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ON TOPOLINE SET-INDEXERS OF GRAPHS

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ABSTRACT. This paper investigates the topoline set indexers of certain complete graphs and cycles. The topoline numbers of certain classes of triangular books and some complete bipartite graphs are also obtained.

1. INTRODUCTION

The notions of set valuations and set-indexers of graphs were introduced by B.D.Acharya [1]. Set valuation of a graph is an assignment of the subsets of a given nonempty set to the vertices and/or edges of a graph. Acharya also propounded the idea of set indexing number of a graph. Further, introducing the concept of topological set-indexers (t-set indexers) in [2], he established a link between Graph Theory and Point Set Topology. Motivating a lot of investigations in this area, he also introduced the concept of topologically set-graceful graphs in [3].

Later, U. Thomas and S. C. Mathew [5] introduced the concept of topoline set-indexers analogous to that of a topological set-indexer. Unlike topological set-indexers, not all graphs have topoline set-indexers. This caused the origin of topoline graphs admitting a topology on the edge set of the graph. It has been noted in [5] that set-graceful graphs form a proper subfamily of topoline graphs. Further it is conjectured that the complete graph K_n; $n > 1$ is topoline only if it is set-graceful. But, M. Mollard and C. Payan [4] have proved that the

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complete graph K_n is set-graceful if and only if $n \in \{2, 3, 6\}$. So the conjecture states that, among complete graphs, only K_2, K_3 and K_6 are topoline. This paper investigates the topoline set indexers of the complete graphs K_4, K_5, K_7 and K_8 and establishes that they are not topoline thereby affirming the above conjecture. It has already been proved in [5] that cycles C_3, C_4 and C_7 are topoline. Here it is proved that C_5 and C_6 are not topoline. Further the topoline numbers of certain classes of triangular books and complete bipartite graphs are also obtained.

2. PRELIMINARIES

B. D. Acharya introduced the notion of a set-indexer of a graph as follows: Let $G = (V, E)$ be a graph and X be a nonempty set. Then a mappping $f : V \to 2^X$ or $f : E \to 2^X$ or $f : V \cup E \to 2^X$ is called a set-valuation or set-assignment of the vertices or edges or both. It is proved in [1] that every graph G has a set-valuation.

Definition 2.1. Let $G = (V, E)$ be a graph and X be a nonempty set. Then a set-valuation $f : V \cup E \to 2^X$ is called a set-indexer of G if

(1) $f(uv) = f(u) \oplus f(v)$ where $\oplus$ is the symmetric difference and
(2) the restrictive maps $f|_V$ and $f|_E$ are both injective.

In this case, X is called an indexing set of G. A graph can have many indexing sets and the minimum of the cardinalities of the indexing sets is said to be the set-indexing number of G, denoted by $\gamma(G)$. It is also proved in [1] that every graph G has a set-indexer. A graph G is set-graceful if $\gamma(G) = \log_2(|E| + 1)$ and the corresponding set-indexer is called set-graceful labeling and it is an optimal set-indexer of G.

A set-indexer f of a graph G with indexing set X is said to be a topological set-indexer (t-set indexer) if $f(V)$ is a topology on X and X is called the topological indexing set (t-indexing set) of G. The minimum number among the cardinalities of such topological indexing sets is said to be the topological number (t - number) of G and is denoted by $\tau(G)$.

Definition 2.2. [5] A set-indexer $f : V \cup E \to 2^X$ of a nonempty graph G is said to be a topoline set-indexer, if $f(E) \cup \{\emptyset\}$ is a topology on X and X is called the topoline indexing set of G.

The minimum among the cardinalities of such topoline indexing sets is caled the topoline number of G and is denoted by $\tau_e(G)$. A nonempty graph G is said to be topoline if it has a topoline set-indexer. The following results can be seen in [5].

Theorem 2.1. For a topoline graph $G = (V, E)$, $\lceil log_2(|E| + 1)\rceil \leq \tau_e(G)$.

Theorem 2.2. The complete bipartite graph $K_{m,n}$ is topoline.

Theorem 2.3. Let G be a set graceful tree. Then $G \vee N_n$ is topoline.

It is proved in [6] that

Theorem 2.4. For $n \geq 3$, there is no topology on n points having k open sets, where $3 \cdot 2^{(n-2)} < k < 2^n$.

Let $m(k)$ denote the minimum number of points needed to make a topology having k open sets. Then from [7] we have $m(11) = 5$, $m(13) = 5$, $m(19) = 6$, $m(21) = 6$ and $m(35) = 7$.

3. Topoline Set-indexers of Complete Graphs

Among complete graphs, K_2, K_3 and K_6 are known to be topoline. The fact that K_4 is not topoline is proved in [5]. In that proof it is assumed that $(A_1 \oplus A_3) \cap (A_2 \oplus A_3) = (A_1 \cap A_2) \oplus A_3$. But, this is wrong as can be easily verified by taking $A_1 = \{1, 2, 3, 4\}, A_2 = \{2, 3, 5, 6\}$ and $A_3 = \{3, 4, 5, 7\}$. However, the result K_4 is not topoline is true and we give an alternate proof for the same in Theorem 3.1.

Lemma 3.1. Let X be a nonempty set and A, B be subsets of X such that $\tau = \{X, \emptyset, A, B, A^c, B^c, A \oplus B\}$ consists of seven distinct subsets of X. Then τ is not a topology on X.

Proof. Suppose τ is a topology on X. Then $A \cup B$ must be one of the seven elements in τ. Obviously, $A \cup B$ cannot be $\emptyset$, A^c or B^c. Then there arise the following four possibilities:

Case I: $A \cup B = X$. This leads to the conclusion that $A \cap B \notin \tau$, a contradiction.

Case II: $A \cup B = A$. This leads to the conclusion that $B \cup A^c \notin$, a contradiction.

Case III: $A \cup B = B$. This leads to the contradictory conclusion that $A \cup B^c \notin \tau$.

Case IV: $A \cup B = A \oplus B$. This leads to the conclusion that $A^c \cap B^c \notin \tau$, a contradiction.

Hence, τ can not be a topology on X. □

Theorem 3.1. The complete graph K_4 is not topoline.

Proof. Suppose K_4 is topoline. Then there exists a nonempty set X and a set-indexer f which assigns subsets A_1, A_2, A_3 and A_4 of X to the four vertices of K_4 such that $\tau = f(E) \cup \{\emptyset\} = \{A_1 \oplus A_2, A_1 \oplus A_3, A_1 \oplus A_4, A_2 \oplus A_3, A_2 \oplus A_4, A_3 \oplus A_4, \emptyset\}$ is a topology on X.

Without loss of generality, we may assume that $A_1 \oplus A_2 = X$. ie, $A_2 = A_1^c$. Then the edge labels are given by $f(E) = \{X, A_1 \oplus A_3, A_1 \oplus A_4, A_1^c \oplus A_3, A_1^c \oplus A_4, A_3 \oplus A_4\}$ so that τ={X, A, B, A^c, B^c, $A \oplus B$, $\emptyset$} where A = $A_1 \oplus A_3$ and B = $A_1 \oplus A_4$. But by Lemma 3.1, τ can not be a topology on X. Consequently, K_4 is not topoline. □

Analogously it can be shown that the complete graphs, K_5, K_7 and K_8 are not topoline and we state those results below without proof.

Lemma 3.2. Let X be a nonempty set and A, B, C be subsets of X such that $\tau = \{X, \emptyset, A, B, C, A^c, B^c, C^c, A \oplus B, A \oplus C, B \oplus C\}$ consists of 11 distinct subsets of X. Then τ is not a topology on X.

Theorem 3.2. The complete graph K_5 is not topoline.

Lemma 3.3. Let X be a nonempty set and A, B, C, D, E be five subsets of X such that τ={ X, A, B, C, D, E, A^c, B^c, C^c, D^c, E^c, $A \oplus B$, $A \oplus C$, $A \oplus D$, $A \oplus E$, $B \oplus C$, $B \oplus D$, $B \oplus E$, $C \oplus D$, $C \oplus E$, $D \oplus E$, $\emptyset$ } consists of 22 distinct subsets of X. Then τ is not a topology on X.

Theorem 3.3. The complete graph K_7 is not topoline.

Lemma 3.4. Let X be a nonempty set and A, B, C, D, E, F be six subsets of X such that τ = {X, A, B, C, D, E, F, A^c, B^c, C^c, D^c, E^c,F^c, $A \oplus B$, $A \oplus C$, $A \oplus D$, $A \oplus E$, $A \oplus F$, $B \oplus C$, $B \oplus D$, $B \oplus E$, $B \oplus F$, $C \oplus D$, $C \oplus E$, $C \oplus F$, $D \oplus E$, $D \oplus F$, $E \oplus F$, $\emptyset$} consists of 29 distinct subsets of X. Then τ is not a topology on X.

Theorem 3.4. The complete graph K_8 is not topoline.

4. TOPOLINE SET-INDEXERS OF CYCLES

It is already noticed that the cycles C_3, C_4 and C_7 are topoline and here we prove that C_5 and C_6 are not topoline. The following lemma is straightforward.

Lemma 4.1. If A_1, A_2, A_3, $\cdots$, A_n are the labels assigned by set-indexer f to the vertices of a cycle C_n, then the symmetric difference of all the edge labels is $\emptyset$.

Theorem 4.1. C_5 is not topoline.

Proof. If possible, let C_5 be topoline. Then we can find a topoline set-indexer$f$ of C_5 and a set X such that $\tau = f(E) \cup \{\emptyset\}$ is a topology on X. The edge labels will be of the form $f(E) = \{A_1,\ A_2,\ A_3,\ A_4,\ X\}$ and by Lemma 4.1, $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = \emptyset$.

Claim: If $\tau = \{\emptyset,\ A_1,\ A_2,\ A_3,\ A_4,\ X\}$ be a point set topology on X such that $\mid A_1 \mid \leq \mid A_2 \mid \leq \mid A_3 \mid \leq \mid A_4 \mid < \mid X \mid$ then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X \neq \emptyset$.

To prove the claim we consider the following cases.

Case I: Let $\mid A_1 \mid = \mid A_2 \mid = \mid A_3 \mid = \mid A_4 \mid$. Then $A_1 \cap A_2 = \emptyset$. For, if $A_1 \cap A_2 \neq \emptyset$, then $\mid A_1 \cap A_2 \mid < \mid A_1 \mid$, which is a contradiction. Similarly, $A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$. Again, $A_1 \cup A_2$ cannot be $\emptyset, A_1, A_2, A_3$ and A_4 by assumption. Now, if $A_1 \cup A_2 = X$, then $A_3 \subseteq A_1 \cup A_2$, which is a contradiction to the fact that both $A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$. Thus this case is impossible.

Case II: Let $\mid A_1 \mid = \mid A_2 \mid = \mid A_3 \mid < \mid A_4 \mid$. This case is also impossible by the reason given in CaseI.

Case III: Let $\mid A_1 \mid = \mid A_2 \mid < \mid A_3 \mid = \mid A_4 \mid$. Then $A_1 \cap A_2 = \emptyset$. (If $A_1 \cap A_2 \neq \emptyset$, then $\mid A_1 \cap A_2 \mid < \mid A_1 \mid$, which is a contradiction). Also $A_1 \cup A_2$ must be in τ and cannot be $\emptyset$, A_1 or A_2 by assumption.

If $A_1 \cup A_2 = A_3$ then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_4^C \neq \emptyset$.

If $A_1 \cup A_2 = A_4$ then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_3^C \neq \emptyset$.

If $A_1 \cup A_2 = X$ then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_3 \oplus A_4 \neq \emptyset$.

Case IV: Let $\mid A_1 \mid = \mid A_2 \mid < \mid A_3 \mid < \mid A_4 \mid$. Then $A_1 \cap A_2 = \emptyset$. (If $A_1 \cap A_2 \neq \emptyset$, then $\mid A_1 \cap A_2 \mid < \mid A_1 \mid$, which is a contradiction). Also $A_1 \cup A_2$ must be in τ and cannot be $\emptyset$, A_1 or A_2 by assumption.

If $A_1 \cup A_2 = A_3$, A_3 or X then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X \neq \emptyset$, as described in Case III.

Case V: Let $| A_1 |<| A_2 |=| A_3 |=| A_4 |$.
Then $A_2 \cap A_3$ must be in τ and cannot be A_2, A_3, A_4 or X by assumption.
If $A_2 \cap A_3 = \emptyset$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_1 \oplus A_4 \neq \emptyset$.
If $A_2 \cap A_3 = A_1$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_4 \neq \emptyset$.
Case VI: Let $| A_1 |<| A_2 |=| A_3 |<| A_4 |$. Then $A_2 \cup A_3$ must be in τ and cannot be $\emptyset$, A_1, A_2 or A_3 by assumption.
Also $A_2 \cap A_3$ must be in τ and cannot be A_2, A_3, A_4 or X by assumption.
Case VI (A): $A_2 \cup A_3 = A_4$.
If $A_2 \cap A_3 = A_1$, then $A_4 = X$, which is a contradiction.
If $A_2 \cap A_3 = \emptyset$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_1^C \neq \emptyset$.
Case VI (B): $A_2 \cup A_3 = X$.
If $A_2 \cap A_3 = \emptyset$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_1 \oplus A_4 \neq \emptyset$.
If $A_2 \cap A_3 = A_1$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_4 \neq \emptyset$.
Case VII: Let $| A_1 |<| A_2 |=| A_3 |<| A_4 |$.
Then $A_3 \cup A_4 = X$. Also $A_3 \cap A_4$ must be in τ and cannot be A_3, A_4 or X by assumption.
If $A_3 \cap A_4 = \emptyset$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_1 \oplus A_2 \neq \emptyset$.
If $A_3 \cap A_4 = A_1$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_2 \neq \emptyset$.
If $A_3 \cap A_4 = A_2$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_1 \neq \emptyset$.
Case VIII: Let $| A_1 |<| A_2 |<| A_3 |<| A_4 |$.
Then $A_1 \cap A_2$ must be in τ and cannot be A_2, A_3,A_4 or X by assumption.
Case VIII (A): $A_1 \cap A_2 = \emptyset$.
Then $A_1 \cup A_2$ must be in τ and cannot be $\emptyset$, A_1 or A_2.
If $A_1 \cup A_2 = A_3$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_4^C \neq \emptyset$.
If $A_1 \cup A_2 = A_4$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_3^C \neq \emptyset$.
If $A_1 \cup A_2 = X$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_3 \oplus A_4 \neq \emptyset$.
Case VIII (B): $A_1 \cap A_2 = A_1$.
Then $A_1 \cup A_2 = A_2$. Also $A_2 \cap A_3$ must be in τ and cannot be A_3, A_4 or X.
Case VIII (B) (a): $A_2 \cap A_3 = \emptyset$.
If $A_2 \cup A_3 = A_4$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_1^C \neq \emptyset$.
If $A_2 \cup A_3 = X$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_1 \oplus A_4 \neq \emptyset$.
Case VIII (B) (b): $A_2 \cap A_3 = A_1$.
If $A_2 \cup A_3 = A_3$, then $A_2 \cap A_3 = A_2$, which is a contradiction.
If $A_2 \cup A_3 = A_4$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = X \neq \emptyset$.
If $A_2 \cup A_3 = X$, then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_4^C \neq \emptyset$.

Case VIII (B) (c): $A_2 \cap A_3 = A_2$.
Then $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X = A_2 \oplus A_3 \neq \emptyset$.

Thus in all possible cases $A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus X \neq \emptyset$ and the claim is proved. Hence C_5 is not topoline. □

Analogously it can be proved that

Theorem 4.2. *C_6 is not topoline.*

5. Topoline numbers of some graphs

By Theorem 2.3 it follows that all triangular books are topoline. The topoline numbers of certain classes of triangular books and some complete bipartite graphs are obtained here.

Theorem 5.1. $\tau_e(K_2 \vee N_{2^n-1}) = n + 1$.

Proof. Let $X = \{x_1, x_2, \ldots, x_{n+1}\}$ and $Y = \{x_2, x_3, \ldots, x_{n+1}\}$. Now, a topoline set-indexer for $G = K_2 \vee N_{2^n-1}$ with indexing set X can be obtained as follows: Assign $\emptyset$ and $\{x_1\}$ to the vertices of K_2. Assign all $2^n - 1$ non-empty subsets of Y to $2^n - 1$ vertices of N_{2^n-1}. Thus, we have $\tau_e(G) \leq n + 1$. Since the number of edges of $(K_2 \vee N_{2^n-1})$ is $2^{n+1} - 1$, by Theorem 2.1 we have $\tau_e(G) \geq n + 1$. Combining, $\tau_e(G) = n + 1$. □

Theorem 5.2. $\tau_e(K_2 \vee N_{2^n}) = n + 2$.

Proof. Let $X = \{x_1, x_2, \ldots, x_{n+2}\}$ and $Y = \{x_2, x_3, \ldots, x_{n+1}\}$. We can obtain a topoline set-indexer for $G = K_2 \vee N_{2^n}$ with indexing set X as follows: Assign $\emptyset$ and $\{x_1\}$ to the vertices of K_2. Assign all $2^n - 1$ non-empty subsets of Y to $2^n - 1$ vertices of N_{2^n}. Then label the remaining vertex of N_{2^n} by $Y \cup \{x_{n+2}\}$. Thus we have $\tau_e(G) \leq n + 2$. Since the number of edges of $(K_2 \vee N_{2^n})$ is $2^{n+1} + 1$, by Theorem 2.1 we have $\tau_e(G) \geq n + 2$. Combining, $\tau_e(G) = n + 2$. □

Theorem 5.3. $\tau_e(K_2 \vee N_{2^n+1}) = n + 2$.

Proof. Let $X = \{x_1, x_2, \ldots, x_{n+2}\}$ and $Z = \{x_2, x_3, \ldots, x_{n+1}\}$. A topoline set-indexer for $G = K_2 \vee N_{2^n+1}$ with indexing set X can be obtained as follows: Assign $\emptyset$ and $\{x_1\}$ to the vertices of K_2. Assign all $2^n - 1$ non-empty subsets of Z to $2^n - 1$ vertices of N_{2^n+1}. Then label the remaining two vertices of N_{2^n+1} by

$Z \cup \{x_{n+2}\}$ and $S \cup \{x_{n+2}\}$, where S is an $(n-1)$-element subset of Z. Thus, we have $\tau_e(G) \leq n+2$. Since the number of edges of $(K_2 \vee N_{2^n+1})$ is $2^{n+1}+3$, by Theorem 2.1 we have $\tau_e(G) \geq n+2$. Combining, $\tau_e(G) = n+2$. □

Theorem 5.4. $\tau_e(K_{2,2^n-1}) = n+2$, $n \geq 2$.

Proof. Since the number of edges of $G = K_{2,2^n-1}$ is $2^{n+1}-2$, by Theorem 2.1 and Theorem 2.2, $\tau_e(G) \geq n+1$. Since $3 \cdot 2^{n-1} < 2^{n+1} - 1 < 2^{n+1}$ for $n \geq 2$, by Theorem 2.4, $\tau_e(G) \geq n+2$. Let $X = \{x_1, x_2, \ldots, x_{n+2}\}$. We can obtain a topoline set-indexer for $G = K_{2,2^n-1}$ with indexing set X as follows: Assign $\emptyset$ and $\{x_1\}$ to the vertices of degree $2^n - 1$. Label the vertices of degree 2 by $\{x_2\}$, $X \setminus \{x_1\}$, union of $\{x_2\}$ with any $(n-1)$-element subsets of $X \setminus \{x_1, x_2\}$ until all such subsets are exhausted, union of $\{x_2\}$ with any $(n-2)$-element subsets of $X \setminus \{x_1, x_2\}$ until all such subsets are exhausted and so on sequentially. Thus, $\tau_e(G) \leq n+2$. Combining, $\tau_e(G) = n+2$. □

Theorem 5.5. $\tau_e(K_{2,5}) = 5$.

Proof. Since $m(11) = 5$, $\tau_e(K_{2,5}) \geq 5$. We can obtain a topoline set-indexer for $K_{2,5}$ with indexing set $X = \{x_1, x_2, x_3, x_4, x_5\}$ by assigning $\emptyset$ and $\{x_1\}$ to the vertices of degree 5 and $\{x_2\}$, $\{x_2, x_3, x_4, x_5\}$, $\{x_2, x_3, x_4\}$, $\{x_2, x_3\}$ and $\{x_2, x_4\}$ to the vertices of degree 2. Thus $\tau_e(K_{2,5}) \leq 5$. Combining, $\tau_e(K_{2,5}) = 5$. □

Theorem 5.6. $\tau_e(K_{2,6}) = 5$.

Proof. Since $m(13) = 5$, $\tau_e(K_{2,6}) \geq 5$. We can obtain a topoline set-indexer for $K_{2,6}$ with indexing set $X = \{x_1, x_2, x_3, x_4, x_5\}$ by assigning $\emptyset$ and $\{x_1\}$ to the vertices of degree 6 and $\{x_2\}$, $\{x_2, x_3, x_4\}$, $\{x_2, x_3, x_4, x_5\}$, $\{x_2, x_4, x_5\}$, $\{x_2, x_3\}$ and $\{x_2, x_4\}$ to the vertices of degree 2. Thus, $\tau_e(K_{2,6}) \leq 5$. Combining, $\tau_e(K_{2,6}) = 5$. □

Theorem 5.7. $\tau_e(K_{2,9}) = 6$.

Proof. Since $m(19) = 6$, $\tau_e(K_{2,9}) \geq 6$. We can obtain a topoline set-indexer for $K_{2,9}$ with indexing set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ by assigning $\emptyset$ and $\{x_1\}$ to the vertices of degree 9 and $\{x_2\}$, $\{x_2, x_3\}$, $\{x_2, x_3, x_4\}$, $\{x_2, x_3, x_5\}$, $\{x_2, x_3, x_6\}$, $\{x_2, x_3, x_4, x_5\}$, $\{x_2, x_3, x_4, x_6\}$, $\{x_2, x_3, x_5, x_6\}$ and $\{x_2, x_3, x_4, x_5, x_6\}$ to the vertices of degree 2. Thus $\tau_e(K_{2,9}) \leq 6$. Combining, $\tau_e(K_{2,9}) = 6$. □

Theorem 5.8. $\tau_e(K_{2,10}) = 6$.

Proof. Since $m(21) = 6$, $\tau_e(K_{2,10}) \geq 6$. We can obtain a topoline set-indexer for $K_{2,10}$ with indexing set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ by assigning $\emptyset$ and $\{x_1\}$ to the vertices of degree 10 and $\{x_2\}$, $\{x_2, x_3\}$, $\{x_2, x_4\}$, $\{x_2, x_3, x_4\}$, $\{x_2, x_3, x_5\}$, $\{x_2, x_3, x_6\}$, $\{x_2, x_3, x_4, x_5\}$, $\{x_2, x_3, x_4, x_6\}$, $\{x_2, x_3, x_5, x_6\}$ and $\{x_2, x_3, x_4, x_5, x_6\}$ to the vertices of degree 2. Thus, $\tau_e(K_{2,10}) \leq 6$. Combining, $\tau_e(K_{2,10}) = 6$. $\square$

Theorem 5.9. $\tau_e(K_{2,17}) = 7$.

Proof. Since $m(35) = 7$, $\tau_e(K_{2,17}) \geq 7$. We can obtain a topoline set-indexer for $K_{2,17}$ with indexing set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ by assigning $\emptyset$ and $\{x_1\}$ to the vertices of degree 17 and $\{x_2\}$, $\{x_2, x_3\}$, $\{x_2, x_4\}$, $\{x_2, x_5\}$, $\{x_2, x_6\}$, $\{x_2, x_3, x_4\}$, $\{x_2, x_3, x_5\}$, $\{x_2, x_3, x_6\}$, $\{x_2, x_4, x_5\}$, $\{x_2, x_4, x_6\}$, $\{x_2, x_5, x_6\}$, $\{x_2, x_3, x_4, x_5\}$, $\{x_2, x_3, x_4, x_6\}$, $\{x_2, x_3, x_5, x_6\}$, $\{x_2, x_4, x_5, x_6\}$, $\{x_2, x_3, x_4, x_5, x_6\}$ and $\{x_2, x_3, x_4, x_5, x_6, x_7\}$ to the vertices of degree 2. Thus, $\tau_e(K_{2,17}) \leq 7$. Combining, $\tau_e(K_{2,17}) = 7$. $\square$

6. Conclusion

The study has obtained some results favouring the conjecture that the complete graph K_n; $n > 1$ is topoline only if it is set-graceful. The topoline numbers of certain classes of triangular books and some complete bipartite graphs are also obtained.

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