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# BOUNDS ON AG TOPOLOGICAL INDICES OF SOME GRAPH OPERATIONS

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ABSTRACT. The AG index of a connected graph $G$ is $AG(G) = \sum_{uv\epsilon E(G)} \frac{du+dv}{2\sqrt{du.dv}}$ where $du$ and $dv$ represent the degrees of the vertices of the edge $uv$. In this paper some bounds of AG index are presented

## 1. INTRODUCTION

The topological indices are numerical values associated with molecular graphs. These graph invariants are called molecular descriptors. They play a vital role in chemical documentation, isomer discrimination, relationship analysis like QSAR and QSPR. In 1947, [7] Weiner used his topological index named as Weiner index to calculate the boiling point of paraffins.Then in 1972, [5] Gutman and Trinajstic defined the Zagreb indices which are popular. Thereafter many indices are defined namely [1] [4] Randic index, topological index etc.In 2016 [6] V.S. Shigehalli and Rachanna Kanavur introduced arithmetic-geometric indices.

Throughout this paper we consider only connected graphs without loops or multiple edges called simple connected graphs.For a graph $G$, V(G) and $E(G)$ denote the set of all vertices and edges respectively.For a graph $G$ the degree of a vertex $v$ is the number of edges incident to $v$ and is denoted by $d(v)$. The composition (also called Lexicographic product) of graphs $G_1$ and $G_2$ with disjoint vertex set $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph with

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vertex set $V(G_1) \times V(G_2)$ and $(u_i, v_j)$ is adjacent with $u_k$ or $u_i = u_k$ and $v_j$ is adjacent with $v_l$.

The Cartesian product [2] of $G_1 \times G_2$ of graphs $G_1$ and $G_2$ has the vertex set $V(G_1) \times V(G_2)$ and $(u_i, v_j)$, $(u_k, v_l)$ is an edge of $G_1 \times G_2$ if $u_i = u_k$ and $(v_j, v_l) \in E(G_2)$ or $(u_i, u_k) \in E(G_1)$ and $v_j = v_l$.

In this paper bounds for the AG indices of Corona product,Cartesian product and Composition of graphs are derived.

**Definition 1.1.** *Arithmetico-Geometrico topological index for a non-empty graph G is denoted by AG(G) and is defined as* $AG(G) = \sum_{uv\epsilon E(G)} \frac{du+dv}{2\sqrt{du.dv}}$*, where* $du$ *and* $dv$ *represent the degrees of the vertices of the edge* $uv$*.*

## 2. AG Indices of Graph Operations

**Definition 2.1.** *The eccentricity* $eec_G(v)$ *of a vertex* $v$ *in a connected graph* $G$ *is the greatest geodesic distance between* $v$ *and any other vertex. The diameter* $D(G)$ *of* $G$ *is defined as* $d(G) = max\{eec_G(v)|v \in V(G)\}$*.Also the radius* $rad(G)$ *is defined as the* $d(G) = min\{eec_G(v)|v \in V(G)\}$*.*

**Definition 2.2.** *The Cartesian product* $G_1 \times G_2$ *of* $G_1$ *and* $G_2$ *is a graph with vertex set* $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ *and* $(u_i, v_j), (u_k, v_l)$ *are adjacent in* $G_1 \times G_2$ *if* $u_i = u_k$ *and* $v_j v_l \in E(G_2)$ *or* $u_i u_k \in E(G_1)$*and* $v_j = v_l$*.*

It can be seen that $|E(G_1 \times G_2) = |E(G_1)||V(G_2)| + |E(G_2)||V(G_1)|$ and

$$d_{G_1 \times G_2}(u, v) = d_{G_1}(u) + d_{G_2}(v).$$

**Theorem 2.1.** *Let* $G_1$ *and* $G_2$ *be two graphs with orders* $n_1$ *and* $n_2$ *and size* $m_1$ *and* $m_2$ *respectively. Then*

$$AG(G_1 \times G_2) \leq \frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2}(m_2 n_1 + m_1 n_2),$$

*where* $\delta_1$ *and* $\delta_2$ *are the minimum degrees of the vertices of* $G_1$ *and* $G_2$ *and* $\Delta_1$ *and* $\Delta_2$ *are their maximum degrees.*

*Proof.*

$$
\begin{aligned}
AG(G_1\times G_2) &= \\
&\sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1\times G_2),\\ (u_i,v_j)\neq(u_k,v_l)}} \frac{d_{G_1\times G_2}(u_i,v_j)+d_{G_1\times G_2}(u_k,v_l)}{2\sqrt{d_{G_1\times G_2}(u_i,v_j).d_{G_1\times G_2}(u_k,v_l)}} \\
&= \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\ (v_j,v_l)\in E(G_2)}} \frac{d_{G_1\times G_2}(u_i,v_j)+d_{G_1\times G_2}(u_i,v_l)}{2\sqrt{d_{G_1\times G_2}(u_i,v_j).d_{G_1\times G_2}(u_i,v_l)}} \\
&\quad + \sum_{\substack{(u_i,v_j),(u_k,v_j)\in E(G_1\times G_2),\\ (u_i,u_k)\in E(G_1)}} \frac{d_{G_1\times G_2}(u_i,v_j)+d_{G_1\times G_2}(u_k,v_j)}{2\sqrt{d_{G_1\times G_2}(u_i,v_j).d_{G_1\times G_2}(u_k,v_j)}} \\
&= \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\ (v_j,v_l)\in E(G_2)}} \frac{d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_1}(u_i)+d_{G_2}(v_l)}{2\sqrt{(d_{G_1}(u_i)+d_{G_2}(v_j))(d_{G_1}(u_i)+d_{G_2}(v_l))}} \\
&\quad + \sum_{\substack{(u_i,v_j),(u_k,v_j)\in E(G_1\times G_2),\\ (u_i,u_k)\in E(G_1)}} \frac{d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_1}(u_k)+d_{G_2}(v_j)}{2\sqrt{(d_{G_1}(u_i)+d_{G_2}(v_j))(d_{G_1}(u_k)+d_{G_2}(v_j))}} \\
&= \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\ (v_j,v_l)\in E(G_2)}} \frac{2d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_2}(v_l)}{2\sqrt{(d_{G_1}(u_i)+d_{G_2}(v_j))(d_{G_1}(u_i)+d_{G_2}(v_l))}} \\
&\quad + \sum_{\substack{(u_i,v_j),(u_k,v_j)\in E(G_1\times G_2),\\ (u_i,u_k)\in E(G_1)}} \frac{d_{G_1}(u_i)+d_{G_1}(u_k)+2d_{G_2}(v_j)}{2\sqrt{(d_{G_1}(u_i)+d_{G_2}(v_j))(d_{G_1}(u_k)+d_{G_2}(v_j))}}
\end{aligned} \tag{2.1}
$$

Suppose $\delta_1$ and $\delta_2$ be minimum degrees of the vertices of $G_1$ and $G_2$ and $\Delta_1$ and $\Delta_2$ be their maximum degrees. Then $\delta_1 \leq d_{G_1}(u_i) \leq \Delta_1$ and $\delta_2 \leq d_{G_1}(u_i) \leq \Delta_2$. So,

$$
\begin{aligned}
&AG(G_1\times G_2) \\
&\leq \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\ (v_j,v_l)\in E(G_2)}} \frac{2\Delta_1+2\Delta_2}{2\sqrt{(\delta_1+\delta_2)^2}} + \sum_{\substack{(u_i,v_j),(u_k,v_j)\in E(G_1\times G_2),\\ (u_i,u_k)\in E(G_1)}} \frac{2\Delta_1+2\Delta_2}{2\sqrt{(\delta_1+\delta_2)^2}} \\
&\leq \frac{\Delta_1+\Delta_2}{\delta_1+\delta_2} \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\ (v_j,v_i)\in E(G_2)}} 1 + \frac{\Delta_1+\Delta_2}{\delta_1+\delta_2} \sum_{\substack{(u_i,v_j),(u_k,v_j)\in E(G_1\times G_2),\\ (u_i,u_k)\in E(G_1)}} 1
\end{aligned}
$$

$$= \frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2}|E(G_2)||V(G_1)| + \frac{\Delta_1 + \Delta_2}{\delta_1 + \delta_2}|E(G_1)||V(G_2)|.$$

Hence, $AG(G_1 \times G_2) \leq \frac{\Delta_1+\Delta_2}{\delta_1+\delta_2}(m_2 n_1 + m_1 n_2)$. □

**Theorem 2.2.** *Let $G_1$ and $G_2$ be two graphs with orders $n_1$ and $n_2$ and size $m_1$ and $m_2$ respectively. Then*

$$AG(G_1 \times G_2) \leq (n_1 m_2 + n_2 m_1)\left(\frac{n_1 + n_2 - rad(G_1) - rad(G_2)}{\delta_1 + \delta_2}\right).$$

*Proof.* From (2.1), we have

$$\begin{aligned}
&AG(G_1 \times G_2) \\
&= \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\ (v_j,v_l)\in E(G_2)}} \frac{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_l)}{2\sqrt{(d_{G_1}(u_i) + d_{G_2}(v_j))(d_{G_1}(u_i) + d_{G_2}(v_l))}} \\
&\quad + \sum_{\substack{(u_i,v_j),(u_k,v_j)\in E(G_1\times G_2),\\ (u_i,u_k)\in E(G_1)}} \frac{d_{G_1}(u_i) + d_{G_1}(u_k) + 2d_{G_2}(v_j)}{2\sqrt{(d_{G_1}(u_i) + d_{G_2}(v_j))(d_{G_1}(u_k) + d_{G_2}(v_j))}} \\
&= A_1 + A_2.
\end{aligned}$$

Now,

$$\begin{aligned}
&A_1 \\
&= \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\ (v_j,v_l)\in E(G_2)}} \frac{2d_{G_1}(u_i) + d_{G_2}(v_j) + d_{G_2}(v_l)}{2\sqrt{(d_{G_1}(u_i) + d_{G_2}(v_j))(d_{G_1}(u_i) + d_{G_2}(v_l))}} \\
&\leq \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\ (v_j,v_l)\in E(G_2)}} \frac{2(n_1 - ecc_{G_1}(u_i)) + (n_2 - ecc_{G_2}(v_j)) + (n_2 - ecc_{G_2}(v_l))}{2\sqrt{(\delta_1 + \delta_2)^2}} \\
&\leq \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\ (v_j,v_l)\in E(G_2)}} \frac{2(n_1 - rad(G_1)) + (n_2 - rad(G_2)) + (n_2 - rad(G_2))}{2(\delta_1 + \delta_2)}
\end{aligned}$$

$$= \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\(v_j,v_l)\in E(G_2)}} \frac{n_1+n_2-rad(G_1)-rad(G_2)}{\delta_1+\delta_2}$$

$$= \frac{n_1+n_2-rad(G_1)-rad(G_2)}{\delta_1+\delta_2} \sum_{\substack{(u_i,v_j),(u_i,v_l)\in E(G_1\times G_2),\\(v_j,v_l)\in E(G_2)}} 1$$

$$= \frac{n_1+n_2-rad(G_1)-rad(G_2)}{\delta_1+\delta_2}|V(G_1)||E(G_2)|$$

$$= n_1m_2\left(\frac{n_1+n_2-rad(G_1)-rad(G_2)}{\delta_1+\delta_2}\right)$$

Similarly,

$$A_2$$

$$= \sum_{\substack{(u_i,v_j),(u_k,v_j)\in E(G_1\times G_2),\\(u_i,u_k)\in E(G_1)}} \frac{d_{G_1}(u_i)+d_{G_1}(u_k)+2d_{G_2}(v_j)}{2\sqrt{(d_{G_1}(u_i)+d_{G_2}(v_j))(d_{G_1}(u_k)+d_{G_2}(v_j))}}$$

$$\leq \sum_{\substack{(u_i,v_j),(u_k,v_j)\in E(G_1\times G_2),\\(u_i,u_k)\in E(G_1)}} \frac{(n_1-ecc_{G_1}(u_i))+(n_1-ecc_{G_1}(u_k))+2(n_2-ecc_{G_2}(v_j))}{2\sqrt{(\delta_1+\delta_2)^2}}$$

$$\leq \sum_{\substack{(u_i,v_j),(u_k,v_j)\in E(G_1\times G_2),\\(u_i,u_k)\in E(G_1)}} \frac{(n_1-rad(G_1))+(n_1-rad(G_1))+2(n_2-rad(G_2))}{2(\delta_1+\delta_2)}$$

$$= n_2m_1\left(\frac{n_1+n_2-rad(G_1)-rad(G_2)}{\delta_1+\delta_2}\right).$$

Hence the conclusion,

$$A(G_1\times G_2) \leq n_1m_2\left(\frac{n_1+n_2-rad(G_1)-rad(G_2)}{\delta_1+\delta_2}\right) + n_2m_1\left(\frac{n_1+n_2-rad(G_1)-rad(G_2)}{\delta_1+\delta_2}\right).$$

□

**Definition 2.3.** *The Corona product [3] $G_1 \circ G_2$ and $G_1$ and $G_2$ is a graph obtained by taking $|V(G_1)|$ copies of $G_2$ and joining each vertex of the $i^{th}$ copy with vertex $v_i \in V(G_1)$. Then*

$$|V(G_1\circ G_2)| = |V(G_1)|(1+|V(G_2)|)$$

*and*

$$|E(G_1 \circ G_2)| = |E(G_1)| + |V(G_1)|(|V(G_2)| + |E(G_2)|).$$

*Also, for a vertex in $V(G_1 \circ G_2)$,*

$$d_{G_1 \circ G_2}(u) = \begin{cases} d_{G_1}(u) + |V(G_2)| & ; u \in V(G_1) \\ d_{G_2}(u) + 1 & ; u \in V(G_2) \end{cases}.$$

**Theorem 2.3.** *Let $G_{2_i}(i = 1, 2, \ldots, |V(G_1)|)$ represent the $i^{th}$ copy of $G_2$ attached to the $i^{th}$ vertex of $G_1$ and $\delta_i$ and $\Delta_i$ are minimum and maximum degrees of the vertices of $G_i, i = 1, 2$. Then for the corona product $G_1 \circ G_2$ of $G_1$ and $G_2$,*

$$AG(G_1 \circ G_2) \leq \frac{m_1(\Delta_1 + n_2)}{\delta_1 + n_2} + \frac{(\Delta_2 + 1)n_1 m_2}{\delta_2 + 1} + \frac{\Delta_2 + \Delta_1 + n_1 + 1}{2\sqrt{(\delta_2 + 1)(\delta_1 + n_1)}} n_1 n_2$$

*Proof.* The edge sets of $G_1 \circ G_2$ can be partitioned into three sets,

$$\begin{aligned} E_1 &= \{e = uv \in E(G_1 \circ G_2), e \in E(G_1)\}, \\ E_2 &= \{e = uv \in E(G_1 \circ G_2), e \in E(G_{2_i}), i = 1, 2, \ldots, |V(G_1)|\}, \\ E_3 &= \{e = uv \in E(G_1 \circ G_2), u \in V(G_{2_i}), i = 1, 2, \ldots, |V(G_1)| \text{and}, v \in V(G_1)\}. \end{aligned}$$

Now,

$$\begin{aligned} AG(G_1 \circ G_2) &= \sum_{uv \in E(G_1 \circ G_2)} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u) d_{G_1 \circ G_2}(v)}} \\ &= \sum_{uv \in E_1} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u) d_{G_1 \circ G_2}(v)}} + \sum_{uv \in E_2} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u) d_{G_1 \circ G_2}(v)}} \\ &\quad + \sum_{uv \in E_3} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u) d_{G_1 \circ G_2}(v)}} \\ &\quad = A_1 + A_2 + A_3, \end{aligned}$$

$$\begin{aligned}
A_1 &= \sum_{uv \in E_1} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u) d_{G_1 \circ G_2}(v)}} \\
&= \sum_{uv \in E_1} \frac{d_{G_1}(u) + |V(G_2)| + d_{G_1}(v) + |V(G_2)|}{2\sqrt{(d_{G_1}(u) + |V(G_2)|)(d_{G_1}(v) + |V(G_2)|)}} \\
&= \sum_{uv \in E_1} \frac{d_{G_1}(u) + d_{G_1}(v) + 2n_2}{2\sqrt{(d_{G_1}(u) + n_2)(d_{G_1}(v) + n_2)}} \\
&\leq \sum_{uv \in E_1} \frac{\Delta_1 + \Delta_1 + 2n_2}{2\sqrt{(\delta_1 + n_2)(\delta_1 + n_2)}} \\
&\leq \sum_{uv \in E_1} \frac{\Delta_1 + n_2}{\delta_1 + n_2} \\
&= \frac{\Delta_1 + n_2}{\delta_1 + n_2} \sum_{uv \in E_1} 1 = \frac{m_1(\Delta_1 + n_2)}{\delta_1 + n_2}.
\end{aligned} \tag{2.2}$$

Hence, $A_1 \leq \dfrac{m_1(\Delta_1 + n_2)}{\delta_1 + n_2}$.

$$\begin{aligned}
A_2 &= \sum_{uv \in E_2} \frac{d_{G_1 \circ G_2}(u) + d_{G_1 \circ G_2}(v)}{2\sqrt{d_{G_1 \circ G_2}(u) d_{G_1 \circ G_2}(v)}} \\
&= \sum_{uv \in E_2} \frac{d_{G_2}(u) + 1 + d_{G_2}(v) + 1}{2\sqrt{(d_{G_2}(u) + 1)(d_{G_2}(v) + 1)}} \\
&\leq \sum_{uv \in E_2} \frac{\Delta_2 + 1 + \Delta_2 + 1}{2\sqrt{(\delta_2 + 1)(\delta_2 + 1)}} \\
&= \sum_{uv \in E_2} \frac{\Delta_2 + 1}{\delta_2 + 1} \\
&= \frac{\Delta_2 + 1}{\delta_2 + 1} \sum_{uv \in E_2} 1 \\
&= \frac{\Delta_2 + 1}{\delta_2 + 1} (|V(G_1)||E(G_2)|)
\end{aligned} \tag{2.3}$$

Hence, $A_2 \leq \dfrac{n_1 m_2(\Delta_2 + 1)}{\delta_2 + 1}$.

$$\begin{aligned} A_3 &= \sum_{uv\in E_3} \frac{d_{G_1\circ G_2}(u)+d_{G_1\circ G_2}(v)}{2\sqrt{d_{G_1\circ G_2}(u)d_{G_1\circ G_2}(v)}} \\ &= \sum_{uv\in E_3} \frac{d_{G_2}(u)+1+d_{G_1}(v)+|V(G_2)|}{2\sqrt{(d_{G_2}(u)+1)(d_{G_1}(v)+|V(G_2)|)}} \\ &\leq \sum_{uv\in E_3} \frac{\Delta_2+1+\Delta_1+n_2}{2\sqrt{(\delta_2+1)(\delta_1+n_2)}} \qquad (2.4) \\ &= \frac{\Delta_2+1+\Delta_1+n_2}{2\sqrt{(\delta_2+1)(\delta_1+n_2)}} \sum_{uv\in E_3} 1 \\ &= \frac{\Delta_2+1+\Delta_1+n_2}{2\sqrt{(\delta_2+1)(\delta_1+n_2)}} |V(G_1)||V(G_2)| \end{aligned}$$

i.e.,

$$A_3 \leq \frac{\Delta_2+\Delta_1+n_2+1}{2\sqrt{(\delta_2+1)(\delta_1+n_2)}} n_1n_2.$$

Hence we conclude:

$$AG(G_1\circ G_2) \leq \frac{m_1(\Delta_1+n_2)}{\delta_1+n_2} + \frac{(\Delta_2+1)n_1m_2}{\delta_2+1} + \frac{\Delta_2+\Delta_1+n_2+1}{2\sqrt{(\delta_2+1)(\delta_1+n_2)}} n_1n_2.$$

□

**Theorem 2.4.** *Let $G_{2_i}(i=1,2,\ldots,|V(G_1)|)$ represent the $i^{th}$ copy of $G_2$ attached to the $i^{th}$ vertex of $G_1$ and $\delta_i$ and $\Delta_i$ are minimum and maximum degrees of the vertices of $G_i, i=1,2$. Then for the corona product $G_1\circ G_2$ of $G_1$ and $G_2$,*

$$AG(G_1\circ G_2) \geq \frac{(\delta_1+n_2)m_1}{\Delta_1+n_2} + \frac{(\delta_2+1)n_1m_2}{\Delta_2+1} + \frac{(\delta_2+\delta_1+n_1+1)n_1n_2}{2\sqrt{(\Delta_2+1)(\Delta_1+n_1)}} n_1n_2.$$

*Proof.* We have from (2.2)

$$\begin{aligned} A_1 &= \sum_{uv\in E_1} \frac{d_{G_1}(u)+|V(G_2)|+d_{G_1}(u)+|V(G_2)|}{2\sqrt{(d_{G_1}(u)+|V(G_2)|)(d_{G_1}(u)+|V(G_2)|)}} \\ &\geq \sum_{uv\in E_1} \frac{\delta_1+n_2+\delta_1+n_2}{2\sqrt{(\Delta_1+n_2)(\Delta_1+n_2)}} \\ &= \frac{\delta_1+n_2}{\Delta_1+n_2}\sum_{uv\in E_1} 1 \geq \frac{m_1(\delta_1+n_2)}{\Delta_1+n_2}. \end{aligned}$$

Again, from (2.3)

$$\begin{aligned} A_2 &= \sum_{uv\in E_2} \frac{d_{G_2}(u)+1+d_{G_2}(v)+1}{2\sqrt{(d_{G_2}(u)+1)(d_{G_2}(v)+1)}} \geq \sum_{uv\in E_2} \frac{\delta_2+1+\delta_2+1}{2\sqrt{(\Delta_2+1)(\Delta_2+1)}} \\ &= \frac{\delta_2+1}{\Delta_2+1}\sum_{uv\in E_2} \geq \frac{(\delta_2+1)n_1m_2}{\Delta_2+1}. \end{aligned}$$

From (2.4)

$$\begin{aligned} A_3 &= \sum_{uv\in E_3} \frac{d_{G_2}(u)+1+d_{G_1}(v)+|V(G_2)|}{2\sqrt{(d_{G_2}(u)+1)(d_{G_1}(v)+|V(G_2)|)}} \geq \sum_{uv\in E_3} \frac{\delta_2+1+\delta_1+n_2}{2\sqrt{(\Delta_2+1)(\Delta_1+n_2)}} \\ &= \frac{\delta_2+1+\delta_1+n_1}{2\sqrt{(\Delta_2+1)(\Delta_1+n_1)}}\sum_{uv\in E_3} 1 = \frac{n_1n_2(\delta_2+1+\delta_1+n_2)}{2\sqrt{(\Delta_2+1)(\Delta_1+n_2)}}. \end{aligned}$$

Hence,

$$AG(G_1\circ G_2) \geq \frac{(\delta_1+n_2)m_1}{\Delta_1+n_2} + \frac{(\delta_2+1)n_1m_2}{\Delta_2+1} + \frac{(\delta_2+\delta_1+n_2+1)n_1n_2}{2\sqrt{(\Delta_2+1)(\Delta_1+n_2)}}n_1n_2.$$

□

**Definition 2.4.** *The composition or lexicographic product $G = G_1[G_2]$ of graphs $G_1$ and $G_2$ with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is a graph with vertex set $V(G_1)\times V(G_2)$ and $(u_i, v_j)$ is adjacent with $(u_k, v_l)$ whenever $u_i$ is adjacent with $u_k$ or $u_i = u_k$ and $v_j$ adjacent with $v_l$.*

By this definition, one can see that

$$\begin{aligned} |E(G_1[G_2]) &= |E(G_1)||V(G_2)^2| + |E(G_2)||V(G_1)| \\ d_{G_1[G_2]}(u,v) &= |V(G_2)|d_{G_1}(u) + d_{G_2}(v). \end{aligned} \tag{2.5}$$

**Theorem 2.5.** *Let $G_1$ and $G_2$ be two connected graphs with order $n_1$ and $n_2$, size $m_1$ and $m_2$,$\delta_i$ and $\Delta_i$ are minumum and maximum degrees of the vertices $G_i, i = 1,2$ respectively. Then $AG(G_1[G_2]) \leq \dfrac{(n_2\Delta_1+\Delta_2)(n_1m_2+m_1n_2^2)}{n_2\delta_1+\delta_2}$.*

*Proof.*

$$\begin{aligned}
AG(G_1[G_2]) = & \sum_{\substack{(u_i,v_j),(u_k,v_l)\in E(G_1[G_2]]),\\ (u_i,v_j)\neq(u_k,v_l)}} \frac{d_{G_1[G_2)]}(u_i,v_j)+d_{G_1[G_2)]}(u_k,v_l)}{2\sqrt{(d_{G_1[G_2)]}(u_i,v_j).d_{G_1[G_2)]}(u_k,v_l)}}\\
= & \sum_{\substack{(u_i,v_j),(u_i,v_l)\in\\ E(G_1[G_2]]),j\neq l}} \frac{d_{G_1[G_2)]}(u_i,v_j)+d_{G_1[G_2)]}(u_i,v_l)}{2\sqrt{(d_{G_1[G_2)]}(u_i,v_j).d_{G_1[G_2)]}(u_i,v_l)}}\\
+ & \sum_{\substack{(u_i,v_j),(u_k,v_j)\in\\ E(G_1[G_2]]),i\neq k}} \frac{d_{G_1[G_2)]}(u_i,v_j)+d_{G_1[G_2)]}(u_k,v_j)}{2\sqrt{(d_{G_1[G_2)]}(u_i,v_j).d_{G_1[G_2)]}(u_k,v_j)}}\\
= & A_1 + A_2.
\end{aligned}$$

Consider

$$\begin{aligned}
A_1 = & \sum_{\substack{(u_i,v_j),(u_i,v_l)\in\\ E[G_1[G_2]]),j\neq l}} \frac{d_{[G_1[G_2]}(u_i,v_j)+d_{[G_1[G_2]}(u_i,v_l)}{2\sqrt{d_{[G_1[G_2]}(u_i,v_j).d_{[G_1[G_2]}(u_i,v_l)}}\\
= & \sum_{\substack{(u_i,v_j),(u_i,v_l)\in\\ E[G_1[G_2]]),j\neq l}} \frac{|V(G_2)|d_{G_1}(u_i)+d_{G_2}(v_j)+|V(G_2)|d_{G_1}(u_i)+d_{G_2}(v_l)}{2\sqrt{(|V(G_2)|d_{G_1}(u_i)+d_{G_2}(v_j))(|V(G_2)|d_{G_1}(u_i)+d_{G_2}(v_l))}}\\
\leq & \sum_{\substack{(u_i,v_j),(u_i,v_l)\in\\ E[G_1[G_2]]),j\neq l}} \frac{2(n_2\Delta_1+\Delta_2)}{2\sqrt{(n_2\delta_1+\delta_2)(n_2\delta_1+\delta_2)}}\\
\leq & \frac{(n_2\Delta_1+\Delta_2)}{(n_2\delta_1+\delta_2)} \sum_{\substack{(u_i,v_j),(u_i,v_l)\in\\ E[G_1[G_2]]),j\neq l}} 1 \leq \frac{(n_2\Delta_1+\Delta_2)n_1m_2}{(n_2\delta_1+\delta_2)}.
\end{aligned}$$

Now consider

$$\begin{aligned}
A_2 = & \sum_{\substack{(u_i,v_j),(u_k,v_j)\in\\ E[G_1[G_2]]),i\neq k}} \frac{d_{[G_1[G_2]}(u_i,v_j)+d_{[G_1[G_2]}(u_k,v_j)}{2\sqrt{d_{[G_1[G_2]}(u_i,v_j).d_{[G_1[G_2]}(u_k,v_j)}}\\
= & \sum_{\substack{(u_i,v_j),(u_k,v_j)\in\\ E[G_1[G_2]]),i\neq k}} \frac{|V(G_2)|d_{G_1}(u_i)+d_{G_2}(v_j)+|V(G_2)|d_{G_1}(u_k)+d_{G_2}(v_j)}{2\sqrt{(|V(G_2)|d_{G_1}(u_i)+d_{G_2}(v_j))(|V(G_2)|d_{G_1}(u_k)+d_{G_2}(v_j))}}\\
\leq & \sum_{\substack{(u_i,v_j),(u_i,v_l)\in\\ E[G_1[G_2]]),j\neq l}} \frac{2(n_2\Delta_1+\Delta_2)}{2\sqrt{(n_2\delta_1+\delta_2)(n_2\delta_1+\delta_2)}}
\end{aligned}$$

$$\leq \frac{(n_2\Delta_1 + \Delta_2)}{(n_2\delta_1 + \delta_2)} \sum_{\substack{(u_i,v_j),(u_k,v_j)\in \\ E[G_1[G_2]),i\neq k}} 1$$
$$= \frac{(n_2\Delta_1 + \Delta_2)}{(n_2\delta_1 + \delta_2)} \sum_{(u_i u_k)\in E(G_1)} \sum_{v_j\in V(G_2)} 1$$
$$= \frac{(n_2\Delta_1 + \Delta_2)m_1 n_2^2}{(n_2\delta_1 + \delta_2)}.$$

Hence,

$$AG(G_1[G_2]) \leq \frac{(n_2\Delta_1 + \Delta_2)n_1 m_2}{(n_2\delta_1 + \delta_2)} + \frac{(n_2\Delta_1 + \Delta_2)m_1 n_2^2}{(n_2\delta_1 + \delta_2)}$$
$$= \frac{(n_2\Delta_1 + \Delta_2)(n_1 m_2 + m_1 n_2^2)}{n_2\delta_1 + \delta_2}.$$

□

**Theorem 2.6.** *Let $G_1$ and $G_2$ be two connected graphs with order $n_1$ and $n_2$, size $m_1$ and $m_2$,$\delta_i$ and $\Delta_i$ are minimum and maximum degrees of the vertices $G_i, i = 1, 2$, respectively. Then*

$$AG(G_1[G_2]) \geq \frac{(n_2\delta_1 + \delta_2)(n_1 m_2 + m_1 n_2^2)}{n_2\Delta_1 + \Delta_2}.$$

*Proof.* Same as above. □

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