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NUMERICAL QUENCHING FOR A SLOW DIFFUSION SYSTEM COUPLED AT THE BOUNDARY

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ABSTRACT. This paper concerns the study of a numerical approximation for the following problem, $u_t = u_{xx}$, $v_t = v_{xx}$, $0 < x < 1$, $0 < t < T$; $u_x(0,t) = (u^{-p_1}v^{-q_1})(0,t)$, $v_x(0,t) = (u^{-p_2}v^{-q_2})(0,t)$ and $u_x(1,t) = v_x(1,t) = 0$, $0 < t < T$, with p_1, q_1, p_2 and q_2 real parameters. We show that the solution of the semidiscrete scheme, obtained by the finite differences method quenches in a finite time only at first node of the mesh. We also prove that the time derivative of the solution blows up at quenching node and establish some conditions under which occurs the non-simultaneous or simultaneous quenching for the solution of the semidiscrete problem. After show the convergence of the quenching time, we finally present some numerical results to illustrate certain point of our work.

1. INTRODUCTION

Consider the following Newton filtration equations

$$u_t(x,t) = u_{xx}(x,t),\ v_t(x,t) = v_{xx}(x,t),\ (x,t) \in (0,1) \times (0,T), \tag{1.1}$$

coupled with the boundary singularities at the left border

$$u_x(0,t) = (u^{-p_1}v^{-q_1})(0,t), \quad u_x(1,t) = 0, \quad t \in (0,T), \tag{1.2}$$

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$$v_x(0,t) = (u^{-p_2}v^{-q_2})(0,t), \quad v_x(1,t) = 0, \quad t \in (0,T), \tag{1.3}$$

subject to smooth initial data

$$u(x,0) = u_0(x) > 0, \quad v(x,0) = v_0(x) > 0, \quad x \in [0,1], \tag{1.4}$$

where p_2, $q_1 \geq 0$, p_1, $q_2 > 0$, $u_0'(x)$, $v_0'(x) > 0$, $u_0''(x)$, $v_0''(x) \leq 0$, $x \in (0,1)$, and u_0, v_0 compatible on the boundary.

The equations in (1.1) describe the heat-conduction of electron in the plasma body and the radiation heat-conduction, where the thermal conductivity increases while the temperature is decreasing. The nonlinear boundary conditions can be understood as that the heat convection occurs on the surfaces of bodies [24].

The problem (1.1)–(1.4) is said to be quench at time T if the two components u and v of the solution (u,v) of (1.1)–(1.4) are nonnegative for all $(x,t) \in [0,1] \times [0,T)$ and

$$\liminf_{t \to T^-} \min\{u(\cdot,t), v(\cdot,t)\} = 0^+.$$

The quenching phenomenon was first observed by Kawarada [16]. Since then, it has attracted a lot of attention, both for scalar and coupling cases. Many studies have concentrated on the quenching solutions, including quenching criteria, quenching locations, quenching rates, and quenching profiles, see [3]- [7], [10]-[18], [22], [25] and references therein. Ji and Zheng [15] studied the problem (1.1)-(1.4), they obtain that if $p_2 \geq p_1 + 1$, $q_1 \geq q_2 + 1$, then quenching is always simultaneous, while for $p_2 < p_1 + 1$ with $q_1 \geq q_2 + 1$, or $q_1 < q_2 + 1$ with $p_2 \geq p_1 + 1$, the quenching must be non-simultaneous. When $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$, both simultaneous and non-simultaneous quenching is possible.

Unless we are mistaken, the studies carried out so far do not concern the numerical approximation of the problem (1.1)–(1.4). We will therefore deal in this paper with the numerical study using a semidiscrete form of (1.1)–(1.4), particularly in study of simultaneous and non-simultaneous quenching. We start by the construction of the semidiscrete scheme. Let $I \geq 3$ be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$ with $i = 0, \ldots, I$. Approximate the solution (u,v) of (1.1)–(1.4) by the solution $(U_h(t) = (U_0(t), \ldots, U_I(t))^T, V_h(t) = (V_0(t), \ldots, V_I(t))^T)$ and approximate the initial data (u_0, v_0) of the same problem by $\left(\varphi_{1,h} = (\varphi_{1,0}, \ldots, \varphi_{1,I})^T, \varphi_{2,h} = (\varphi_{2,0}, \ldots, \varphi_{2,I})^T\right)$ of the following system of ODEs whose is obtain using the finite difference method

$$U_i'(t) = \delta^2 U_i(t) - b_i(U^{-p_1}V^{-q_1})_i(t), \quad i = 0, \dots, I, \ t \in (0, T_h), \tag{1.5}$$

$$V_i'(t) = \delta^2 V_i(t) - b_i(U^{-p_2}V^{-q_2})_i(t), \quad i = 0, \dots, I, \ t \in (0, T_h), \tag{1.6}$$

$$U_i(0) = \varphi_{1,i}, \quad V_i(0) = \varphi_{2,i}, \quad i = 0, \dots, I, \tag{1.7}$$

where

$$p_2, q_1 \geq 0, \ p_1, q_2 > 0, \ 0 < \varphi_{1,i} \leq M, \ 0 < \varphi_{2,i} \leq M, \ i = 0, \dots, I,$$
$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \ \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}, \ t \in (0, T_h),$$
$$\delta^2 U_i(t) = \frac{U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)}{h^2}, \ 1 \leq i \leq I-1, \ t \in (0, T_h),$$
$$b_0 = \frac{2}{h}, \text{ and } b_i = 0, \ i = 1, \dots, I.$$

Here $[0, T_h)$ is the maximal time interval such that

$$\forall t \in [0, T_h), \ \inf \min_{0 \leq i \leq I} \{U_i(t), V_i(t)\} > 0.$$

For $0 \leq i \leq I$, we have

$$\liminf_{t \to T_h^-} \min\{U_i(t), V_i(t)\} = 0^+.$$

The time T_h can be finite or infinite. On the one hand, if T_h is finite, we say that the solution (U_h, V_h) of (1.5)–(1.7) quenches in a finite time and T_h is called the semidiscrete quenching time of (U_h, V_h). We say on the other hand that the solution (U_h, V_h) quenches globally when T_h is infinite.

Numerical approximations of heat equations with non-linear boundary conditions have been the focus of many authors in recent years. We refer to [1], [2], [8], [19]– [21], [23] and the references cited therein for our work.

The paper is organized as follows. In the next section, we give some properties concerning our semidiscrete scheme. In Section 3, under some conditions, we prove that the solution of the semidiscrete scheme (1.5)–(1.7) quenches in a finite time, we give a result on numerical quenching set. We also show that the time derivative of the solution blows up at quenching node. A criterion to identify simultaneous and non-simultaneous quenching is proposed in section 4. In Section 5, we show the convergence of the solution of the semidiscrete scheme and the convergence of the quenching times to the theoretical one when the mesh size goes to zero. Finally, in last section, we give some numerical experiments.

2. PROPERTIES OF THE SEMIDISCRETE SCHEME

In this section, we give some auxiliary results for the problem (1.5)–(1.7).

Definition 2.1. *We say that* $(\underline{U_h}, \underline{V_h}) \in \left(C^1([0,T_h),\mathbb{R}^{I+1})\right)^2$ *is a lower solution of* (1.5)–(1.7) *if*

$$\begin{aligned} \underline{U_i}'(t) &\le \delta^2 \underline{U_i}(t) - b_i(\underline{U_i}^{-p_1}(t)\underline{V_i}^{-q_1}(t)), \quad i=0,\dots,I,\ t\in(0,T_h),\\ \underline{V_i}'(t) &\le \delta^2 \underline{V_i}(t) - b_i(\underline{U_i}^{-p_2}(t)\underline{V_i}^{-q_2}(t)), \quad i=0,\dots,I,\ t\in(0,T_h),\\ 0<\underline{U_i}(0) &\le \varphi_{1,i},\ 0<\underline{V_i}(0)\le\varphi_{2,i}, \qquad i=0,\dots,I, \end{aligned}$$

where (U_h, V_h) *is the solution of* (1.5)–(1.7). *On the other hand, we say that* $(\overline{U_h}, \overline{V_h}) \in \left(C^1([0,T_h),\mathbb{R}^{I+1})\right)^2$ *is an upper solution of* (1.5)–(1.7) *if these inequalities are reversed.*

The following lemma is a discrete form of the maximum principle.

Lemma 2.1. *Let* $e_h, c_h, \alpha_h, \beta_h \in C^0([0,T_h),\mathbb{R}^{I+1})$ *and* $U_h, V_h \in C^1([0,T_h),\mathbb{R}^{I+1})$ *such that*

$$\begin{aligned} U_i'(t) - \delta^2 U_i(t) + e_i(t)U_i(t) + c_i(t)V_i(t) &\ge 0, \quad i=0,\dots,I,\ t\in(0,T_h),\\ V_i'(t) - \delta^2 V_i(t) + \alpha_i(t)U_i(t) + \beta_i(t)V_i(t) &\ge 0, \quad i=0,\dots,I,\ t\in(0,T_h),\\ U_i(0)\ge 0,\ V_i(0) &\ge 0, \qquad i=0,\dots,I. \end{aligned}$$

Then we have

$$U_i(t)\ge 0,\ V_i(t)\ge 0, \quad i=0,\dots,I,\ t\in(0,T_h).$$

Proof. Let $T_0 < T_h$ and let $(Z_h(t), W_h(t)) = (\mathrm{e}^{\lambda t}U_h(t), \mathrm{e}^{\lambda t}V_h(t))$ where λ is a real. We find that $(Z_h(t), W_h(t))$ satisfies the following inequalities :

$$Z_i'(t) - \delta^2 Z_i(t) + (e_i(t)-\lambda)Z_i(t) + c_i(t)W_i(t) \ge 0, \tag{2.1}$$

$$i=0,\dots,I,\ t\in(0,T_h),$$

$$W_i'(t) - \delta^2 W_i(t) + \alpha_i(t)Z_i(t) + (\beta_i(t)-\lambda)W_i(t) \ge 0,$$

$$i=0,\dots,I,\ t\in(0,T_h),$$

$$Z_i(0) \geq 0,\ W_i(0) \geq 0, \quad i = 0, \dots, I. \tag{2.2}$$

Set $m = \min\left\{\min_{0\leq i\leq I, t\in[0,T_0]} Z_i(t), \min_{0\leq i\leq I, t\in[0,T_0]} W_i(t)\right\}$. Since for $i \in \{0, \dots, I\}$, Z_i and W_i are continuous functions on a compact, we can assume that $m = Z_{i_0}(t_{i_0})$ for a certain $i_0 \in \{0, \dots, I\}$.

Assume $m < 0$. Taking λ negative such that

$$(e_{i_0}(t_{i_0}) - \lambda)Z_{i_0}(t_{i_0}) + c_{i_0}(t_{i_0})W_{i_0}(t_{i_0}) < 0.$$

If $t_{i_0} = 0$, then $Z_{i_0}(0) < 0$, which contradicts (2.2), hence $t_{i_0} \neq 0$; if $0 \leq i_0 \leq I$, we have

$$Z'_{i_0}(t_{i_0}) = \lim_{k\to 0} \frac{Z_{i_0}(t_{i_0}) - Z_{i_0}(t_{i_0} - k)}{k} \leq 0.$$

Moreover by a straightforward computation we get

$$Z'_{i_0}(t_{i_0}) - \delta^2 Z_{i_0}(t_{i_0}) + (e_{i_0}(t_{i_0}) - \lambda)Z_{i_0}(t_{i_0}) + c_{i_0}(t_{i_0})W_{i_0}(t_{i_0}) < 0,$$

but this inequalitie contradicts (2.1) and the proof is completed. □

Lemma 2.2. *Let* $(\underline{U_h}, \underline{V_h}), (\overline{U_h}, \overline{V_h}) \in \left(C^1([0,T_h), \mathbb{R}^{I+1})\right)^2$ *be lower and upper solutions of* (1.5)–(1.7) *respectively such that,* $(\underline{U_h}(0), \underline{V_h}(0)) \leq (\overline{U_h}(0), \overline{V_h}(0))$ *then*

$$(\underline{U_h}(t), \underline{V_h}(t)) \leq (\overline{U_h}(t), \overline{V_h}(t)).$$

Proof. Let us define $(Z_h(t), W_h(t)) = (\overline{U_h}(t), \overline{V_h}(t)) - (\underline{U_h}(t), \underline{V_h}(t))$. By a straightforward computation and using the Mean value theorem, we obtain

$$Z'_i(t) - \delta^2 Z_i(t) - p_1 b_i(\mu_i(t))^{-p_1-1} Z_i(t) - q_1 b_i(\nu_i(t))^{-q_1-1} W_i(t) \geq 0, \tag{2.3}$$

$$i = 0, \dots, I$$

$$W'_i(t) - \delta^2 W_i(t) - p_2 b_i(\mu_i(t))^{-p_2-1} Z_i(t) - q_2 b_i(\nu_i(t))^{-q_2-1} W_i(t) \geq 0, \tag{2.4}$$

$$i = 0, \dots, I$$

$$Z_i(0) \geq 0,\ W_i(0) \geq 0, \quad i = 0, \dots, I$$

where $\mu_i(t), \nu_i(t)$ lie, respectively, between $\underline{U_i}(t)$ and $\overline{U_i}(t)$, and between $\underline{V_i}(t)$ and $\overline{V_i}(t)$, for $i \in \{0, \dots, I\}$.

We can rewrite (2.3)–(2.4) as

$$Z'_i(t) - \delta^2 Z_i(t) + e_i(t) Z_i(t) + c_i(t) W_i(t) \geq 0, \quad i = 0, \dots, I,\ t \in (0, T_h),$$

$$W'_i(t) - \delta^2 W_i(t) + \alpha_i(t) Z_i(t) + \beta_i(t) W_i(t) \geq 0, \quad i = 0, \dots, I,\ t \in (0, T_h),$$

where $e_i(t) = -p_1 b_i(\mu_i(t))^{-p_1-1}$, $c_i(t) = -q_1 b_i(\nu_i(t))^{-q_1-1}$, $\alpha_i(t) = -p_2 b_i(\mu_i(t))^{-p_2-1}$ and $\beta_i(t) = -q_2 b_i(\nu_i(t))^{-q_2-1},\ i = 0, \dots, I\ \forall t \in (0, T_h)$. According to Lemma 2.1, $Z_i(t) \geq 0$, $W_i(t) \geq 0$, for $i = 0, \dots, I$, $\forall t \in (0, T_h)$ and the proof is completed. □

The next lemma gives the properties of the semidiscrete solution.

Lemma 2.3. *Let* $(U_h, V_h) \in \left(C^1([0, T_h), \mathbb{R}^{I+1})\right)^2$ *be the solution of* (1.5)–(1.7) *with an initial data* $(\varphi_{1,h}, \varphi_{2,h})$ *upper solution such that* $0 < \varphi_{1,i} < \varphi_{1,i+1} \leq M$ *and* $0 < \varphi_{2,i} < \varphi_{2,i+1} \leq M$ *for* $i = 0, \dots, I-1$. *Then we have*

(i) $0 < U_i(t) \leq \varphi_{1,i} \leq M$ *and* $0 < V_i(t) \leq \varphi_{2,i} \leq M$ *for* $i = 0, \dots, I$, $t \in [0, T_h)$;

(ii) $(U_{i+1}(t), V_{i+1}(t)) > (U_i(t), V_i(t))$, $i = 0, \dots, I-1$, $t \in (0, T_h)$;

(iii) $(U_i'(t), V_i'(t)) \leq 0$, $i = 0, \dots, I$, $t \in (0, T_h)$.

Proof.

(i) Since $(\varphi_{1,h}, \varphi_{2,h})$ is an upper solution of (1.5)–(1.7), by the Lemma 2.1 and 2.2 we have $0 < U_i(t) \leq \varphi_{1,i} \leq M$ and $0 < V_i(t) \leq \varphi_{2,i} \leq M$ for $i = 0, \dots, I$, $t \in [0, T_h)$.

(ii) We argue by contradiction. Assume that t_0 is the first $t > 0$, such that $(K_i, L_i)(t) = (U_{i+1} - U_i, V_{i+1} - V_i)(t) > 0$, for $0 \leq i \leq I-1$, but $\min\{K_{i_0}(t_0), L_{i_0}(t_0)\} = 0$ for a certain $i_0 \in \{0, \dots, I-1\}$. Assume that $K_{i_0}(t_0) = U_{i_0+1}(t_0) - U_{i_0}(t_0) = 0$. Without lost of generality, we can suppose that i_0 is the smallest integer which satisfies the above equality. We have

$$K_0'(t) = \frac{U_2(t) - 2U_1(t) + U_0(t)}{h^2} - \left(\frac{2U_1(t) - 2U_0(t)}{h^2} - \frac{2}{h}\left(U_0^{-p_1}(t) V_0^{-q_1}(t)\right)\right)$$

$$K_i'(t) = \frac{U_{i+2}(t) - 2U_{i+1}(t) + U_i(t)}{h^2} - \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2},\ 1 \leq i \leq I-2$$

$$K_{I-1}'(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} - \frac{U_I(t) - 2U_{I-1}(t) + U_{I-2}(t)}{h^2}$$

$$\begin{cases} K_0'(t) = \dfrac{K_1(t) - 3K_0(t)}{h^2} + \dfrac{2}{h}\left(U_0^{-p_1}(t) V_0^{-q_1}(t)\right) \\ K_i'(t) = \dfrac{K_{i+1}(t) - 2K_i(t) + K_{i-1}(t)}{h^2},\ 1 \leq i \leq I-2 \\ K_{I-1}'(t) = \dfrac{K_{I-2}(t) - 3K_{I-1}(t)}{h^2} \end{cases} \tag{2.5}$$

According to the hypotheses on t_0, we have the following inequalities:

$$K'_{i_0}(t_0) = \lim_{\epsilon \to 0} \frac{K_{i_0}(t_0) - K_{i_0}(t_0 - \epsilon)}{\epsilon} \leq 0,$$

$$\frac{K_{i_0+1}(t_0) - 2K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} > 0 \text{ if } 1 \leq i_0 \leq I-2,$$

$$\frac{K_{i_0+1}(t_0) - 3K_{i_0}(t_0)}{h^2} > 0 \text{ if } i_0 = 0,$$

$$\frac{-3K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} > 0 \text{ if } i_0 = I-1,$$

which implies,

$$K'_{i_0}(t_0) - \frac{K_{i_0+1}(t_0) - 2K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} < 0 \text{ if } 1 \leq i_0 \leq I-2,$$

$$K'_{i_0}(t_0) - \frac{K_{i_0+1}(t_0) - 3K_{i_0}(t_0)}{h^2} - \frac{2}{h}\left(U_0^{-p_1}(t)V_0^{-q_1}(t)\right) < 0 \text{ if } i_0 = 0,$$

$$K'_{i_0}(t_0) - \frac{-3K_{i_0}(t_0) + K_{i_0-1}(t_0)}{h^2} < 0 \text{ if } i_0 = I-1.$$

Thus, we have a contradiction with (2.5), which leads to the desired result.

(iii) Denote $F_i(t) = U_i(t) - U_i(t+\varepsilon)$ and $G_i(t) = V_i(t) - V_i(t+\varepsilon)$, for $i = 0, \dots, I$, using (i) and (1.7) we obtain $F_i(0) \geq 0$, $G_i(0) \geq 0$ for $i = 0, \dots, I$. It is not hard to see that

$$F'_i(t) - \delta^2 F_i(t) + p_1 b_i(\xi_i(t))^{-p_1-1} F_i(t) + q_1 b_i(\eta_i(t))^{-q_1-1} G_i(t) \geq 0,$$
$$G'_i(t) - \delta^2 G_i(t) + p_2 b_i(\xi_i(t))^{-p_2-1} F_i(t) + q_2 b_i(\eta_i(t))^{-q_2-1} G_i(t) \geq 0,$$

where $\xi_i(t)$, $\eta_i(t)$ lie, respectively, between $U_i(t+\varepsilon)$ and $U_i(t)$ and between $V_i(t+\varepsilon)$ and $V_i(t)$. From Lemma 2.1 we get

$$F_i(t) \geq 0 \text{ and } G_i(t) \geq 0 \text{ for } i = 0, \dots, I, \ t \in (0, T_h).$$

This fact implies the desired result. $\square$

3. Semidiscrete quenching solution

Let (U_h, V_h) be the solution of (1.5)–(1.7) with $0 < \varphi_{1,i} \leq M$, $0 < \varphi_{2,i} \leq M$ for $i = 0, \ldots, I$. Using [4] and [9], we show that (U_h, V_h) quenches in a finite time and (U_h', V_h') blows up at quenching node.

Theorem 3.1. *For every initial data, the solution* (U_h, V_h) *of the system* (1.5)–(1.7) *quenches in finite time with the only quenching node* $\{i = 0\}$.

Proof. Integrating (1.5) in time we have

$$U_i(t) - U_i(0) = \int_0^t \left(\delta^2 U_i(\tau) - b_i \left(U_i^{-p_1}(\tau) V_i^{-q_1}(\tau)\right)\right) d\tau$$

summing up the above equality we arrive at

$$\sum_{i=0}^{I} hU_i(t) = \sum_{i=0}^{I} hU_i(0) + \int_0^t \left(\frac{U_{I-1}(\tau) - U_I(\tau)}{h} + \frac{U_1(\tau) - U_0(\tau)}{h} - 2\left(U_0^{-p_1}(\tau) V_0^{-q_1}(\tau)\right)\right) d\tau.$$

(1.5) implies that

$$\frac{h}{2}U_I(t) - \frac{h}{2}U_I(0) = \int_0^t \frac{U_{I-1}(\tau) - U_I(\tau)}{h} d\tau, \text{ and}$$

$$\frac{h}{2}U_0(t) - \frac{h}{2}U_0(0) = \int_0^t \left(\frac{U_1(\tau) - U_0(\tau)}{h} - \left(U_0^{-p_1}(\tau) V_0^{-q_1}(\tau)\right)\right) d\tau.$$

Thus

$$\frac{h}{2}U_I(t) + \sum_{i=1}^{I-1} hU_i(t) + \frac{h}{2}U_0(t) = \frac{h}{2}U_I(0) + \sum_{i=1}^{I-1} hU_i(0) + \frac{h}{2}U_0(0) - \int_0^t \left(U_0^{-p_1}(\tau) V_0^{-q_1}(\tau)\right) d\tau,$$

therefore

$$\frac{h}{2}U_I(t) + \sum_{i=1}^{I-1} hU_i(t) + \frac{h}{2}U_0(t) \leq M - \left(M^{-p_1} M^{-q_1}\right) t.$$

By the same way, we also prove that

$$\frac{h}{2}V_I(t) + \sum_{i=1}^{I-1} hV_i(t) + \frac{h}{2}V_0(t) \leq M - \left(M^{-p_2} M^{-q_2}\right) t,$$

which yield a contradiction because U_h and V_h are positive for all times in $[0,T_h)$. Then there exists $0 < T_h < \infty$ such that

$$\lim_{t \to T_h^-} \min\{U_0(t), V_0(t)\} = 0^+.$$

Now we will show that $\{i = 0\}$ is the unique quenching node. In everything that follows $i \in \{0,\dots,I-1\}$ and $t \in (0,T_h)$. Set $g(U_i(t)) = U_i^{-p_1}(t)$, $f(V_i(t)) = V_i^{-q_1}(t)$, $d(U_i(t)) = U_i^{-p_2}(t)$, $j(V_i(t)) = V_i^{-q_2}(t)$, and

$$Z_i(t) = \frac{U_{i+1}(t) - U_i(t)}{h} - \phi_i(g(U_i(t)) + f(V_i(t)))$$
$$W_i(t) = \frac{V_{i+1}(t) - V_i(t)}{h} - \phi_i(d(U_i(t)) + j(V_i(t)))$$

where $\phi_i, \delta^2\phi_i \geq 0$, $\delta^+\phi_i \leq 0$, $\phi_I = 0$, $\phi_0 = 1$, $\phi_i(g(U_i(0)) + f(V_i(0))) \leq \delta^+ U_i(0)$ et $\phi_i(d(U_i(0)) + j(V_i(0))) \leq \delta^+ V_i(0)$.

By means of Taylor expansions inspired by [9] we have

$$\begin{aligned}\delta^2(\phi_i k(J_i(t))) = \phi_i k'(J_i(t))\delta^2 J_i(t) + k(J_i(t))\delta^2\phi_i + k'(J_i(t))\delta^+\phi_i\delta^+ J_i(t) + \\ k'(J_i(t))\delta^-\phi_i\delta^- J_i(t) + \phi_i\frac{(\delta^+ J_i(t))^2}{2}k''(\rho_i(t)) + \\ \phi_i\frac{(\delta^- J_i(t))^2}{2}k''(\lambda_i(t)),\ i = 1,\dots,I-1,\end{aligned}$$

$$\begin{aligned}\delta^2(\phi_0 k(J_0(t))) = \phi_0 k'(J_0(t))\delta^2 J_0(t) + k(J_0(t))\delta^2\phi_0 + 2k'(J_0(t))\delta^+\phi_0\delta^+ J_0(t) + \\ \phi_0(\delta^+ J_0(t))^2 k''(\rho_0(t)).\end{aligned}$$

If we use the fact that J_i, $\delta^+ J_i(t)$ and $\delta^2 J_i(t)$ are nonnegative and the hypothesis on ϕ_h, we arrive at

$$\delta^2(\phi_i k(J_i(t))) \geq \phi_i k'(J_i(t))\delta^2 J_i(t),\ i = 0,\dots,I-1. \tag{3.1}$$

By using (3.1) we can get

$$\begin{aligned}Z_i'(t) - \delta^2 Z_i(t) \geq \frac{b_i}{h}(g(U_i) + f(V_i)) + b_i\phi_i g'(U_i)(g(U_i) + f(V_i)) + \\ b_i\phi_i f'(U_i)(d(U_i) + j(V_i)).\end{aligned}$$

The above inequalities implies that

$$\begin{aligned}Z_i'(t) - \delta^2 Z_i(t) + b_i g'(U_i(t))Z_i(t) + b_i f'(V_i(t))W_i(t) \geq b_i[\frac{1}{h}(g(U_i(t)) + f(V_i(t))) + \\ f'(U_i(t))(d(U_i(t)) + j(V_i(t))) + g'(U_i(t))(g(U_i(t)) + f(V_i(t)))].\end{aligned}$$

We obtain

$$Z_i'(t) - \delta^2 Z_i(t) + b_i g'(U_i(t))Z_i(t) + b_i f'(V_i(t))W_i(t) \geq 0,$$

for the parameter h small enough. Thus we have

$$\begin{aligned} Z_i'(t) - \delta^2 Z_i(t) + b_i g'(U_i(t))Z_i(t) + b_i f'(V_i(t))W_i(t) &\geq 0, \\ W_i'(t) - \delta^2 W_i(t) + b_i d'(U_i(t))Z_i(t) + b_i j'(V_i(t))W_i(t) &\geq 0, \\ Z_i(0) \geq 0,\ W_i(0) &\geq 0. \end{aligned}$$

Using the Lemma 2.1 we have $Z_i(t) \geq 0$ and $W_i(t) \geq 0$, for $i = 0, \ldots, I-1$ and $t \in (0, T_h)$. This implies that $\dfrac{U_{i+1}(t) - U_i(t)}{h} \geq \phi_i(g(U_i(t)) + f(V_i(t))) \geq \dfrac{1}{2}\left(\dfrac{1}{M^{p_1}} + \dfrac{1}{M^{q_1}}\right)$ for $i = 0, \ldots, J$, with $\phi_J = \dfrac{1}{2}$, where $J \in \{1, \ldots, I-1\}$. Thus by summing we get

$$U_i(t) \geq U_0 + \frac{ih}{2}\left(\frac{1}{M^{p_1}} + \frac{1}{M^{q_1}}\right) \geq \frac{ih}{2}\left(\frac{1}{M^{p_1}} + \frac{1}{M^{q_1}}\right) \text{ whenever } i > 0.$$

We deal with V_h by the same way. □

Theorem 3.2. *If* $\lim_{t \to T_h^-} U_0(t) = 0$ $\left(\lim_{t \to T_h^-} V_0(t) = 0\right)$, *then* $U_h'(t)$ *blows up* ($V_h'(t)$ *blows up*).

Proof. Suppose that $U_h'(t)$ is bounded. Then, there exists a nonnegative constant C such that $U_h'(t) > C$ and we have

$$\sum_{i=0}^{I-1}\sum_{j=0}^{i} h^2 U_j'(t) > \sum_{i=0}^{I-1}\sum_{j=0}^{i} h^2 C.$$

$$\sum_{i=0}^{I-1}\sum_{j=0}^{i} h^2 C = \sum_{i=0}^{I-1} (i+1)h^2 C = \frac{I}{2}\left(h^2 C + I h^2 C\right) = \frac{hC}{2} + \frac{C}{2}.$$

$$\sum_{i=0}^{I-1}\sum_{j=0}^{i} h^2 U_j'(t) = \sum_{i=1}^{I-1}\left(\sum_{j=1}^{i} h^2 U_j'(t) + h^2 U_0'(t)\right) + h^2 U_0'(t).$$

From (1.5) we arrive at

$$\sum_{i=0}^{I-1}\sum_{j=0}^{i} h^2 U_j'(t) = U_I(t) - U_0(t) - \left(U_0^{-p_1}(t) V_0^{-q_1}(t)\right) + \frac{h}{2} U_0'(t)$$

and using the Lemma 2.3 we obtain

$$U_I(t) - U_0(t) - \left(U_0^{-p_1}(t)V_0^{-q_1}(t)\right) > hC + \frac{C}{2}.$$

As $t \to T_h^-$, the left-hand side tends to infinity while the right-side is finite. This contradiction proves that U_h' blows up. $\square$

4. Simultaneous versus non-simultaneous quenching

We identify simultaneous and non-simultaneous quenching in this section. We consider (U_h, V_h) the solution of (1.5)–(1.7) with h fixed.

Theorem 4.1. *If U_h quenches and V_h does not quench in* (1.5)–(1.7) *then* $p_2 < p_1 + 1$.

Proof. We suppose that V_h does not quench. By (1.5) there exists $c > 0$ such that

$$U_0'(t) \geq -cU_0^{-p_1}(t),$$

integrating this inequality from t to T_h, we obtain

$$U_0(t) \leq C(T_h - t)^{\frac{1}{p_1+1}}, \text{ where } C = ((p_1+1)c)^{\frac{1}{p_1+1}}. \tag{4.1}$$

Now we use (4.1) and (1.6) and we arrive at

$$V_0'(t) \leq \delta^2 V_0(t) - b_0 \left(V_0^{-q_2}(t)C(T_h - t)^{\frac{-p_2}{p_1+1}}\right).$$

Thus $V_0(T_h) \leq C_1 - C_2 \int_0^{T_h} (T_h - t)^{\frac{-p_2}{p_1+1}} dt$. We remark that this integral diverges if $p_2 \geq p_1 + 1$, which is a contradiction and the proof is completed. $\square$

Theorem 4.1 implies the following corollary:

Corollary 4.1. *If $p_2 \geq p_1 + 1$ and $q_1 \geq q_2 + 1$, then any quenching in* (1.5)–(1.7) *must be simultaneous.*

Lemma 4.1. *Let (U_h, V_h) be the solution of* (1.5)–(1.7). *Assume that U_h quenches at time T_h (V_h quenches at time T_h), $0 < \varphi_{1,i} \leq M$, $0 < \varphi_{2,i} \leq M$ for $i = 0, \ldots, I$ and*

$$\delta^2 \varphi_{1,i} - b_i \left(\varphi_{1,i}^{-p_1} \varphi_{2,i}^{-q_1}\right) + c\left(\varphi_{1,i}^{-p_1} \varphi_{2,i}^{-q_1}\right) \leq 0, \tag{4.2}$$

$$\delta^2 \varphi_{2,i} - b_i \left(\varphi_{1,i}^{-p_2} \varphi_{2,i}^{-q_2}\right) + c\left(\varphi_{1,i}^{-p_2} \varphi_{2,i}^{-q_2}\right) \leq 0. \tag{4.3}$$

Then there exists a positive constant C such that for $t \in (0, T_h)$

$$\frac{U_0^{p_1+1}(t)}{C(p_1+1)} \geq T_h - t \quad \left(\frac{V_0^{q_2+1}(t)}{C(q_2+1)} \geq T_h - t\right),$$

$$U_0(t) \geq C(T_h - t)^{\frac{1}{p_1+1}} \quad \left(V_0(t) \geq C(T_h - t)^{\frac{1}{q_2+1}}\right).$$

Proof. Set for $i = 0, \dots, I$, $t \in [0, T_h)$,

$$Z_i(t) = U_i'(t) + c\left(U_i^{-p_1}(t)V_i^{-q_1}(t)\right) \text{ and } W_i(t) = V_i'(t) + c\left(U_i^{-p_2}(t)V_i^{-q_2}(t)\right).$$

A straightforward calculation and also the Mean value theorem give

$$\begin{aligned} Z_i'(t) - \delta^2 Z_i(t) + \alpha_i(t)Z_i(t) + \beta_i(t)W_i(t) \leq 0,\ i = 0, \dots, I,\ t \in (0, T_h), \\ W_i'(t) - \delta^2 W_i(t) + a_i(t)Z_i(t) + b_i(t)W_i(t) \leq 0,\ i = 0, \dots, I,\ t \in (0, T_h), \\ Z_i(0) \leq 0, \quad W_i(0) \leq 0, \quad i = 0, \dots, I. \end{aligned}$$

Using the Lemma 2.1, we have

$$Z_i(t) \leq 0,\ W_i(t) \leq 0, \quad i = 0, \dots, I,\ t \in (0, T_h).$$

Thus we get

(4.4)$U_i'(t) \leq -cU_i^{-p_1}(t)$ and $V_i'(t) \leq -cV_i^{-q_2}(t),\ i = 0, \dots, I,\ t \in (0, T_h)$.

Using the fact that U_h quenches (V_h quenches) and integrating (4.4) from t to T_h, we arrive at

$$\frac{U_0^{p_1+1}(t)}{C(p_1+1)} \geq T_h - t \quad \left(\frac{V_0^{q_2+1}(t)}{C(q_2+1)} \geq T_h - t\right),$$

and we have so

$$U_0(t) \geq C(T_h - t)^{\frac{1}{p_1+1}} \quad \left(V_0(t) \geq C(T_h - t)^{\frac{1}{q_2+1}}\right).$$

□

Theorem 4.2. *If $q_1 < q_2 + 1$ ($p_2 < p_1 + 1$), then there exist initial data such that V_h (U_h) quenches but U_h (V_h) doesn't.*

Proof. Here we argume by contradiction. Assuming that U_h and V_h quench simultaneously at time T_h for any initial data. We can write

$$\int_0^t U_0'(s)ds \geq \int_0^{T_h} U_0'(s)ds = \frac{2}{h^2}\int_0^{T_h}(U_1(s)-U_0(s))ds - \frac{2}{h}\int_0^{T_h}(U_0^{-p_1}(s)V_0^{-q_1}(s))ds.$$

Using the Lemma 4.1, we have

$$\begin{aligned} U_0(t) &\geq U_0(0)+\frac{2}{h^2}\int_0^{T_h}(U_1(s)-U_0(s))ds - \frac{2C}{h}\int_0^{T_h}(T_h-s)^{\frac{-p_1}{p_1+1}}(T_h-s)^{\frac{-q_1}{q_2+1}}ds \\ &\geq U_0(0)+\frac{2}{h^2}\int_0^{T_h}(U_1(s)-U_0(s))ds - \frac{2C}{h}\int_0^{T_h}(T_h-s)^{\frac{-p_1}{p_1+1}+\frac{-q_1}{q_2+1}}ds. \end{aligned}$$

$q_1 < q_2+1$ implies that this integral is converged and

$$U_0(t) \geq C_1 - C_2 T_h^{\frac{1}{p_1+1}-\frac{q_1}{q_2+1}}, \text{ with } C_1, C_2 > 0.$$

By summation of (1.6) we observe that

$$-\frac{h}{2}V_0'(t) - \frac{h}{2}V_I'(t) - \sum_{i=1}^{I-1} hV_i'(t) = U_0^{-p_2}(t)V_0^{-q_2}(t),$$

$$-\frac{h}{2}V_0'(t) - \frac{h}{2}V_I'(t) - \sum_{i=1}^{I-1} hV_i'(t) \geq U_0^{-p_2}(0)V_0^{-q_2}(0) \tag{4.5}$$

integrate (4.5) from 0 to T_h, we can obtain

$$V_I(0)\left(U_0^{-p_2}(0)V_0^{-q_2}(0)\right)^{-1} \geq T_h,$$

then if T_h is sufficiently small (depending on $U_h(0)$ and $V_h(0)$), $U_0(T_h) \geq c_0 > 0$. We have so a contradiction with the hypothesis that U_h quenches and leads us to the desired result. □

Theorem 4.3. *If* $p_2 \leq \dfrac{q_2(p_1+1)}{q_2+1}$ *and* $q_1 \geq q_2+1$ $\left(q_1 \leq \dfrac{p_1(q_2+1)}{p_1+1}\right.$ *and* $\left.p_2 \geq p_1+1\right)$ *then* U_h *(*V_h*) quenches alone under any positive initial data.*

Proof. Assume that there exists initial data such that U_h and V_h quench simultaneously at time T_h. We can suppose without lost of generality that this initial data satisfies (4.2)–(4.3). According to (1.6)

$$V_0'(t) = \delta^2 V_0(t) - b_0(U_0^{-p_2}(t)V_0^{-q_2}(t)),$$
$$V_0'(t) \geq -b_0(U_0^{-p_2}(t)V_0^{-q_2}(t)),$$
$$V_0(t) \leq b_0 \int_t^{T_h} U_0^{-p_2}(s)V_0^{-q_2}(s)ds.$$

We know from Lemma 4.1 that $U_0(t) \geq C_1(T_h-t)^{\frac{1}{p_1+1}}$, $V_0(t) \geq C_2(T_h-t)^{\frac{1}{q_2+1}}$. As $p_2 \leq \dfrac{q_2(p_1+1)}{q_2+1}$, there exists $\alpha > 0$ such that $V_0(t) \leq \alpha(T_h-t)^{\frac{1}{q_2+1}}$. (1.5) implies

$$U_0'(t) = \delta^2 U_0(t) - b_0(U_0^{-p_1}(t)V_0^{-q_1}(t)),$$
$$U_0'(t) \leq \delta^2 U_0(t) - b_0 V_0^{-q_1}(t)),$$
$$U_0'(t) \leq \delta^2 U_0(t) - b_0\alpha^{-q_1}(T_h-t)^{\frac{-q_1}{q_2+1}}.$$

Integrating both sides from 0 to T_h, we obtain $U_0(0) \geq -c_1 + c_2 \int_0^{T_h}(T_h-t)^{\frac{-q_1}{q_2+1}}dt$. It is clear that the integral diverges if $q_1 \geq q_2+1$, which is a contradiction. We have so the desired result. □

Remark 4.1. *We can see of the Lemma 4.1 and the proof of Theorem 4.3 that if* U_h (V_h) *quenches at time* T_h*, then* $U_0(t) \sim (T_h-t)^{\frac{1}{p_1+1}}$ $\left(V_0(t) \sim (T_h-t)^{\frac{1}{q_2+1}}\right)$ *for* t *close enough to* T_h *where* (U_h, V_h) *is the solution of* (1.5)–(1.7) *such that the initial data satisfies* (4.2)–(4.3).

5. Convergence of the semidiscrete quenching time

Under some assumptions, we show in this section that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. To obtain the convergence of semidiscrete quenching time, we firstly prove the following theorem about the convergence of the semidiscrete scheme. Before, we denote

$$u_h(t) = (u(x_0,t),\ldots,u(x_I,t))^T, \quad v_h(t) = (v(x_0,t),\ldots,v(x_I,t))^T,$$

$$\|U_h(t)\|_\infty = \max_{0\leq i\leq I}|U_i(t)|, \quad \|U_h(t)\|_{\inf} = \min_{0\leq i\leq I}|U_i(t)|.$$

Theorem 5.1. *Assume that the problem* (1.1)–(1.4) *has solution* $(u,v) \in \left(C^{4,1}\left([0,1]\times[0,T^*]\right)\right)^2$ *and the initial data* $(\varphi_{1,h},\varphi_{2,h})$ *of* (1.5)–(1.7) *satisfies*

$$\|\varphi_{1,h}-u_h(0)\|_\infty = o(1), \quad \|\varphi_{2,h}-v_h(0)\|_\infty = o(1) \quad h\to 0. \tag{5.1}$$

Then, for h *sufficiently small, the problem* (1.5)–(1.7) *has a unique solution* $(U_h, V_h) \in \left(C^1\left([0,T^*],\mathbb{R}^{I+1}\right)\right)^2$ *such that*

$$\max_{t\in[0,T^*]}\|U_h(t)-u_h(t)\|_\infty = O(\|\varphi_{1,h}-u_h(0)\|_\infty + \|\varphi_{2,h}-v_h(0)\|_\infty + h),\ \textit{as } h\to 0,$$
$$\max_{t\in[0,T^*]}\|V_h(t)-v_h(t)\|_\infty = O(\|\varphi_{1,h}-u_h(0)\|_\infty + \|\varphi_{2,h}-v_h(0)\|_\infty + h),\ \textit{as } h\to 0.$$

Proof. Let $\rho>0$ be such that

$$(\|u\|_\infty, \|v\|_\infty) < \rho,\ t\in[0,T^*]. \tag{5.2}$$

Then the problem (1.5)–(1.7) has for each h, a unique solution $(U_h,V_h) \in \left(C^1\left([0,T^*],\mathbb{R}^{I+1}\right)\right)^2$. Let $t(h)\le T^*$ be the greatest value of $t>0$ such that

$$\max\left\{\|U_h(t)-u_h(t)\|_\infty, \|V_h(t)-v_h(t)\|_\infty\right\} < 1. \tag{5.3}$$

The relation (5.1) implies $t(h)>0$ for h small enough. Using the triangle inequality, we obtain

$$\|U_h(t)\|_\infty \le 1+\rho \text{ and } \|V_h(t)\|_\infty \le 1+\rho \text{ for } t\in(0,t(h)). \tag{5.4}$$

Let $(e_{1,h},e_{2,h})(t) = (U_h-u_h, V_h-v_h)(t)$ $\forall t\in[0,T^*]$ be the discretization error. These error functions verify

$$e'_{1,i}(t) = \delta^2 e_{1,i}(t) + p_1 b_i(\theta_i(t))^{-p_1-1}e_{1,i}(t) + q_1 b_i(\Theta_i(t))^{-q_1-1}e_{2,i}(t) + O(h),$$
$$e'_{2,i}(t) = \delta^2 e_{2,i}(t) + p_2 b_i(\theta_i(t))^{-p_2-1}e_{1,i}(t) + q_2 b_i(\Theta_i(t))^{-q_2-1}e_{2,i}(t) + O(h),$$

where $\theta_i(t)$ and $\Theta_i(t)$ lie, respectively, between $U_i(t)$ and $u(x_i,t)$, and between $V_i(t)$ and $v(x_i,t)$, for $i\in\{0,\dots,I\}$. Using (5.2) and (5.4), there exist K and L positive constants such that

$$e'_{1,i}(t) \le \delta^2 e_{1,i}(t) + b_i L|e_{1,i}(t)| + b_i L|e_{2,i}(t)| + Kh,$$
$$e'_{2,i}(t) \le \delta^2 e_{2,i}(t) + b_i L|e_{1,i}(t)| + b_i L|e_{2,i}(t)| + Kh.$$

Let $(z,w)\in\left(C^{4,1}([0,1],[0,T^*])\right)^2$ be such that

$$z(x,t) = (\|\varphi_{1,h}-u_h(0)\|_\infty + \|\varphi_{2,h}-v_h(0)\|_\infty + Qh)\,\mathrm{e}^{(M+2)t-(1-x)^2}$$

and $w = z$ $\forall (x,t) \in [0,1] \times [0,T^*]$, with M, Q positive constants. By the Lemma 2.2, we can prove that

$$|e_{1,i}(t)| < z(x_i,t) \text{ and } |e_{2,i}(t)| < w(x_i,t) \text{ with } 0 \leq i \leq I \text{ for } t \in (0,t(h)).$$

Thus we get

$$\|U_h(t) - u_h(t)\|_\infty \leq (\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + Qh)\, \mathrm{e}^{(M+2)t},$$
$$\|V_h(t) - v_h(t)\|_\infty \leq (\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + Qh)\, \mathrm{e}^{(M+2)t},$$

where $t \in (0,t(h))$. Suppose that $T^* > t(h)$. From (5.3), we obtain

$$1 = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq (\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + Qh)\, \mathrm{e}^{(M+2)t}.$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $1 \leq 0$, which is impossible. Consequently $t(h) = T^*$ and we conclude the proof. □

Theorem 5.2. *Let* $(u,v) \in \left(C^{4,1}([0,1]\times[0,T[)\right)^2$ *be solution of* (1.1)–(1.4) *with quenches time* T *and the initial data at* (1.5)–(1.7) *satisfies* (4.2)–(4.3) *and* (5.1). *Then the solution* (U_h, V_h) *of* (1.5)–(1.7) *quenches in a finite time* T_h *and we have* $\lim_{h\to 0^+} T_h = T$.

Proof. Set $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\frac{y^{1+p_1}}{C(p_1+1)} \leq \frac{\varepsilon}{2}, \quad 0 \leq y \leq \eta. \tag{5.5}$$

There exists a time $T_0 \in (T - \varepsilon/2; T)$ such that $0 < |u(x_i,t)| \leq \frac{\eta}{2}$ for $i = 0, \dots, I$ and $t \in [T_0, T)$. Denote $T_1 = \dfrac{T_0 + T}{2}$, we obtain easily that $0 < \|u(x_i,t)\|_{\inf}$, for $t \in [0, T_1]$. It follows from Theorem 5.1 that for h sufficiently small

$$\|U_h(t) - u_h(t)\|_\infty \leq \frac{\eta}{2}.$$

Applying the triangle inequality, we get

$$\|U_h(T_1)\|_{\inf} \leq \|U_h(T_1) - u_h(T_1)\|_\infty + \|u_h(T_1)\|_{\inf} \leq \eta.$$

We know from Theorem 3.1 that (U_h, V_h) quenches in a finite time T_h. Assuming that U_h quenches, we can deduce from Lemma 4.1 and (5.5) that

$$|T_h - T| \leq |T_h - T_1| + |T_1 - T| \leq \frac{\|U_h(T_1)\|_{\inf}^{1+p_1}}{C(p_1+1)} + \frac{\varepsilon}{2} \leq \varepsilon.$$

The case where V_h quenches is analogous. □

6. Numerical experiments

In this section, we present some numerical approximations to the quenching time of (1.5)–(1.7) for the initial data $\varphi_{1,i} = \varphi_{2,i} = 1 + \frac{2}{\pi}\sin\left(\frac{\pi}{2}ih\right)$ for $i = 0, \dots, I$, with different values of p_1, p_2, q_1 and q_2.

By setting $W_i(t) = (U_i(t))^{-1}$ and $W_{I+1+i}(t) = (V_i(t))^{-1}$, $i = 0, \dots, I$, we obtain the following differential system

$$\begin{aligned}
W_0'(t) &= \frac{2}{h^2}\left(W_0(t) - \frac{(W_0(t))^2}{W_1(t)}\right) + \frac{2}{h}(W_0(t))^{p_1+2}(W_{I+1}(t))^{q_1}\\
W_i'(t) &= \frac{1}{h^2}\left(2W_i(t) - \frac{(W_i(t))^2}{W_{i-1}(t)} - \frac{(W_i(t))^2}{W_{i+1}(t)}\right),\ i = 1, \dots, I-1\\
W_I'(t) &= \frac{2}{h^2}\left(W_I(t) - \frac{(W_I(t))^2}{W_{I-1}(t)}\right)\\
W_{I+1}'(t) &= \frac{2}{h^2}\left(W_{I+1}(t) - \frac{(W_{I+1}(t))^2}{W_{I+2}(t)}\right) + \frac{2}{h}(W_0(t))^{p_2}(W_{I+1}(t))^{q_2+2}\\
W_{I+i+1}'(t) &= \frac{1}{h^2}\left(2W_{I+i+1}(t) - \frac{(W_{I+i+1}(t))^2}{W_{I+i}(t)} - \frac{(W_{I+i+1}(t))^2}{W_{I+i+2}(t)}\right),\ i = 1, \dots, I-1\\
W_{2I+1}'(t) &= \frac{2}{h^2}\left(W_{2I+1}(t) - \frac{(W_{2I+1}(t))^2}{W_{2I}(t)}\right)
\end{aligned}$$

where $W_i(0) = (\varphi_{1,i})^{-1}$ and $W_{I+i+1}(0) = (\varphi_{2,i})^{-1}$ for $i = 0, \dots, I$. We can see that W_h blows up when (U_h, V_h) quenches. Let η be the arc length of W_h. Considering the variables t and W_h as fonctions of η, we obtain the following system of differential equations

$$(6.1)\begin{cases}
\dfrac{dt}{d\eta} = \dfrac{1}{\sqrt{1 + \sum_{i=0}^{2I+1} f_i^2}},\\
\dfrac{dW_i}{d\eta} = \dfrac{f_i}{\sqrt{1 + \sum_{i=0}^{2I+1} f_i^2}},\ i = 0, \dots, 2I+1,\\
t(0) = 0,\ W_i(0) = (\varphi_{1,i})^{-1},\ W_{I+i+1}(0) = (\varphi_{2,i})^{-1}, \quad i = 0, \dots, I,
\end{cases}$$

where $0 < \eta < \infty$ and $f_i(t) = W_i'(t)$ since $d\eta^2 = dt^2 + dW_0^2 + \cdots + dW_{2I+1}^2$. It is well known (Hirota & Ozawa, 2006) that

$$\lim_{\eta\to\infty} t(\eta) = T_h \text{ and } \lim_{\eta\to\infty} \|W_h(\eta)\|_\infty = \infty.$$

For the numerical computation, let us define $\eta = \eta_l$ by $\eta_l = 2^{16} \cdot 2^l$ ($l = 0, 1, \ldots, 12$). For each value of l, we apply DOP54 (see Hairer, Nørsett & Wanner, 1993) to system (6.1) and we get a linearly convergent sequence to the blow-up time $\left\{t_l^{(k)}\right\}_{k=1}^{l+1}$. We also accelerate the sequence recursively by Aitken method's:

$$t_{l+2}^{(k+1)} = t_{l+1}^{(k)} - \frac{\left(t_{l+2}^{(k)} - t_{l+1}^{(k)}\right)^2}{t_{l+2}^{(k)} - 2t_{l+1}^{(k)} + t_l^{(k)}}, \quad l \geq 2k, \ k = 0, 1, 2, \ldots$$

As in (Hirota & Ozawa, 2006), for our experiments we set RTOL = ATOL = 1.d-15 and ITOL = 0. Where the parameters RTOL and ATOL are the tolerances of the relative and absolute errors, respectively, and ITOL is used to choose the manner in which the errors are controlled.

Tables and graphics: $\varphi_{1,i} = \varphi_{2,i} = 1 + \frac{2}{\pi} \sin\left(\frac{\pi}{2} ih\right), \quad i = 0, \ldots, I.$

In the following tables, in rows, we present the numerical quenching times T_h and the numbers of iterations n corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024.

TABLE 1. Numerical quenching times and numbers of iterations obtained for $p_1 = 2$, $p_2 = 1$, $q_1 = 1.6$ and $q_2 = 0.5$

I	T_h	n
16	0.12669452	2063
32	0.12406180	3791
64	0.12326487	7168
128	0.12302885	13779
256	0.12296021	27181
512	0.12294057	59900
1024	0.12293502	176017

TABLE 2. Numerical quenching times and numbers of iterations obtained for $p_1 = 1$, $p_2 = 2$, $q_1 = 2$ and $q_2 = 1$

I	T_h	n
16	0.12097809	1871
32	0.11817407	3546
64	0.11732426	6782
128	0.11707255	13090
256	0.11699937	25861
512	0.11697843	56990
1024	0.11697252	167374

TABLE 3. Numerical quenching times and numbers of iterations obtained for $p_1 = 0.5$, $p_2 = 2.5$, $q_1 = 0.5$ and $q_2 = 1.5$

I	T_h	n
16	0.14183488	2282
32	0.13941895	4309
64	0.13868637	8238
128	0.13846865	15899
256	0.13840510	31361
512	0.13838685	68551
1024	0.13838169	199191

TABLE 4. Numerical quenching times and numbers of iterations obtained for $p_1 = 2$, $p_2 = 1$, $q_1 = 4$ and $q_2 = 2$

I	T_h	n
16	0.08282272	1649
32	0.08018808	2940
64	0.07938374	5479
128	0.07914533	10446
256	0.07907608	20407
512	0.07905629	43379
1024	0.07905071	119081

FIGURE 1. On the left, quenching of U_h and on the right, no quenching of V_h for $p_1 = 2$, $p_2 = 1$, $q_1 = 1.6$ and $q_2 = 0.5$.

FIGURE 2. On the left, quenching of U_h and on the right, quenching of V_h for $p_1 = 1$, $p_2 = 2$, $q_1 = 2$ and $q_2 = 1$.

FIGURE 3. On the left, no quenching of U_h and on the right, quenching of V_h for $p_1 = 0.5$, $p_2 = 2.5$, $q_1 = 0.5$ and $q_2 = 1.5$.

FIGURE 4. On the left, quenching of U_h and on the right, no quenching of V_h for $p_1 = 2$, $p_2 = 1$, $q_1 = 4$ and $q_2 = 2$.

Remark 6.1. *We observe that, the solution of our problem quenches in a finite time and the convergence of quenching time T_h is given in differents tables. Moreover, we can see that of the figure 1, U_h quenches while V_h doesn't when $p_2 < p_1 + 1$, of the figure 2, U_h and V_h quench simultaneously when $p_2 \geq p_1 + 1$ and $q_1 \geq q_2 + 1$, of the figure 3, V_h quenches while U_h doesn't when $q_1 < q_2 + 1$ and of the figure 4, U_h quenches alone under any positive initial data when $p_2 \leq \frac{q_2(p_1+1)}{q_2+1}$ and $q_1 \geq q_2 + 1$. These numerical results are in fact consistent with the Theorem 4.1, Corollary 4.1, Theorem 4.2 and Theorem 4.3.*

REFERENCES

[1] K. A. ADOU, K. A. TOURÉ, A. COULIBALY: *Numerical study of the blow-up time of positive solutions of semilinear heat equations*, Far East J. Appl. Math., **4** (2018), 91–308.

[2] T. K. BONI: *Extinction for discretizations of some semilinear parabolic equations*, C. R. Acad. Sci. Paris, **333** (2001), 79–800.

[3] C. Y. CHAN: *New results in quenching*, In: *Proceedings of the first world congress on World congress of nonlinear analysts*, Walter de Gruyter, Berlin, 1996.

[4] Q. U. CHENG-YUAN, W. WEI, Z. SI-NING: *Non-simultaneous quenching for a slow diffusion system coupled at the boundary*, Appl. Math. J. Chinese Univ., **27** (2012), 139–149.

[5] K. DENG, M. X. XU: *On solutions of a singular diffusion equation*, Nonlinear Anal., **41** (2000), 489–500.

[6] K. DENG, M. X. XU: *Quenching for a nonlinear diffusion equation with a singular boundary condition*, Z. Angew Math. Phys., **50** (1999), 574–584.

[7] Z. W. DUAN, C.H. XIE, W. M. LU: *Quenching on boundary to the Newton filtration equation(I)*, Acta Math. Sci., **23B**(3) (2003), 361–368.

[8] K. B. EDJA, K. A. TOURÉ, B. J.-C. KOUA: *Numerical Blow-up for a Heat Equation with Nonlinear Boundary Conditions*, J. Math. Res., **10** (2018), 119–128.

[9] K. B. EDJA, K. N'GUESSAN, B. J. -C. KOUA, K. A. TOURÉ: *Numerical quenching for heat equations with coupled nonlinear boundary fux*, Int. J. Anal. Appl., **17** (2019), 1034–1051.

[10] R. FERREIRA, A. DE PABLO, F. QUIRÓS, J. D. ROSSI: *Nonsimultaneous quenching in a system of heat equations coupled at the boundary*, Z. Angew Math. Phys., **57** (2006), 586–594.

[11] M. FILA, J. HULSHOF: *A note on the quenching rate*, Proc. Amer. Math. Soc., **112** (1991), 473–477.

[12] M. FILA, H. A. LEVINE: *Quenching on the boundary*, Nonlinear Anal., **21** (1993), 795–802.

[13] J. S. GUO: *On the quenching behavior of the solution of a semilinear parabolic equations*, J. Math. Anal. Appl., **151** (1990), 58–79.

[14] J. S. GUO, B. HU: *Quenching problem for a singular semilinear heat equation*, Nonlinear Anal., **30** (1997), 905–910.

[15] R. H. JI, S. N. ZHENG: *Quenching behavior of solutions to heat equations with coupled boundary singularities*, Appl. Math. Comput., **206** (2008), 403–412.

[16] H. KAWARADA: *On the solutions of initial boundary value problem for* $u_t = u_{xx} + \frac{1}{1-u}$, RIMS Kyoto University, **10** (1975), 729–736.

[17] H. A. LEVINE: *Advances in quenching*, In: *Proceedings of the International Conference on ReactionDiffusion Equations and their Equilibrium States*, Birkhäuser, Boston, 1992.

[18] H. A. LEVINE, G. M. LIEBERMAN: *Quenching of solutions of parabolic equations with nonlinear boundary conditions in several dimensions*, J. Reine Angew Math., **345** (1983), 23–38.

[19] K. W. LIANG, A. P. LIN, R. C. E. TAN: *Numerical Solution of Quenching Problems Using Mesh-Dependent Variable Temporal Steps*, Appl. Numer. Math., **57** (2007), 791–800.

[20] D. Nabongo, T. K. Boni: *Quenching for semidiscretizations of a heat equation with a singular boundary condition*, Asymptotic Anal., **59** (2008), 27–38.

[21] K. C. N'dri, K. A. Touré, G. Yoro: *Numerical blow-up time for a parabolic equation with nonlinear boundary conditions*, Int. J. Numer. Methods Appl., **17** (2018), 141–160.

[22] A. de Pablo, F. Quirós, J. D. Rossi: *Nonsimultaneous quenching*, Appl. Math. Lett., **15** (2002), 265–269.

[23] M. Taha, K. Toure, E. Mensah: *Numerical approximation of the blow-up time for a semilinear parabolic equation with nonlinear boundary equation*, Far East J. Appl. Math., **60** (2012), 125–167.

[24] Z. Q. Wu, J. N. Zhao, J. X. Yin, H. L. Li: *Nonlinear Diffusion Equations*, World Scientific Publishing Company, 2001.

[25] S. N. Zheng, X. F. Song: *Quenching rates for heat euqations with coupled singular boundary flux*, Sci. China Ser. A., **51** (2008), 1631-1643.

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