

SOLVING MIXED BOUNDARY VALUE PROBLEM ON SIMPLY CONNECTED REGIONS

MOHSEN AGHAEIBOORKHEILI AND AEZEDEN MOHAMED¹

ABSTRACT. A numerical method for solving mixed condition on simply-connected regions will be presented in this paper. Accuracy of suggested method is proved by some examples that already compared with exact solution.

1. INTRODUCTION

There are about three main types of boundary value problems. The first one that the values of a function in a boundary are given is called Dirichlet problem. In the second type the values for the normal - derivative of a function on the boundary are given. This type of BVP is called Neumann boundary condition. The third type is a combination of the last two BVP. In this type, on a part of the boundary the values of the function are given, while for the rest part of the boundary the normal-derivative values of the function is given. This type of BVP is called mixed boundary value problem [1].

All types of BVP's arise in so many problems in physics, engineering and the other science. Since mixed boundary value problem included both Dirichlet and Neumann BVP, so it has more applications in other area.

So far so many methods have been developed for solving mixed BVP including analytical and numerical method. But since the most problems that appear

¹*corresponding author*

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in the real life can not be solved analytically so numerical methods are more interested and preferred.

The most popular numerical methods for solving BVP are BIE, FEM and FDM. Boundary integral equation method is the more recent and effective (in term of accuracy) method comparing the other methods [2]. In this paper a special type of a BIE method has been applied for solving mixed boundary value problems.

2. NOTATIONS AND PRELIMINARIES

Suppose that $\overline{\mathbb{C}}$ represents the extended complex plane. Then G will be a bounded simply connected area at $\overline{\mathbb{C}}$ with boundary Γ . Note that the direction of Γ is counterclockwise and also α will be a fixed point in G (see Figure 1).

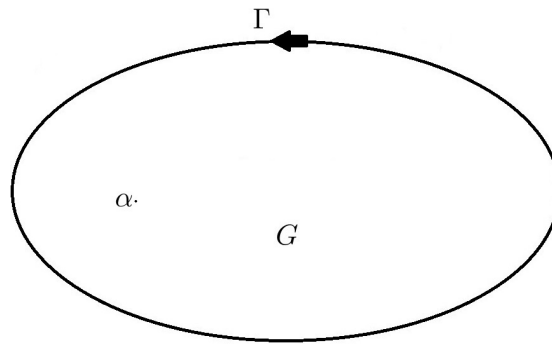


FIGURE 1. A graph for region G

Let $\delta(t)$ be a 2π -periodic twice continuously differentiable complex function that represent parametric equation for the curve Γ such that

$$\dot{\delta}(y) = \frac{d(\delta(y))}{dy} \neq 0, \quad y \in J := [0, 2\pi].$$

Suppose that p and q to be two arbitrary functions in H (Hölder space) defined on Γ with $y \in J$. For convenience, $p(\delta(y))$ will be presented by $p(y)$ and also $q(\delta(y))$ by $q(y)$.

Definition 2.1. Assume that p is a real Hölder continuous function. Dirichlet problem is looking for a function w such that [3–5],

- (1) w should be harmonic in G^+ ,
- (2) $w(\delta(y)) = p(y)$.

Definition 2.2. Let ϕ to be a real Hölder continuous function such that

$$\int_J \phi(y) |\dot{\delta}(y)| dy = 0.$$

Moreover suppose that \mathbf{n} represent the exterior normal to Γ . Neumann problem is looking for a function w such that

- (1) w should be harmonic in G^+ ,
- (2) w should be continuous on the closure \overline{G} ,
- (3) $w(\alpha) = 0$,
- (4) $\left. \frac{\partial w}{\partial \mathbf{n}} \right|_{\delta(y)} = \phi(y), \quad \delta(y) \in \Gamma.$

For convenience, normal vector with exterior direction that has been unitize, will be mention just by normal vector in this paper. Suppose that $\mathbf{n}(\iota)$ is the normal vector to Γ at $\iota \in \Gamma$. Moreover assume that $T(\iota)$ is the unit tangent vector. Suppose also the angle connecting $\mathbf{n}(\iota)$ and the positive direction of the real axis will be presented by $w(\iota)$, i.e., $\mathbf{n}(\iota) = e^{i\Omega(\iota)}$. Then

$$e^{i\Omega(\iota)} = -iT(\delta(y)) = -i \frac{\dot{\delta}(y)}{|\dot{\delta}(y)|}.$$

Thus

$$\frac{\partial w}{\partial \mathbf{n}} = \cos(\Omega) \frac{\partial w}{\partial X} + \sin(\Omega) \frac{\partial w}{\partial Y} = \Re \left[e^{i\Omega} \left(\frac{\partial w}{\partial X} - i \frac{\partial w}{\partial Y} \right) \right].$$

Let to write $w(z)$ as a real part of an analytic function such $w(z) = \Re[F(z)]$ then using the Cauchy-Riemann equations

$$F'(z) = \frac{\partial w(z)}{\partial X} - i \frac{\partial w(z)}{\partial Y}.$$

Thus

$$\Re[-i\dot{\delta}F'] = |\dot{\delta}| \frac{\partial w}{\partial \mathbf{n}}.$$

Assume that F is an analytic function with boundary values of

$$F = U + iV,$$

then the derivative function $F'(z)$ will be also an analytic function with

$$\dot{\delta}F' = U' + iV'.$$

So for the Neumann problem

$$\begin{cases} \Re[-iF(\delta(y))] = \psi(y), \\ \psi'(y) = \Re[-i\dot{\delta}(y)F'(\delta(y))] = |\dot{\delta}(y)|\frac{\partial w}{\partial \mathbf{n}}(y), \end{cases} \implies \begin{cases} \Re[-i(p(y) + iq(y))] = \psi(y), \\ \psi'(y) = |\dot{\delta}(y)|\frac{\partial w}{\partial \mathbf{n}}(y), \end{cases} \implies \begin{cases} q(y) = \psi(y), \\ \psi'(y) = |\dot{\delta}(y)|\frac{\partial w}{\partial \mathbf{n}}(y). \end{cases}$$

Which means for the Neumann problem the following equation is hold

$$q'(y) = |\dot{\delta}(y)|\frac{\partial w}{\partial \mathbf{n}}(y)$$

where

$$\frac{\partial w}{\partial \mathbf{n}}(y) = \phi(y).$$

3. GENERALIZED NEUMANN KERNEL

Let B to be defined in the form of $B(y) = \delta(y) - \alpha$. Using this function, the generalized Neumann kernel will be defined by [6]

$$R(x, y) := \frac{1}{\pi} \operatorname{Im} \left(\frac{B(x)}{B(y)} \frac{\dot{\delta}(y)}{\delta(y) - \delta(x)} \right).$$

Also by the following definition the kernel R will be continuous

$$R(y, y) := \frac{1}{\pi} \operatorname{Im} \left(\frac{1}{2} \frac{\ddot{\delta}(y)}{\dot{\delta}(y)} - \frac{\dot{B}(y)}{B(y)} \right).$$

The real kernel T will be defined by [6]

$$T(x, y) := \frac{1}{\pi} \Re \left(\frac{B(x)}{B(y)} \frac{\dot{\delta}(y)}{\delta(y) - \delta(x)} \right).$$

Then, the kernel T can be represented by

$$T(x, y) := -\frac{1}{2\pi} \cos \frac{x-y}{2} + T_1(x, y),$$

such that T_1 is a continuous kernel that take the following values on the diagonal

$$T_1(y, y) := \frac{1}{\pi} \Re \left(\frac{1}{2} \frac{\ddot{\delta}(y)}{\dot{\delta}(y)} - \frac{\dot{B}(y)}{B(y)} \right).$$

So the operator [6]

$$\mathbf{R}q(x) := \int_J R(x, y)q(y)dy, \quad x \in J,$$

will be a integral operator of Fredholm type. Moreover the operator

$$\mathbf{T}q(x) := \int_J T(x, y)q(y)dy, \quad x \in J,$$

is a singular integral operator.

Theorem 3.1. [6] Suppose that F to be a solution for the Dirichlet condition $\Re[F(\delta(y))] = p(y)$ provided by the following values on the boundary

$$F = p + iq$$

and $\text{Im } F(0) = 0$, then the following integral equation will be held

$$(3.1) \quad q - \mathbf{R}q = -\mathbf{T}p,$$

i.e.

$$q(x) - \int_0^{2\pi} R(x, y)q(y)dy = - \int_0^{2\pi} T(x, y)p(y)dy.$$

The IE in (3.1) represent a required condition that the solution of Dirichlet problem should satisfy on it. The following theorem will present the relation between q and final answer of Dirichlet problem.

Theorem 3.2. [6] Assume that p to be real Hölder continuous function. Moreover, assume also that q is a solution for equation (3.1) and Φ is

$$\Phi = \frac{1}{2\pi i} \int_{\Gamma} \frac{p + iq}{\delta - z} d\delta, \quad z \notin \Gamma.$$

Then $\Phi = f$ and satisfies

$$f = p + iq.$$

4. DIRICHLET PROBLEM WITH DISCONTINUOUS COEFFICIENTS

Let p is a function on the region G such that it is discontinuous at the point $y \in G$. If $\lim_{d \rightarrow y} p(d)$ does not exist, but both $\lim_{d \rightarrow y^+} p(d)$ and $\lim_{d \rightarrow y^-} p(d)$ exist, then y is called a discontinuity of the first kind, or jump discontinuity [3].

Suppose that there exist finite number of first kind discontinuities for a given function p . Dirichlet problem (with discontinuous coefficients) is looking for a function w such that [3–5],

(1) w should be harmonic in G^+ ,

(2)

$$(4.1) \quad w(\delta(y)) = p(y).$$

For Dirichlet problem with discontinuous coefficients (4.1), the unique solution can be considered as a real part of an analytic function in G called $F(z)$. Let for $k = 1, 2, \dots, l$ the discontinuities of the given function $p(y)$ occur at $y_k \in [0, 2\pi)$. Then the limit values of the given function $p(y)$ at the points of discontinuities can be represented by

$$a_k^- = p(y_k - 0), \quad a_k^+ = p(y_k + 0), \quad k = 1, 2, \dots, l.$$

We allow the boundary Γ to have corner points at $\delta(t_k)$ for $k = 1, 2, \dots, l$. Let $b_k^- = \arg_{y_k}(\delta(y_k - 0) - \delta(y_k))$, $b_k^+ = \arg_{y_k}(\delta(y_k + 0) - \delta(y_k))$, $k = 1, 2, \dots, l$, such that each value of the argument \arg_{y_k} is selected in a way that the branch cut form the point $\delta(y_k)$ to the point at ∞ is outside of the domain G . After that the value $b_k^+ - b_k^-$ denotes the interior angle between the left and right tangents to the boundary Γ at $\delta(y_k)$. Suppose that $0 < b_k^+ - b_k^- < 2\pi$. In case the boundary Γ be smooth at $\delta(y_k)$, then it can be considered that $b_k^+ - b_k^- = \pi$. With all this, the function $f(z)$ is defined in the following way

$$(4.2) \quad f(z) = F(z) + i \sum_{k=1}^l \frac{a_k^- - a_k^+}{a_k^- - a_k^+} \log_{y_k}(z - \delta(y_k)).$$

The representation (4.2) is different from the one used in [7]. The branch cut of \log_{y_k} is selected such that to be outside of the domain G from the point $\delta(y_k)$ to the point at ∞ . Hence by these facts, the function $f(z)$ will be analytic in G such that the boundary values are

$$(4.3) \quad \Re[f(\delta(y))] = \lambda(y)$$

where

$$(4.4) \quad \lambda(y) = p(y) - \sum_{k=1}^l \frac{a_k^- - a_k^+}{b_k^- - b_k^+} p_k(y)$$

and

$$p_k(y) = \operatorname{Im} \log_{y_k} (\delta(y) - \delta(y_k)) = \arg_{y_k} (\delta(y) - \delta(y_k)).$$

Theorem 4.1. $\lambda(y)$, given in (4.4), is continuous every where on the boundary Γ .

Proof. Continuity of $\lambda(y)$ on $[0, 2\pi] - \{y_1, \dots, y_l\}$ is clear. So just need to prove it is continuous at y_j for $j = 1, 2, \dots, m$. Then it will be enough to show that

$$\lambda(y_j - 0) - \lambda(y_j + 0) = 0.$$

Using this fact that the function $p_k(y)$ is continuous at all points y_j with $j \neq k$, it can be shown that

$$\begin{aligned} \lambda(y_j - 0) - \lambda(y_j + 0) &= p(y_j - 0) - \frac{a_j^- - a_j^+}{b_j^- - b_j^+} p_j(y_j - 0) - p(y_j + 0) + \frac{a_j^- - a_j^+}{b_j^- - b_j^+} p_j(y_j + 0) \\ &= a_j^- - \frac{a_j^- - a_j^+}{b_j^- - b_j^+} b_j^- - a_j^+ + \frac{a_j^- - a_j^+}{b_j^- - b_j^+} b_j^+ \\ &= a_j^- - a_j^+ - \frac{a_j^- - a_j^+}{b_j^- - b_j^+} (b_j^- - b_j^+) \\ &= a_j^- - a_j^+ - (a_j^- - a_j^+) \\ &= 0. \end{aligned}$$

□

So by all of these, now solving Dirichlet problem with discontinuous coefficients (4.1) is reduced to solve a Dirichlet problem with continuous coefficients (4.3).

Let $\operatorname{Im}[f(0)] = 0$ (without lose the generality). Moreover, suppose that $\varphi = \operatorname{Im}[f]$, i.e.,

$$(4.5) \quad f(\delta(y)) = \lambda(y) + i\varphi(y)$$

then by Theorem 3.1 we have

$$(4.6) \quad \varphi - \mathbf{R}\varphi = -\mathbf{T}\lambda.$$

By solving the system of equations obtained from (4.6), φ will be found, then $f(z)$ will resulted from (4.5), so $F(z)$ will be deducted from (4.2) and the final solution $w(z)$, the real part of the $F(z)$, will be found.

5. MIXED BOUNDARY VALUE PROBLEM ON BOUNDED SIMPLY CONNECTED REGIONS

Suppose that p_0, q_* are two given continuous functions on the boundary $\Gamma = \Gamma_1 \dot{\cup} \Gamma_2$. Mixed boundary value problem is looking for a function w such that (see Figure 2),

- (1) w should be harmonic in G^+ ,
- (2)

$$(5.1) \quad \begin{cases} w(y) = p_0(y), & y \in \Gamma_1, \\ \frac{\partial w}{\partial \mathbf{n}}(y) = q_*(y), & y \in \Gamma_2. \end{cases}$$

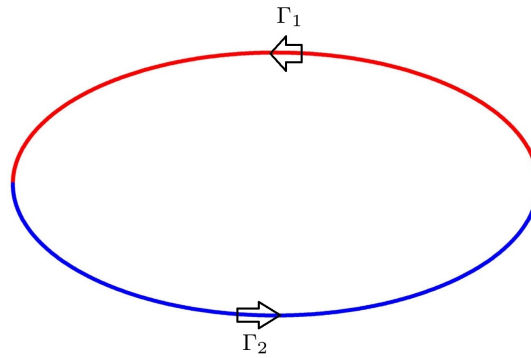


FIGURE 2. Mixed boundary value problem on simply connected region

Now we define the two following functions

$$p^*(y) = \begin{cases} p_0(y), & y \in \Gamma_1, \\ p_1(y) \text{ (Unknown)}, & y \in \Gamma_2, \end{cases}$$

and

$$(5.2) \quad q^*(y) = \begin{cases} q_0(y) \text{ (Unknown)}, & y \in \Gamma_1, \\ q_1(y), & y \in \Gamma_2, \end{cases}$$

where $q'_1(t) = q_*(y)|\dot{\delta}(y)|$. By using the Theorem 3.1 functions

$$(5.3) \quad p(y) = p^*(y) - \sum_{m \in S} \frac{a_m^{p^-} - a_m^{p^+}}{b_m^{p^-} - b_m^{p^+}} \arg_{y_m} (\delta(y) - \delta(y_m))$$

and

$$q(y) = q^*(y) - \sum_{m \in S} \frac{a_m^{q^-} - a_m^{q^+}}{b_m^{q^-} - b_m^{q^+}} \arg_{y_m}(\delta(y) - \delta(y_m))$$

are continuous on the boundary Γ such that $S = \{j, k\}$. In the equations (5.2) and (5.3), the values $a_m^{p^-}, a_m^{p^+}, b_m^{p^-}, b_m^{p^+}, a_m^{q^-}, a_m^{q^+}, b_m^{q^-}$ and $b_m^{q^+}$ are define as below:

$$\begin{cases} a_m^{p^-} = p^*(y_m - 0), \\ a_m^{p^+} = p^*(y_m + 0), \\ b_m^{p^-} = \arg_{y_m}(\delta(y_m - 0) - \delta(y_m)), \\ b_m^{p^+} = \arg_{y_m}(\delta(y_m + 0) - \delta(y_m)), \end{cases} \quad \begin{cases} a_m^{q^-} = q^*(y_m - 0), \\ a_m^{q^+} = q^*(y_m + 0), \\ b_m^{q^-} = \arg_{y_m}(\delta(y_m - 0) - \delta(y_m)), \\ b_m^{q^+} = \arg_{y_m}(\delta(y_m + 0) - \delta(y_m)). \end{cases}$$

Then application of the Theorem 3.2 for the function $f(\delta(y)) = p(y) + iq(y)$ on the boundary derive that

$$(5.4) \quad q - \mathbf{R}q = -\mathbf{T}p.$$

6. NUMERICAL IMPLIMENTATION

Suppose that Γ is divided into Γ_1 and Γ_2 in the following way:

$$\begin{cases} \delta(y_i) \in \Gamma_1, & i = j, j+1, \dots, k-1, k \\ \delta(y_i) \in \Gamma_2, & i = k+1, k+2, \dots, n-1, n, 1, 2, \dots, j-2, j-1 \end{cases}$$

By this construction, the following values can be define numerically:

$$\begin{cases} a_j^{p^-} = p^*(y_j - 0) = p^*(y_{j-1}), \\ a_j^{p^+} = p^*(y_j + 0) = p^*(y_j), \end{cases} \quad \begin{cases} a_j^{q^-} = q^*(y_j - 0) = q^*(y_{j-1}), \\ a_j^{q^+} = q^*(y_j + 0) = q^*(y_j), \end{cases}$$

$$\begin{cases} a_k^{p^-} = p^*(y_k - 0) = p^*(y_k), \\ a_k^{p^+} = p^*(y_k + 0) = p^*(y_{k+1}), \end{cases} \quad \begin{cases} a_k^{q^-} = q^*(y_k - 0) = q^*(y_k), \\ a_k^{q^+} = q^*(y_k + 0) = q^*(y_{k+1}). \end{cases}$$

Now by these definitions there will be $(k - j + 1)$ unknowns for p and $(n - k + j - 1)$ unknowns for q in the equation (5.4). It means that we have a system of equation with n equations and n unknowns that can be solved for unknowns.

Since we find p and q from (5.4), the function $f(z)$ can be find by

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{p + iq}{\delta - z} d\delta, \quad z \notin \Gamma.$$

After that the final solution of the mixed problem (5.1), $w(z)$, will be the real part of the following function

$$F(z) = f(z) + \sum_{m \in S} \frac{a_m^{p^-} - a_m^{p^+}}{b_m^{p^-} - b_m^{p^+}} \arg_{y_m} (z - \delta(y_m)) + i \sum_{m \in S} \frac{a_m^{q^-} - a_m^{q^+}}{b_m^{q^-} - b_m^{q^+}} \arg_{y_m} (z - \delta(y_m)).$$

7. NUMERICAL EXAMPLES

When the functions in any integral equations are 2π -periodic, one of the best numerical method for applying is Nyström method applying together with the trapezoidal rule [8]. Here, the functions A_j and δ_j are 2π -periodic. After applying this method equation (3.1) will reduced to solve linear system [8]

$$(7.1) \quad (I - K)\mathbf{x} = -Q\mathbf{y},$$

such that I is the identity matrix, Q is the $n \times n$ matrix related to the operator \mathbf{T} , K is the matrix related to the operator \mathbf{R} with size of $n \times n$, \mathbf{x} is a vector that predicts the values of unknown function ψ at the nodes with size of $n \times 1$ and finally \mathbf{y} is also a vector that predicts the values of unknown function ϕ at the nodes with size of $n \times 1$. See [7, 9] for details about Solvability of these problem.

In this paper, the linear systems (7.1) will be solved by the Wolfram Mathematica 10.4. From now on, $w(z)$ will represent the exact answer of mixed condition, $w_n(z)$ will represent the solution that founded numerically for the mixed condition with n nodes on the boundary resulted from presented method.

Example 1. Obtain a harmonic function w in the following region (see Figure 3) such that satisfy on:

$$\begin{aligned} \Gamma : \delta(y) &= \exp(iy) \\ w(y) &= \cos(y), \quad 0 \leq y < \pi, \\ \frac{\partial w}{\partial \mathbf{n}}(y) &= \cos(y), \quad \pi \leq y < 2\pi. \end{aligned}$$

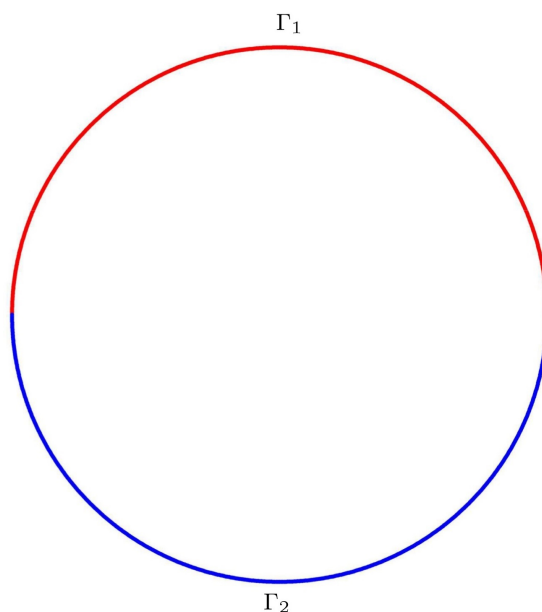


FIGURE 3. The region for Example 1

For this example, the exact answer is given by

$$w(s, t) = \Re[F(s, t)] = \Re[s + it] = s.$$

So we have two following functions

$$p^*(y) = \begin{cases} \cos(y), & y \in \Gamma_1 : [0, \pi), \\ p_1(y)(\text{Unknown}), & y \in \Gamma_2 : [\pi, 2\pi), \end{cases}$$

and

$$q^*(y) = \begin{cases} q_0(y)(\text{Unknown}), & y \in \Gamma_1 : [0, \pi), \\ \sin(y), & y \in \Gamma_2 : [\pi, 2\pi). \end{cases}$$

By using the Theorem 3.1 functions

$$(7.2) \quad p(y) = p^*(y) - \sum_{i=1}^2 \frac{a_i^{p^-} - a_i^{p^+}}{\pi} \arg_{y_i} (\delta(y) - \delta(y_i))$$

and

$$(7.3) \quad q(y) = q^*(y) - \sum_{i=1}^2 \frac{a_i^{q^-} - a_i^{q^+}}{\pi} \arg_{y_i} (\delta(y) - \delta(y_i))$$

are continuous on the boundary Γ such that $y_i = \{0, \pi\}$. In the equations (7.2) and (7.3), the values $a_i^{p^-}$, $a_i^{p^+}$, $a_i^{q^-}$ and $a_i^{q^+}$ are define as below:

$$\begin{cases} a_1^{p^-} = p^*(y_1 - 0) = p^*(y_n), \\ a_1^{p^+} = p^*(y_1 + 0) = p^*(y_1) = \cos(y_1), \\ a_2^{p^-} = p^*(y_2 - 0) = p^*(y_{\frac{n}{2}}) = \cos(y_{\frac{n}{2}}), \\ a_2^{p^+} = p^*(y_2 + 0) = p^*(y_{\frac{n}{2}} + 1), \\ a_1^{q^-} = q^*(y_1 - 0) = q^*(y_n) = \sin(y_n), \\ a_1^{q^+} = q^*(y_1 + 0) = q^*(y_1), \\ a_2^{q^-} = q^*(y_2 - 0) = q^*(y_{\frac{n}{2}}), \\ a_2^{q^+} = q^*(y_2 + 0) = q^*(y_{\frac{n}{2}} + 1) = \sin(y_{\frac{n}{2}} + 1). \end{cases}$$

Then application of the Theorem 3.2 for the function $f(\delta(y)) = p(y) + iq(y)$ on the boundary derive that

$$q - \mathbf{R}q = -\mathbf{T}p,$$

such that for the function q the values of $q(y_1), q(y_2), \dots, q(y_{\frac{n}{2}-1}), q(y_{\frac{n}{2}})$ are unknown and for the function p the values of $p(y_{\frac{n}{2}+1}), p(y_{\frac{n}{2}+2}), \dots, p(y_{n-1}), p(y_n)$ are unknown. Solving of this system of equations leads to the following results.

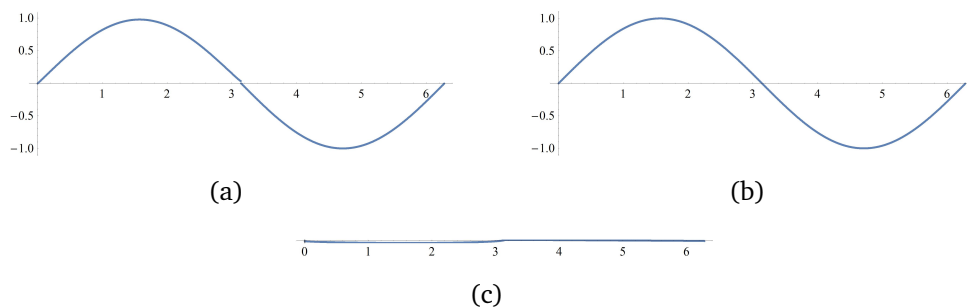


FIGURE 4. (a): The approximate function $q_n(y)$ obtained by the presented method with $n = 2048$ boundary points, (b): The exact function $q(y) = \sin(y)$ and (c): The error function $q_n(y) - q(y)$ from 0 to 2π .

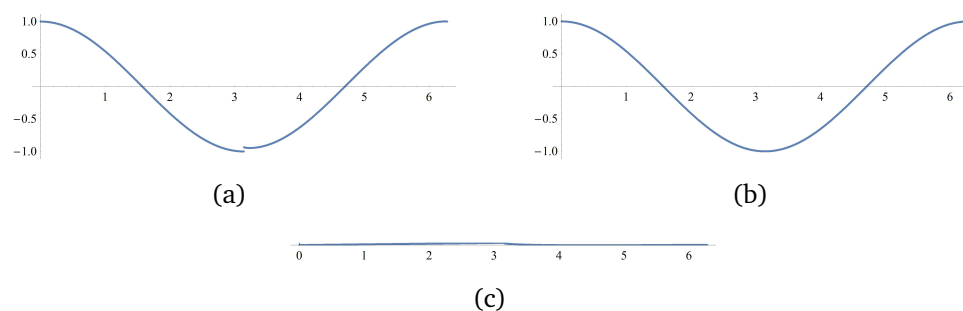


FIGURE 5. (a): The approximate function $p_n(y)$ obtained by the presented method with $n = 2048$ boundary points, (b): The exact function $p(y) = \cos(y)$ and (c): The error function $p_n(y) - p(y)$ from 0 to 2π .

z	$w_n(z)$	$ w(z) - w_n(z) $
$+0.80 + 0.55i$	$+0.7880$	3.1927×10^{-2}
$-0.50 + 0.70i$	-0.4463	5.3679×10^{-2}
$-0.40 - 0.40i$	-0.4093	9.3832×10^{-2}
$+0.60 + 0.05i$	$+0.5728$	2.7128×10^{-2}
$-0.20 + 0.90i$	-0.1617	3.8240×10^{-2}
$-0.10 - 0.10i$	-0.1027	2.7285×10^{-2}
$+0.99 + 0.10i$	$+0.9730$	1.6958×10^{-2}
$-0.10 + 0.90i$	-0.0669	3.3059×10^{-2}
$+0.40 - 0.84i$	$+0.3438$	5.6118×10^{-2}
$+0.10 + 0.99i$	$+0.1291$	2.9121×10^{-2}
$+0.10 - 0.89i$	$+0.0484$	5.1566×10^{-2}
$+0.30 - 0.49i$	$+0.2616$	3.8381×10^{-2}

TABLE 1. Absolute error $|w(z) - w_n(z)|$ at some selected points for Example 1 with $n = 128$

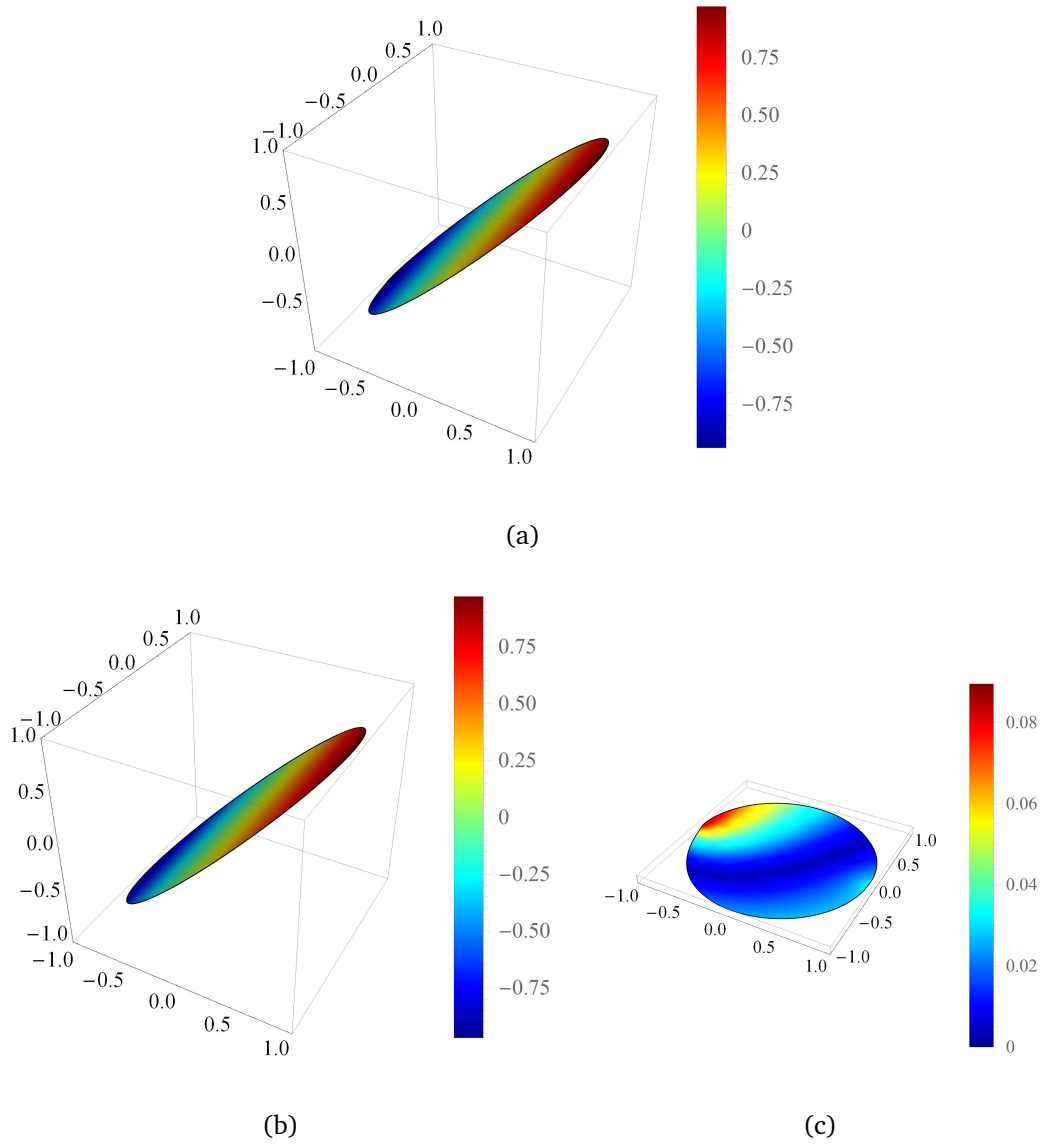


FIGURE 6. (a): The 3-D graph of the solution using presented method, (b): The 3-D graph of the exact answer and (c): $|w_n(z) - w(z)|$

The approximate function $q_n(y)$ obtained by the presented method with $n = 2048$ boundary points is shown in 4(a), the exact function $q(y) = \sin(y)$ is shown in 4(b) and the error function $q_n(y) - q(y)$ from 0 to 2π is shown in the Figure 4(c). The approximate function $p_n(y)$ obtained by the presented method with

$n = 2048$ boundary points is shown in 5(a), the exact function $p(y) = \cos(y)$ is shown in 5(b) and the error function $p_n(y) - p(y)$ from 0 to 2π is shown in the Figure 5(c). Also a 3-D graph for $w(z)$ using presented method with $n = 1024$ nodes is presented at Figure 6(a), 3-D plot of the exact solution is presented at Figure 6(b) and the $|w_n(z) - w(z)|$ is presented at Figure 6(c). Absolute error $|w(z) - w_n(z)|$ at some selected points for this example is shown in the table 1.

Example 2. Obtain a harmonic function w in the ellipse $\frac{s^2}{4} + t^2 = 1$ (see Figure 7) such that satisfy on:

$$\begin{aligned}\Gamma : \delta(y) &= 2 \cos(y) + i \sin(y), \\ w(y) &= 1.5 + 2.5 \cos(2y), \quad 0 \leq y < \frac{\pi}{2}, \\ \frac{\partial w}{\partial \mathbf{n}}(y) &= \frac{4 \cos(2y)}{\sqrt{2.5 + 1.5 \cos(2y)}}, \quad \frac{\pi}{2} \leq y < \frac{3\pi}{2}, \\ w(y) &= 1.5 + 2.5 \cos(2y), \quad \frac{3\pi}{2} \leq y < 2\pi.\end{aligned}$$

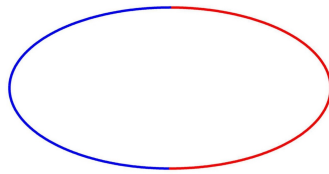


FIGURE 7. The region for Example 2

The exact solution of this example is given by

$$w(s, t) = \Re[F(s, t)] = \Re[(s + i t)^2] = s^2 - t^2.$$

So we have two following functions

$$p^*(y) = \begin{cases} 1.5 + 2.5 \cos(2y), & y \in \Gamma_1 : [0, \frac{\pi}{2}) \cup [\frac{3\pi}{2}, 2\pi), \\ p_1(y) \text{ (Unknown)}, & y \in \Gamma_2 : [\frac{\pi}{2}, \frac{3\pi}{2}), \\ 1.5 + 2.5 \cos(2y), & y \in \Gamma_1 : [0, \frac{\pi}{2}) \cup [\frac{3\pi}{2}, 2\pi), \end{cases}$$

and

$$q^*(y) = \begin{cases} q_0(y)(\text{Unknown}), & y \in \Gamma_1 : [0, \frac{\pi}{2}) \cup [\frac{3\pi}{2}, 2\pi), \\ 2 \sin(2y), & y \in \Gamma_2 : [\frac{\pi}{2}, \frac{3\pi}{2}), \\ q_2(y)(\text{Unknown}), & y \in \Gamma_1 : [0, \frac{\pi}{2}) \cup [\frac{3\pi}{2}, 2\pi). \end{cases}$$

By using the Theorem 3.1 functions

$$(7.4) \quad p(y) = p^*(y) - \sum_{i=1}^2 \frac{a_i^{p^-} - a_i^{p^+}}{\pi} \arg_{y_i} (\delta(y) - \delta(y_i))$$

and

$$(7.5) \quad q(y) = q^*(y) - \sum_{i=1}^2 \frac{a_i^{q^-} - a_i^{q^+}}{\pi} \arg_{y_i} (\delta(y) - \delta(y_i))$$

are continuous on the boundary Γ such that $y_i = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}$. In the equations (7.4) and (7.5), the values $a_i^{p^-}$, $a_i^{p^+}$, $a_i^{q^-}$ and $a_i^{q^+}$ are define as below:

$$\begin{cases} a_1^{p^-} = p^*(y_1 - 0) = p^*(y_{\frac{n}{4}}) = 1.5 + 2.5 \cos(2y_{\frac{n}{4}}), \\ a_1^{p^+} = p^*(y_1 + 0) = p^*(y_{\frac{n}{4}+1}), \\ a_2^{p^-} = p^*(y_2 - 0) = p^*(y_{\frac{3n}{4}}), \\ a_2^{p^+} = p^*(y_2 + 0) = p^*(y_{\frac{3n}{4}+1}) = 1.5 + 2.5 \cos(2y_{\frac{3n}{4}+1}), \\ a_1^{q^-} = q^*(y_1 - 0) = q^*(y_{\frac{n}{4}}) = 2 \sin(y_{\frac{n}{4}}), \\ a_1^{q^+} = q^*(y_1 + 0) = q^*(y_{\frac{n}{4}+1}), \\ a_2^{q^-} = q^*(y_2 - 0) = q^*(y_{\frac{3n}{4}}), \\ a_2^{q^+} = q^*(y_2 + 0) = q^*(y_{\frac{3n}{4}+1}) = 2 \sin(y_{\frac{3n}{4}+1}). \end{cases}$$

Then application of the Theorem 3.2 for the function $f(\delta(y)) = p(y) + iq(y)$ on the boundary derive that

$$q - \mathbf{R}q = -\mathbf{T}p,$$

such that for the function q the values of $q(y_1), q(y_2), \dots, q(y_{\frac{n}{4}-1}), q(y_{\frac{n}{4}})$ and $q(y_{\frac{3n}{4}+1}), q(y_{\frac{3n}{4}+2}), \dots, q(y_{n-1}), q(y_n)$ are unknown and for the function p the values of $p(y_{\frac{n}{4}+1}), p(y_{\frac{n}{4}+2}), \dots, p(y_{\frac{3n}{4}-1}), p(y_{\frac{3n}{4}})$ are unknown. Solving of this system of equations leads to the following results.

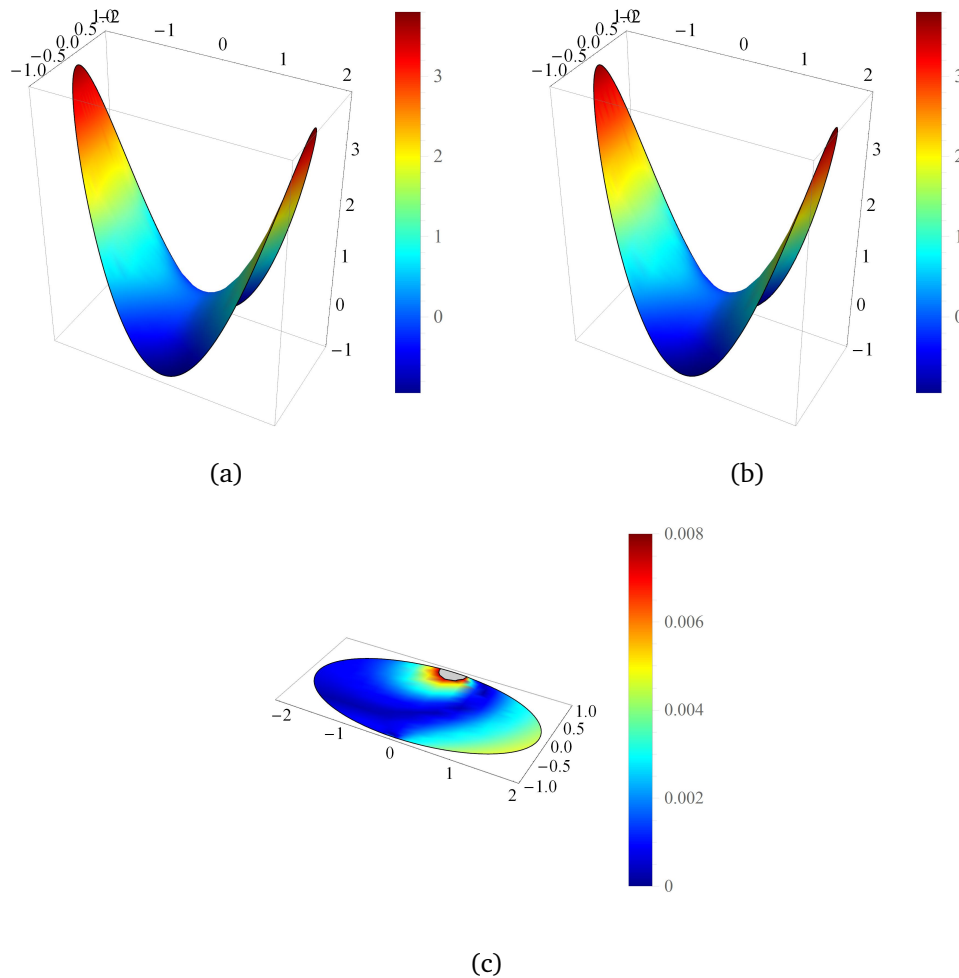


FIGURE 8. (a): The surface plot of the solution by presented method, (b): The surface plot of the solution by exact answer and (c): The absolute error $|w_n(z) - w(z)|$ for the entire region.

The approximate function $q_n(y)$ obtained by the presented method with $n = 2048$ boundary points is shown in 9(a), the exact function $q(y) = 2 \sin(2y)$ is shown in 9(b) and the error function $q_n(y) - q(y)$ from 0 to 2π is shown in the Figure 9(c). The approximate function $p_n(y)$ obtained by the presented method with $n = 2048$ boundary points is shown in 10(a), the exact function $p(y) = \cos(y)$ is shown in 10(b) and the error function $p_n(y) - p(y)$ from 0 to 2π is shown in the Figure 10(c). Also the 3-D graph found by suggested method with $n = 2048$ nodes is presented at Figure 8(a), 3-D graph of the exact solution

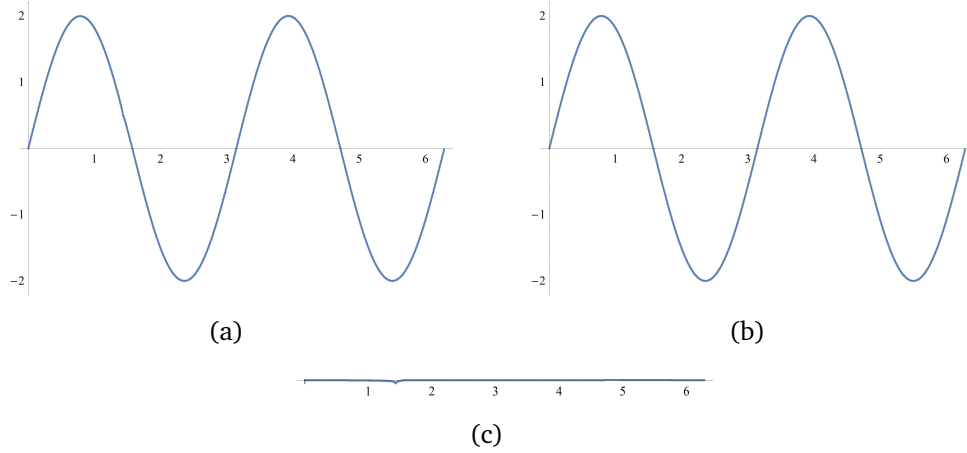


FIGURE 9. (a): The approximate function $q_n(y)$ obtained by the presented method with $n = 2048$ boundary points, (b): The exact function $q(y) = 2 \sin(2y)$ and (c): The error function $q_n(y) - q(y)$ from 0 to 2π .

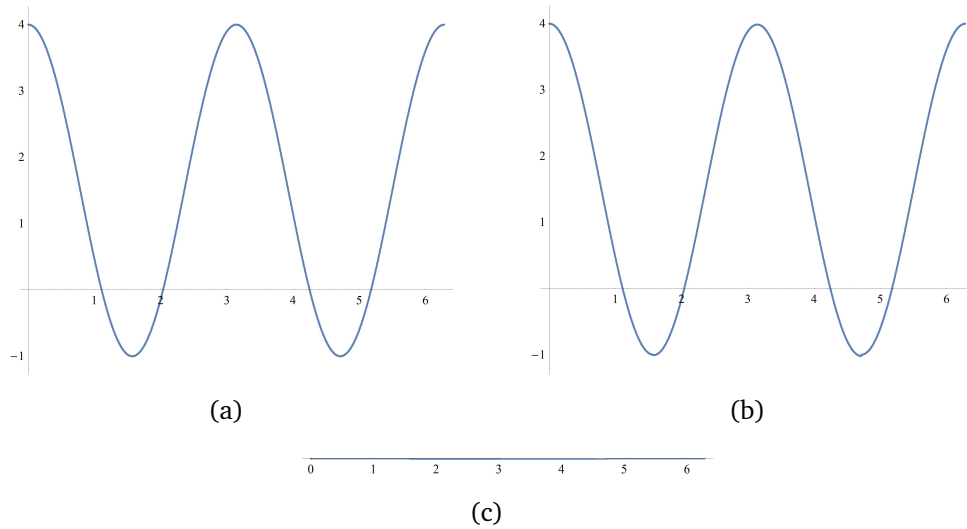


FIGURE 10. (a): The approximate function $p_n(y)$ obtained by the presented method with $n = 2048$ boundary points, (b): The exact function $p(y) = 1.5 + 2.5 \cos(2y)$ and (c): The error function $p_n(y) - p(y)$ from 0 to 2π .

is shown in Figure 8(b) and $|w_n(z) - w(z)|$ is presented at Figure 8(c). Absolute error $|w(z) - w_n(z)|$ at some selected points for this example is shown in the table 2.

z	$w_n(z)$	$ w(z) - w_n(z) $
$+1.50 + 0.00i$	$+1.3501$	8.9988×10^{-1}
$-1.50 + 0.00i$	$+1.3766$	8.7335×10^{-1}
$+1.01 - 0.50i$	-0.0400	8.1010×10^{-1}
$+1.60 + 0.05i$	$+1.6483$	9.0919×10^{-1}
$-1.20 + 0.90i$	-9.4747	1.0104×10^{-1}
$-1.10 - 0.10i$	$+0.3354$	8.6452×10^{-1}
$+1.99 + 0.10i$	$+2.9941$	9.5595×10^{-1}
$-0.10 + 0.90i$	-1.7443	9.4435×10^{-1}
$+1.04 - 0.84i$	-0.4006	7.7668×10^{-1}
$+1.10 + 0.70i$	-0.2739	9.9399×10^{-1}
$+1.10 - 0.70i$	-0.0659	7.8593×10^{-1}
$+1.30 - 0.49i$	$+0.6239$	8.2596×10^{-1}

TABLE 2. Absolute error $|w(z) - w_n(z)|$ at some selected points for Example 2 with $n = 128$

Example 3. This example will illustrate that the suggested method works effectively for problems with unknown solution. Obtain a harmonic function w in the following region(unit disk) such that satisfy on:

$$\begin{aligned}\Gamma : \delta(y) &= \exp(iy) \\ w(y) &= 1 + 4 \cos(3y), \quad 0 \leq y < \frac{\pi}{4}, \\ \frac{\partial w}{\partial \mathbf{n}}(y) &= 15 \cos(3y), \quad \frac{\pi}{4} \leq y < 2\pi.\end{aligned}$$

So we have two following functions

$$p^*(y) = \begin{cases} 1 + 4 \cos(3y), & y \in \Gamma_1 : [0, \frac{\pi}{4}), \\ p_1(y) \text{ (Unknown)}, & y \in \Gamma_1 : [\frac{\pi}{4}, 2\pi), \end{cases}$$

and

$$q^*(y) = \begin{cases} q_0(y)(\text{Unknown}), & y \in \Gamma_1 : [0, \frac{\pi}{4}), \\ 5 \sin(3y), & y \in \Gamma_1 : [\frac{\pi}{4}, 2\pi). \end{cases}$$

By using the Theorem 3.1 functions

$$(7.6) \quad p(y) = p^*(y) - \sum_{i=1}^2 \frac{a_i^{p^-} - a_i^{p^+}}{\pi} \arg_{y_i} (\delta(t) - \delta(y_i))$$

and

$$(7.7) \quad q(y) = q^*(y) - \sum_{i=1}^2 \frac{a_i^{q^-} - a_i^{q^+}}{\pi} \arg_{y_i} (\delta(y) - \delta(y_i))$$

are continuous on the boundary Γ such that $y_i = \left\{0, \frac{\pi}{4}\right\}$. In the equations (7.6) and (7.7), the values $a_i^{p^-}$, $a_i^{p^+}$, $a_i^{q^-}$ and $a_i^{q^+}$ are define as below:

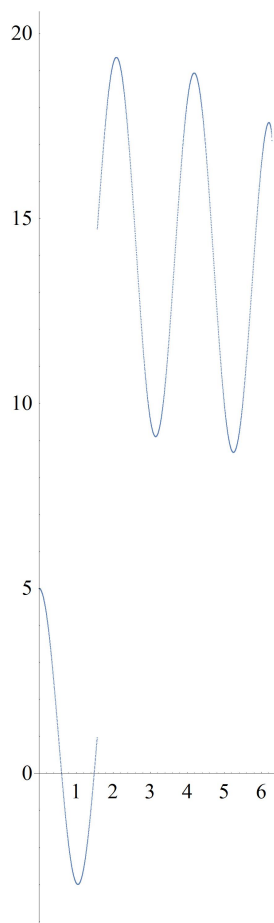
$$\begin{cases} a_1^{p^-} = p^*(y_1 - 0) = p^*(y_n), \\ a_1^{p^+} = p^*(y_1 + 0) = p^*(y_1) = 1 + 4 \cos(3y_1), \\ a_2^{p^-} = p^*(y_2 - 0) = p^*(y_{\frac{n}{4}}) = 1 + 4 \cos(3y_{\frac{n}{4}}), \\ a_2^{p^+} = p^*(y_2 + 0) = p^*(y_{\frac{n}{4}+1}), \\ a_1^{q^-} = q^*(y_1 - 0) = q^*(y_n), \\ a_1^{q^+} = q^*(y_1 + 0) = q^*(y_1) = 5 \sin(3y_1), \\ a_2^{q^-} = q^*(y_2 - 0) = q^*(y_{\frac{n}{4}}) = 5 \sin(3y_{\frac{n}{4}}), \\ a_2^{q^+} = q^*(y_2 + 0) = q^*(y_{\frac{n}{4}+1}). \end{cases}$$

Then application of the Theorem 3.2 for the function $f(\delta(y)) = p(y) + iq(y)$ on the boundary derive that

$$q - \mathbf{R}q = -\mathbf{T}p,$$

such that for the function q the values of $q(y_1), q(y_2), \dots, q(y_{\frac{n}{4}-1}), q(y_{\frac{n}{4}})$ are unknown and for the function p the values of $p(y_{\frac{n}{4}+1}), p(y_{\frac{n}{4}+2}), \dots, p(y_{n-1}), p(y_n)$ are unknown. Solving of this system of equations leads to the following results.

n	z	$w_n(z)$
256	$+0.80 + 0.55i$	18.060838031
	$-0.50 + 0.70i$	22.258975272
	$-0.40 - 0.40i$	19.658067453
512	$+0.60 + 0.05i$	19.576364835
	$-0.20 + 0.90i$	21.582214815
	$-0.10 - 0.10i$	18.888932099
1024	$+0.99 + 0.10i$	22.341987453
	$-0.10 + 0.90i$	20.638285909
	$+0.40 - 0.84i$	15.025243598
2048	$+0.10 + 0.99i$	18.455687961
	$+0.10 - 0.89i$	17.989434561
	$+0.30 - 0.49i$	18.147305126

TABLE 3. The value of $w_n(z)$ at some selected points for Example 3FIGURE 11. The approximate function $p_n(t)$ obtained by the presented method with $n = 2048$ boundary points

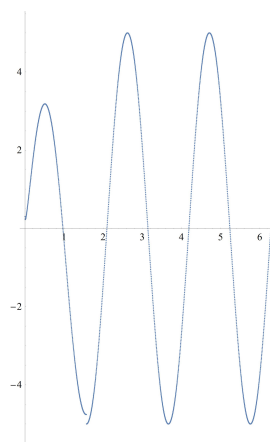


FIGURE 12. The approximate function $q_n(t)$ obtained by the presented method with $n = 2048$ boundary points

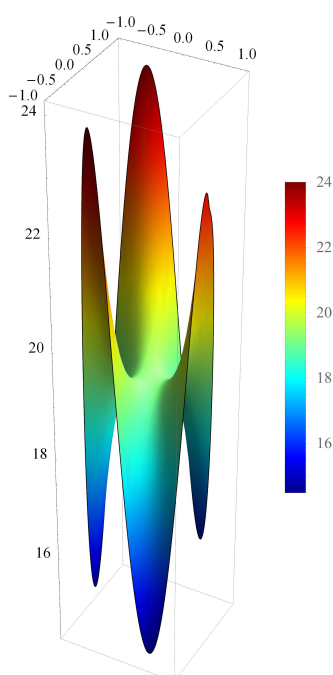


FIGURE 13. The surface plot of the solution by presented method

The approximate function $p_n(t)$ obtained by the presented method with $n = 2048$ boundary points is shown in 11, the approximate function $q_n(t)$ obtained by the presented method with $n = 2048$ boundary points is shown in 12 and

also the 3–D graph found by suggested method is presented at Figure 13. The value of $w_n(z)$ at some selected points for this example is shown in the table 3.

8. CONCLUSIONS

A method for solving mixed condition on bounded simply connected regions has been presented in this paper. For the future work, this method can be extended to solve mixed condition on unbounded(or bounded) simply(or multiply) connected regions. Accuracy of method is proved by some examples that already compared with exact solution.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCENCE

UNIVERSITY OF TECHNOLOGY

PAPUA NEW GUINEA

Email address: mohsen.aghaeiboorkheili@pnguot.ac.pg

DEPARTMENT OF MECHANICAL ENGINEERING

UNIVERSITY OF TECHNOLOGY

PAPUA NEW GUINEA

Email address: aezeden.mohamed@pnguot.ac.pg