(\sqrt{8-\sqrt{3}-\sqrt{15}+\sqrt{10-2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8+\sqrt{3}+\sqrt{15}-\sqrt{10-2\sqrt{5}}}\right)$
13	36°	$\frac{\pi}{5}$	$\frac{1}{4}(\sqrt{10-2\sqrt{5}})$	$\frac{1}{4}(\sqrt{5}+1)$
14	39°	$\frac{13\pi}{60}$	$\frac{1}{4}\left(\sqrt{8-\sqrt{3}+\sqrt{15}-\sqrt{10+2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8+\sqrt{3}-\sqrt{15}+\sqrt{10+2\sqrt{5}}}\right)$
15	42°	$\frac{7\pi}{30}$	$\frac{1}{8}\left(1-\sqrt{5}+\sqrt{30+6\sqrt{5}}\right)$	$\frac{1}{4}\left(\sqrt{7-\sqrt{5}+\sqrt{30-6\sqrt{5}}}\right)$
16	45°	$\frac{\pi}{4}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$
17	48°	$\frac{4\pi}{15}$	$\frac{1}{4}\left(\sqrt{7-\sqrt{5}+\sqrt{30-6\sqrt{5}}}\right)$	$\frac{1}{8}\left(1-\sqrt{5}+\sqrt{30+6\sqrt{5}}\right)$
18	51°	$\frac{17\pi}{60}$	$\frac{1}{4}\left(\sqrt{8+\sqrt{3}-\sqrt{15}+\sqrt{10+2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8-\sqrt{3}+\sqrt{15}-\sqrt{10+2\sqrt{5}}}\right)$
19	54°	$\frac{3\pi}{10}$	$\frac{1}{4}(\sqrt{5}+1)$	$\frac{1}{4}(\sqrt{10-2\sqrt{5}})$
20	57°	$\frac{10\pi}{60}$	$\frac{1}{4}\left(\sqrt{8+\sqrt{3}+\sqrt{15}-\sqrt{10-2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8-\sqrt{3}-\sqrt{15}+\sqrt{10-2\sqrt{5}}}\right)$
21	60°	$\frac{\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$
22	63°	$\frac{7\pi}{20}$	$\frac{1}{4}\left(\sqrt{8+2\sqrt{10-2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8-2\sqrt{10-2\sqrt{5}}}\right)$
23	66°	$\frac{11\pi}{30}$	$\frac{1}{8}\left(1+\sqrt{5}+\sqrt{30-6\sqrt{5}}\right)$	$\frac{1}{4}\left(\sqrt{7+\sqrt{5}-\sqrt{30+6\sqrt{5}}}\right)$
24	69°	$\frac{23\pi}{60}$	$\frac{1}{4}\left(\sqrt{8-\sqrt{3}+\sqrt{15}+\sqrt{10+2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8+\sqrt{3}-\sqrt{15}-\sqrt{10+2\sqrt{5}}}\right)$
25	72°	$\frac{2\pi}{5}$	$\frac{1}{4}(\sqrt{10+2\sqrt{5}})$	$\frac{1}{4}(\sqrt{5}-1)$
26	75°	$\frac{5\pi}{12}$	$\frac{1}{4}(\sqrt{6}+\sqrt{2})$	$\frac{1}{4}(\sqrt{6}-\sqrt{2})$
27	78°	$\frac{13\pi}{30}$	$\frac{1}{8}\left(-1+\sqrt{5}+\sqrt{30+6\sqrt{5}}\right)$	$\frac{1}{4}\left(\sqrt{7-\sqrt{5}-\sqrt{30-6\sqrt{5}}}\right)$
28	81°	$\frac{9\pi}{20}$	$\frac{1}{4}\left(\sqrt{8+2\sqrt{10+2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8-2\sqrt{10+2\sqrt{5}}}\right)$
29	84°	$\frac{7\pi}{15}$	$\frac{1}{4}\left(\sqrt{7+\sqrt{5}+\sqrt{30+6\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{9-\sqrt{5}-\sqrt{30+6\sqrt{5}}}\right)$
30	87°	$\frac{29\pi}{60}$	$\frac{1}{4}\left(\sqrt{8+\sqrt{3}+\sqrt{15}+\sqrt{10-2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8-\sqrt{3}-\sqrt{15}-\sqrt{10-2\sqrt{5}}}\right)$
31	90°	$\frac{\pi}{2}$	1	0

Table 3: Exact Ratios in terms of the Golden Ratio, $\phi = \frac{1}{2}(\sqrt{5}+1)$

	Angle, θ			
S. No.	**Deg.**	**Rad.**	$\sin\theta$	$\cos\theta$
1	9°	$\frac{\pi}{20}$	$\frac{1}{4}\left(\sqrt{8-2\sqrt{10+2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8+2\sqrt{10+2\sqrt{5}}}\right)$
			$\frac{1}{2}\sqrt{2-2\sqrt{2+\phi}}$	$\frac{1}{2}\sqrt{2+2\sqrt{2+\phi}}$

Table 3: Exact Ratios using the Golden Ratio $\phi = \frac{1}{2}(\sqrt{5}+1)$, *continued*

S. No.	**Deg.**	**Rad.**	$\sin\theta$	$\cos\theta$
2	18°	$\frac{\pi}{10}$	$\frac{1}{4}(\sqrt{5}-1)$	$\frac{1}{4}(\sqrt{10+2\sqrt{5}})$
			$\frac{1}{2}(\phi-1)$	$\frac{1}{2}\sqrt{2+\phi}$
3	27°	$\frac{3\pi}{20}$	$\frac{1}{4}\left(\sqrt{8-2\sqrt{10-2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8+2\sqrt{10-2\sqrt{5}}}\right)$
			$\frac{1}{2}\sqrt{2-\sqrt{3-\phi}}$	$\frac{1}{2}\sqrt{2+\sqrt{3-\phi}}$
4	36°	$\frac{\pi}{5}$	$\frac{1}{4}(\sqrt{10-2\sqrt{5}})$	$\frac{1}{4}(\sqrt{5}+1)$
			$\frac{1}{2}\sqrt{3-\phi}$	$\frac{1}{2}\phi$
5	54°	$\frac{3\pi}{10}$	$\frac{1}{4}(\sqrt{5}+1)$	$\frac{1}{4}(\sqrt{10-2\sqrt{5}})$
			$\frac{1}{2}\phi$	$\frac{1}{2}\sqrt{3-\phi}$
6	63°	$\frac{7\pi}{20}$	$\frac{1}{4}\left(\sqrt{8+2\sqrt{10-2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8-2\sqrt{10-2\sqrt{5}}}\right)$
			$\frac{1}{2}\sqrt{2+\sqrt{3-\phi}}$	$\frac{1}{2}\sqrt{2-\sqrt{3-\phi}}$
7	72°	$\frac{2\pi}{5}$	$\frac{1}{4}(\sqrt{10+2\sqrt{5}})$	$\frac{1}{4}(\sqrt{5}-1)$
			$\frac{1}{2}\sqrt{2+\phi}$	$\frac{1}{2}(\phi-1)$
8	81°	$\frac{9\pi}{20}$	$\frac{1}{4}\left(\sqrt{8+2\sqrt{10+2\sqrt{5}}}\right)$	$\frac{1}{4}\left(\sqrt{8-2\sqrt{10+2\sqrt{5}}}\right)$
			$\frac{1}{2}\sqrt{2+2\sqrt{2+\phi}}$	$\frac{1}{2}\sqrt{2-2\sqrt{2+\phi}}$

Table 4: Exact Ratios: Multiples of 0.5625° ($\pi/32$) using nested square-roots of 2

	Angle, θ			
S. No.	**Deg.**	**Rad.**	$2\sin\theta$	$2\cos\theta$
1	0.5625°	$\frac{\pi}{32}$	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$
2	11.25°	$\frac{\pi}{16}$	$\sqrt{2-\sqrt{2+\sqrt{2}}}$	$\sqrt{2+\sqrt{2+\sqrt{2}}}$
3	16.875°	$\frac{3\pi}{32}$	$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}$	$\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2}}}}$
4	22.5°	$\frac{\pi}{8}$	$\sqrt{2-\sqrt{2}}$	$\sqrt{2+\sqrt{2}}$
5	28.125°	$\frac{5\pi}{32}$	$\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}$	$\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2}}}}$
6	33.75°	$\frac{3\pi}{16}$	$\sqrt{2-\sqrt{2-\sqrt{2}}}$	$\sqrt{2+\sqrt{2-\sqrt{2}}}$
7	39.375°	$\frac{7\pi}{32}$	$\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2}}}}$	$\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2}}}}$
8	45°	$\frac{\pi}{4}$	$\sqrt{2}$	$\sqrt{2}$
9	50.625°	$\frac{9\pi}{32}$	$\sqrt{2+\sqrt{2-\sqrt{2+\sqrt{2}}}}$	$\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2}}}}$
10	56.25°	$\frac{5\pi}{16}$	$\sqrt{2+\sqrt{2-\sqrt{2}}}$	$\sqrt{2-\sqrt{2-\sqrt{2}}}$
11	61.875°	$\frac{11\pi}{32}$	$\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2}}}}$	$\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}$

Table 4: Exact Ratios: Multiples of 0.5625° $(\pi/32)$ using nested square-roots of 2, *continued*

S. No.	**Deg.**	**Rad.**	$2\sin\theta$	$2\cos\theta$
12	67.5°	$\frac{3\pi}{8}$	$\sqrt{2+\sqrt{2}}$	$\sqrt{2-\sqrt{2}}$
13	73.125°	$\frac{13\pi}{32}$	$\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{2}}}}$	$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}$
14	78.75°	$\frac{7\pi}{16}$	$\sqrt{2+\sqrt{2+\sqrt{2}}}$	$\sqrt{2-\sqrt{2+\sqrt{2}}}$
15	84.375°	$\frac{15\pi}{32}$	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$	$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2}}}}$

ACKNOWLEDGMENT

The author is indebted to Prof. Ramaswamy Jagannathan (Institute of Mathematical Sciences, Chennai, India) for the training in the field accelerator physics, particularly, in the emerging field of Quantum theory of charged-particle beam optics [49]- [51], which was the topic of Author's doctoral thesis. So, also in the field of *Quantum Algebras* [52]. He gave the brilliant suggestion to employ the Foldy-Wouthuysen transformation technique to investigate the scalar optics (Helmholtz optics) and vector optics (Maxwell optics). The new formalism of optics using quantum methodologies was selected for the *Year in Optics 2016* by the *Optical Society* [of America]. The summary of the two selected papers [53,54] is available in the special issue of the *Optics* & *Photonics News* [55].

REFERENCES

[1] For instance, see *Mathematics Textbook for Class XI and Class XII*, National Council of Educational Research and Training, New Delhi, India, 2020.

[2] S. L. LONEY: *Plane Trigonometry* Part-1, AITBS Publishers, India, 2019.

[3] S. L. LONEY: *Plane Trigonometry* Part-2, AITBS Publishers, India, 2019.

[4] E. MAOR: *Trigonometric Delights*, Princeton University Press, 2013.

[5] M. R. SPIEGEL, J. LIU: *Mathematical Handbook of Formulas and Tables*, Schaum's Outlines, McGraw-Hill, New York, 1999.

[6] I. S. GRADSHTEYN, I. M. RYZHIK: *Table of Integrals, Series, and Products*, Elsevier, Amsterdam, 2007.

[7] M. ABRAMOWITZ, I. A. STEGUN: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 2014.

[8] B. BRADIE: *Exact Values for the Sine and Cosine of Multiples of* 18° *– A Geometric Approach*, College Mathematics Journal. **33** (2002), 318–319.

[9] D. S. AILLES: *Triangles and trigonometry*, Mathematics Teacher, **64** (1971), 562.

[10] J. S. CALCUT: *Grade School Triangles*, The American Mathematical Monthly, **117** (2010), 673–685.

[11] A. WARES: *Geometry and Trigonometry Interplay*, The Mathematics Teacher, **112** (2019), 400–402.

[12] G. H. HARDY, E. M. WRIGHT: *An Introduction to the Theory of Numbers*, Oxford University Press, United Kingdom, 2008.

[13] J. H. CONWAY, R. K. GUY: *Book of Numbers*, Springer, Germany, 1996.

[14] T. M. APOSTOL: *Introduction to Analytic Number Theory*, Springer, Germany, 1995.

[15] N. J. A. SLOANE (EDITOR): *The On-Line Encyclopedia of Integer Sequences*, 2010, http://oeis.org/.

[16] N. J. A. SLOANE: Sequence A000215: 3, 5, 17, 257, 65537, 4294967297, $\ldots$, *Fermat numbers:* $a(n) = 2^{(2^n)} + 1$, OEIS Foundation Inc., 2011, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/A000215.

[17] N. J. A. SLOANE, D. W. WILSON: Sequence A019434: 3, 5, 17, 257, 65537 (only five known terms), *Fermat primes: primes of the form* $2^{(2^k)} + 1$*, for some* $k \geq 0$, OEIS Foundation Inc., 2011, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/A019434.

[18] N. J. A. SLOANE, R. K. GUY: Sequence A003401: 1, 2, 3, 4, 5, 6, 8, 10, $\ldots$, *Numbers of edges of regular polygons constructible with ruler (or, more precisely, an unmarked straightedge) and compass*, OEIS Foundation Inc., 2011, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/A003401.

[19] N. J. A. SLOANE, B. KISACANIN: Sequence A004169: 7, 9, 11, 13, 14, 18, 19, 21, 22, 23, 25, $\ldots$, *Values of* n *for which a regular polygon with* n *sides cannot be constructed with ruler and compass*, OEIS Foundation Inc., 2011, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/A004169.

[20] J. M. H. PETERS: *Some Trigonometric Ratios in Surd Form*, The Mathematical Gazette, **66** (1982), 296–299.

[21] L. D. SERVI: *Nested Square Roots of 2*, The American Mathematical Monthly, **110** (2003), 326–330.

[22] R. A. DUNLAP: *Golden Ratio and Fibonacci Numbers*, World Scientific, 1997.

[23] I. NIVEN: *Irrational Numbers*, John Wiley and Sons, New York, 1956.

[24] I. NIVEN: *Numbers: Rational and Irrational*, The Mathematical Association of America, 1961.

[25] B. PAOLILLO, G. VINCENZI: *An elementary proof of Niven's theorem via the tangent function*, International J. of Mathematical Education in Science and Technology, **51** (2020), 1–6.

[26] J. H. CONWAY, C. RADIN, L. SADUN: *On Angles Whose Squared Trigonometric Functions Are Rational*, Discrete & Computational Geometry, **22** (1999), 321—332.

[27] J. S. CALCUT: *Gaussian Integers and Arctangent Identities for π*, The American Mathematical Monthly, **116** (2009), 515–530.

[28] L. ZHOU, L. MARKOV: *Recurrent Proofs of the Irrationality of Certain Trigonometric Values, The American Mathematical Monthly*, **117** (2010), 360–362.

[29] A. BERGER: *On linear independence of trigonometric numbers*, Carpathian Journal of Mathematics, **34** (2018), 157–166.

[30] *Microsoft EXCEL*, 2015.

[31] F. K. AL-RAWAHI, S. A. KHAN, A. HUQ: *Microsoft Excel in the mathematics classroom: a case study*, The Second Annual Conference for Middle East Teachers of Mathematics, Science and Computing, The Petroleum Institute, Abu Dhabi (2006), 131–134.

[32] S. A. KHAN: *Microsoft Excel in the physics classroom*, The Third Annual Conference for Middle East Teachers of Mathematics, Science and Computing, The Petroleum Institute, Abu Dhabi (2007), 171–175.

[33] S. A. KHAN: Data analysis using Microsoft Excel in the physics laboratory, *Bulletin of the IAPT*, **24** (2007), 184–186.

[34] S. A. KHAN: *Doing numerical calculus using Microsoft EXCEL*, Indian J. of Science and Technology, **9** (2016), 1–5.

[35] S. A. KHAN: *Microsoft Excel for Numerical Calculus*, Focus on Calculus, Nova Science Publishers, New York (2020), 177–201.

[36] *MATHEMATICA*, Wolfgang Research, Inc., 2015.

[37] N. BOCCARA: *Essentials of Mathematica with Applications to Mathematics and Physics*, Springer, Germany, 2007.

[38] S. A. KHAN: *Cylindrometer*, The Physics Teacher, **48** (2010), 607.

[39] S.A. KHAN: *Coordinate geometric approach to spherometer*, Bulletin of the IAPT, **5** (2013), 139–142.

[40] S. A. KHAN: *Coordinate geometric generalization of the spherometer and cylindrometer*, Advances in Engineering Research, **10**, Nova Science Publishers, New York, 2015, 163–190.

[41] S. A. KHAN: *Coordinate geometric generalization of the spherometer*, Far East J. of Mathematical Sciences, **101** (2017), 619–642.

[42] S. A. KHAN: *Farey sequences and resistor networks*, Mathematical Sciences - Proc. of the Indian Academy of Sciences, **122** (2012), 153–182.

[43] S. A. KHAN: *How many equivalent resistances?*, Resonance Journal of Science Education, **17** (2012), 468–475.

[44] S. A. KHAN: *Number theory and resistor networks*, Resistors: Theory of Operation, Behavior and Safety Regulations, Nova Science Publishers, New York, 2013, 99–154.

[45] S. A. KHAN: *Beginning to count the number of equivalent resistances*, Indian J. of Science and Technology, **9** (2016), 1–7.

[46] S. A. KHAN, F. A. KHAN: *Phenomenon of motion of salt along the walls of the container*, International J.of Current Engineering and Technology, **5** (2015), 368–370.

[47] S. A. KHAN: *Primes in geometric-arithmetic progression*, Global J. of Pure and Applied Mathematics, **12** (2016), 1161–1180.

[48] S. A. KHAN: *Sums of the Powers of Reciprocals of Polygonal Numbers*, International Journal of Applied Mathematics, **33** (2020), 265–282.

[49] S. A. KHAN: *E.C.G. Sudarshan and the quantum mechanics of charged-particle beam optics*, Current Science, **115** (2018), 1813–1814.

[50] R. JAGANNATHAN, S. A. KHAN: *Quantum Mechanics of Charged Particle Beam Optics: Understanding Devices from Electron Microscopes to Particle Accelerators*, CRC Press, Taylor & Francis, Boca Raton, 2019.

[51] P. W. HAWKES: *Dirac, c and a Supper date*, Ultramicroscopy, **213** (2020), 112981.

[52] R. JAGANNATHAN, S.A. KHAN: *On the deformed oscillator and the deformed derivative associated with the Tsallis q-exponential*, International Journal of Theoretical Physics, **59**(8), (2020), 2647–2669.

[53] S. A. KHAN: *Quantum methodologies in Helmholtz optics*, Optik-International J. for Light and Electron Optics, **127** (2016), 9798–9809.

[54] S. A. KHAN: *Passage from scalar to vector optics and the Mukunda-Simon-Sudarshan theory for paraxial systems*, J. of Modern Optics, **63** (2016), 1652–1660.

[55] S. A. KHAN: *Quantum methods in light beam optics*, Optics & Photonics News, **27** (2016), 47.

DEPARTMENT OF MATHEMATICS AND SCIENCES
COLLEGE OF ARTS AND APPLIED SCIENCES
DHOFAR UNIVERSITY
SALALAH – 211, SULTANATE OF OMAN
Email address: rohelakhan@yahoo.com
Email address: http://orcid.org/0000-0003-1264-2302