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LUCAS DECOMPOSITION OF GRAPHS

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ABSTRACT. A decomposition $(G_1, G_2, \ldots, G_n)$ of G is said to be Lucas Decomposition (LD) if (i) $q(G_1) = 2$, $q(G_2) = 1$, (ii) $q(G_i) = q(G_{i-1}) + q(G_{i-2})$, $i = 3, 4, \ldots, n$, (iii) $E(G) = E(G_1) \cup E(G_2) \cup \ldots . \cup E(G_n)$ (iv) Each G_i, $i = 1, 2, \ldots, n$ is connected. In this paper, we investigate Lucas Decomposition of some graphs.

1. INTRODUCTION

All basic terminologies from graph theory are used in the sense of Frank Harary [2]. Let $G = (V, E)$ be a simple connected graph with p vertices and q edges. If $G_1, G_2, \ldots, G_n$ are connected edge disjoint subgraphs of G with $E(G) = E(G_1) \cup E(G_2) \cup \ldots . \cup E(G_n)$, then $(G_1, G_2, \ldots, G_n)$ is said to be a decomposition of G. Lucas numbers can be defined recursively. $l_0 = 2$, $l_1 = 1$, $l_n = l_{n-1} + l_{n-2}$, $n > 1$. The first ten lucas numbers are $l_0 = 2$, $l_1 = 1$, $l_2 = 3$, $l_3 = 4$, $l_4 = 7$, $l_5 = 11$, $l_6 = 18$, $l_7 = 29$, $l_8 = 47$, $l_9 = 76$. In [1], Ebin Raja Merly. E and Jeya Jothi. D introduced connected domination path decomposition of Triangular snake graph. A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph.

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2. Lucas Decomposition Of Graphs

Definition 2.1. *A decomposition $(G_1, G_2, \ldots, G_n)$ of G is said to be Lucas Decomposition (LD) if*

(i) $q(G_1) = 2$, $q(G_2) = 1$,
(ii) $q(G_i) = q(G_{i-1}) + q(G_{i-2})$, $i = 3, 4, \ldots, n$,
(iii) $E(G) = E(G_1) \cup E(G_2) \cup \ldots . \cup E(G_n)$
(iv) *Each* G_i, $i = 1, 2, \ldots, n$ *is connected.*

Theorem 2.1. *[3] If G admits LD $(G_1, G_2, \ldots, G_n)$ if and only if* $q(G) = \sum_{i=0}^{n-1} l_i$.

Theorem 2.2. *Let G be a cycle C_y with any of the vertex joined to each nonadjacent vertex. Then G admits LD$(G_1, G_2, \ldots, G_{3t+2})$ if and only if* $y = \frac{\sum_{i=0}^{3t+1} l_i + 3}{2}$.

Proof. Assume that G admits LD$(G_1, G_2, \ldots, G_{3t+2})$. By Theorem 2.1, $q(G) = \sum_{i=0}^{3t+1} l_i$. Since any of the vertex in cycle joined to each nonadjacent vertex, We have $q(G) = 2y - 3$. That is, $y = \frac{\sum_{i=0}^{3t+1} l_i + 3}{2}$. Conversely, assume that $y = \frac{\sum_{i=0}^{3t+1} l_i + 3}{2}$. let u_0 u_1, u_2, …, u_y be the vertices of G with u_0 is a central vertex of G. Then $K_{1, \frac{\sum_{i=0}^{3t+1} l_i + 3}{2}}$ is a star rooted at u_0. Thus $K_{1, \frac{\sum_{i=0}^{3t+1} l_i + 3}{2}} = G_1 \cup (G_5 \cup G_8 \cup \ldots \cup G_{3t+2})$ and each G_i, $i = 1, 5, 8, \ldots, 3t+2$ is a star. Therefore, $K_{1, \frac{\sum_{i=0}^{3t+1} l_i + 3}{2}}$ is decomposed into G_1, $(G_5, G_8, \ldots, G_{3t+2})$. Let $H = G - K_{1, \frac{\sum_{i=0}^{3t+1} l_i + 3}{2}}$. Then H is a path. Thus H $= G_2 \cup (G_3 \cup G_6 \cup \ldots \cup G_{3t}) \cup (G_4 \cup G_7 \cup \ldots \cup G_{3t+1})$, where each $G_i^{'}$s are path. Hence G admits LD$(G_1, G_2, \ldots, G_{3t+2})$. □

Theorem 2.3. *Let G be a graph with each vertex of two copies of P_y joined to a vertex u_0. Then*

(i) *G admits LD$(G_1, G_2, \ldots, G_{3t+1})$ if and only if* $y = \frac{\sum_{i=0}^{3t} l_i + 2}{4}$.
(ii) *If e_1 and e_2 are two edges incident to u_0, then $G \cup \{e_1, e_2\}$ admits LD$(G_1, G_2, \ldots, G_{6t})$ if and only if* $y = \frac{\sum_{i=0}^{6t-1} l_i}{4}$.

Proof. Let u_1, u_2, u_3,…, u_y, $u_1^{'}$, $u_2^{'}$, $u_3^{'}$,…, $u_y^{'}$ be the vertices of two copies of P_y.

(i) Assume that a graph G admits LD$(G_1, G_2, \ldots, G_{3t+1})$. By Theorem 2.1, $q(G) = \sum_{i=0}^{3t} l_i$. Since each vertex of two copies of P_y joined to a vertex u_0. We have $q(G) = 4y + 2$. Therefore, $y = \frac{\sum_{i=0}^{3t} l_i - 2}{4}$. Conversely, assume that $y = \frac{\sum_{i=0}^{3t} l_i - 2}{4}$. Then $u_1 - u_y^{'}$ is the longest path. Therefore $u_1 - u_y^{'}$ path is decomposed into G_1, $(G_2$, G_5,…, $G_{3t-1})$, $(G_3$, G_6,…, $G_{3t})$. Let

$H = G_1 \cup (G_2 \cup G_5 \cup \cdots \cup G_{3t-1}) \cup (G_3 \cup G_6 \cup \ldots \cup G_{3t})$. Then $G - H$ is a star with central vertex u_0. Thus $G - H$ is decomposed into G_4, G_7,..., G_{3t+1}. Hence G admits LD(G_1, G_2,..., G_{3t+1}).

(ii) Assume that $G \cup \{e_1, e_2\}$ admits LD$(G_1, G_2, \ldots, G_{6t})$. By Theorem 2.1, $q(G \cup \{e_1, e_2\}) = \sum_{i=0}^{6t-1} l_i$. Since e_1 and e_2 are incident to a central vertex u_0, we have $q(G \cup \{e_1, e_2\}) = 4y + 4$. That is, $y = \frac{\sum_{i=0}^{6t-1} l_i - 4}{4}$. Conversely, assume that $y = \frac{\sum_{i=0}^{6t-1} l_i - 4}{4}$. Then the longest path $P : u_1 - u'_y$ consists of length $\frac{\sum_{i=0}^{6t-1} l_i}{2}$. Therefore $u_1 - u'_y$ path is decomposed into (G_3, G_9, ..., G_{6t-3}), (G_6, G_{12}, ..., G_{6t}). Let $H = G \cup \{e_1, e_2\} - P$. Then $H = K_{1, \frac{\sum_{i=0}^{6t-1} l_i}{2}}$ is decomposed into $(G_1, G_7, G_{13}, \ldots, G_{6t-5})$, $(G_2, G_8, G_{14}, \ldots, G_{6t-4})$, $(G_4, G_{10}, G_{16}, \ldots, G_{6t-2})$ and $(G_5, G_{11}, G_{17}, \ldots,$ $G_{6t-1})$. Hence $G \cup \{e_1, e_2\}$ admits LD$(G_1, G_2, \ldots, G_{6t})$.

□

Theorem 2.4. *Let G be a graph with each vertex of two copies of P_{y_1} and P_{y_2} are joined to a vertex u_0. If e_1 and e_2 are two edges incident to u_0, then $G \cup \{e_1, e_2\}$ admits LD$(G_1, G_2, \ldots, G_{3t+1})$ if and only if $y_1 = \frac{\sum_{i=0}^{3t} l_i - 2}{4}$ and $y_2 = \frac{\sum_{i=0}^{3t} l_i + 2}{4}$.*

Proof. Assume that $G \cup \{e_1, e_2\}$ admits LD$(G_1, G_2, \ldots, G_{3t+1})$. By Theorem 2.1, $q(G) = \sum_{i=0}^{3t} l_i$. Since each vertex of two copies of P_{y_1} and P_{y_2} are joined to a vertex u_0, We have $q(G) = 2(y_1 + y_2 + 2)$. That is, $2(y_1 + y_2 + 2) = 2[\frac{\sum_{i=0}^{3t} l_i - 2}{4} + \frac{\sum_{i=0}^{3t} l_i + 2}{4} + 2]$. That is, $y_1 = \frac{\sum_{i=0}^{3t} l_i - 2}{4}$ and $y_2 = \frac{\sum_{i=0}^{3t} l_i + 2}{4}$.

Conversely, assume that $y_1 = \frac{\sum_{i=0}^{3t} l_i - 2}{4}$ and $y_2 = \frac{\sum_{i=0}^{3t} l_i + 2}{4}$. let $u_1, u_2, \ldots, u_{y_1}$ and $u'_1, u'_2, \ldots, u'_{y_2}$ be the vertices of P_{y_1} and P_{y_2} respectively.

Case(i): $t = 1$. Then $y_1 = 2$ and $y_2 = 3$. Thus the longest path $P : u_1 - u'_3$ is decomposed into G_1 and G_3. Let $H = G - P$. Then $H = K_{1,5}$ is decomposed into G_2 and G_4. Hence G is decomposed into G_1, G_2, G_3 and G_4.

Case(ii): $t > 1$. Then the longest path $P : u_1 - u'_{y_2}$ consists of length $\frac{\sum_{i=0}^{3t} l_i}{2}$. Therefore $P = G_1 \cup (G_3 \cup G_6 \cup \ldots \cup G_{3t}) \cup (G_5 \cup G_8 \cup \ldots \cup G_{3t-1})$. Thus P is decomposed into G_1, $(G_3, G_6, \ldots, G_{3t})$, $(G_5, G_8, \ldots, G_{3t-1})$. Let $H = G \cup \{e_1, e_2\} - P$. Then $H = K_{1, \frac{\sum_{i=0}^{3t} l_i}{2}}$ is decomposed into $G_2, G_4, G_7, \ldots, G_{3t+1}$. Hence $G \cup \{e_1, e_2\}$ admits LD$(G_1, G_2, \ldots, G_{3t+1})$. □

3. Lucas Star Decomposition Of Graphs

In this section, Lucas star decomposition is forged from Lucas Decomposition.

Definition 3.1. *A decomposition* $(G_1, G_2, \ldots, G_n)$ *of* G *is said to be Lucas Star Decomposition (LSD) if*

(i) G *admits LD.*
(ii) *Each* G_i *,* i *= 1, 2,...,* n *is a star.*

Example 1.

(i) *In a path graph,* P_2 *admits LSD*(G_1) *and* P_3 *admits LSD*(G_1, G_2)*.*
(ii) $S_{\sum_{i=0}^{n-1} l_i}$ *admits LSD.*
(iii) *In a cycle graph,* C_3 *admits LSD*(G_1, G_2) *and* $C_{\sum_{i=0}^{n-1} l_i}$ *does not admit LSD if* $n \geq 3$*.*
(iv) *In a wheel graph,* W_3 *admits LSD*(G_1, G_2, G_3) *and* $W_{\sum_{i=0}^{n-1} l_i}$ *does not admit LSD if* $n \geq 3$*.*

Theorem 3.1. *A Complete graph* K_t *admits LSD*$(G_1, G_2, \ldots, G_{t-1})$*,* $t = 3, 4, 5$*.*

Proof. Choose a vertex $v_1 \in V(K_t)$. Therefore $(t-1)$ edges incident to v_1. Consider two edges incident to v_1 is a star S_2. Therfore, $S_2 = G_1$. Remaining $(t-1) - 2$ edges incident to v_1. We have three cases.

Case(i): No edges incident to v_1. Then $G_2 = K_3 - G_1$. Hence K_3 is decomposed into G_1, G_2.

Case(ii): One edge is incident to v_1. Then one pair of vertices is degree 2 (say (u_1, u_2)) in $K_4 - S_2$. Let $G_2 = \{u_1 u_2\}$. Then $K_4 - (S_2 \cup G_2) = S_3 = G_3$. Hence K_4 is decomposed into G_1, G_2, G_3.

Case(iii): Two edges incident to v_1. Then four edges incident to a vertex v_2 (say) is a star S_4. Therefore, $S_4 = G_4$. Choose a vertex $v_3 \in V(K_5 - (G_1 \cup G_4))$. Then three edges incident to a vertex v_3 is a star S_3. Therefore, $S_3 = G_3$. Now, $G_2 = K_5 - (G_1 \cup G_4 \cup G_3)$. Then $G_2 = S_1$. Hence K_5 is decomposed into G_1, G_2, G_3, G_4. □

Remark 3.1. *A complete graph* K_t*,* $t \geq 6$ *does not admit LSD.*

Theorem 3.2. *Let* G *be a complete graph* K_j*,* $j = 2, 3, 4, 5$ *with the origin and terminus of* y*- copies of* P_2 *is attached to any two vertices of* K_j*. Then the following conditions hold:*

(i) *If $j = 2, 3$, then G has only one LSD$(G_1, G_2, \ldots, G_{3t+2})$.*
(ii) *If $j = 4$, then G admits LSD$(G_1, G_2, \ldots, G_{3t+1})$ and $(G_1, G_2, \ldots, G_{3t+3})$.*
(iii) *If $j = 5$, then G admits LSD$(G_1, G_2, \ldots, G_{3t+3})$ and $(G_1, G_2, \ldots, G_{3t+4})$.*

Proof.

(i) Since $j = 2, 3$, $q(G) = 2y_1 + 1$ or $2y_2 + 3$. By Theorem 2.1, $y_1 = \frac{\sum_{i=0}^{n-1} l_i - 1}{2}$and $y_2 = \frac{\sum_{i=0}^{n-1} l_i - 3}{2}$. Suppose $n = 3t + 2$. Then $y_1 = \frac{\sum_{i=0}^{3t+1} l_i - 1}{2}$and $y_2 = \frac{\sum_{i=0}^{3t+1} l_i - 3}{2}$. choose a pair of vertex (u, v) with $deg(u, v) = \Delta$. Then $G_2 = \{uv\}$ and $q(G_2) = 1$. Let w be a vertex such that $deg(w) = 2$. Then $G_1 = \{uw\} \cup \{wv\}$ and $q(G_1) = 2$. let $H = G - (G_1 \cup G_2)$. Then all the edges of $\frac{\sum_{i=0}^{3t+1} l_i - 3}{2}$ is incident to u and v. Therefore, $K_{1, \frac{\sum_{i=0}^{3t+1} l_i - 3}{2}} = G_5 \cup G_8 \cup \cdots \cup G_{3t+2}$ and $K_{1, \frac{\sum_{i=0}^{3t+1} l_i - 3}{2}} = (G_3 \cup G_6 \cup \cdots \cup G_{3t}) \cup (G_4 \cup G_7 \cup \cdots \cup G_{3t+1})$. Hence G is decomposed into G_1, G_2, $G_3, G_6, \ldots, G_{3t}$, $G_4, G_7, \ldots, G_{3t+1}$, $G_5, G_8, \ldots, G_{3t+2}$.

(ii) Let $j = 4$. Then $q(G) = 2y + 6$. Therefore, $y = \frac{\sum_{i=0}^{n-1} l_i - 6}{2}$. suppose $n = 3t + 1$. we have the following cases.

Case (i): $t = 1$. Then $n = 4$. Thus, $y = \frac{\sum_{i=0}^{3} l_i - 6}{2}$. Therefore, $y = 2$. Hence G is decomposed into G_1, G_2, G_3, G_4.

Case (ii): $t > 1$. Then $y = \frac{\sum_{i=0}^{3t} l_i - 6}{2}$. choose a vertex $u \in V(K_4)$. Therefore three edges incident to u. Then $S_2 = G_1$ and $S_1 = G_2$. choose a vertex $v \in V(K_4 - (G_1 \cup G_2))$ with $deg(v) = \Delta$. Therefore, $K_{1, \frac{\sum_{i=0}^{3t} l_i - 2}{2}}$ is a star decomposed into G_4, $(G_7, \ldots, G_{3t+1})$. Now, $H = G - (K_{1, \frac{\sum_{i=0}^{3t} l_i - 2}{2}} \cup G_1 \cup G_2)$. Then $H = (K_{1, \frac{\sum_{i=0}^{3t} l_i - 4}{2}}$ is a star decomposed into G_3, $(G_5, G_8, \ldots, G_{3t-1})$ and $(G_6, G_9, \ldots, G_{3t})$. Thus G is decomposed into $G_1, G_2, G_3, (G_5, G_8, \ldots, G_{3t-1})$, $(G_4, G_7, \ldots, G_{3t+1})$ and $(G_6, G_9, \ldots, G_{3t})$. Hence G admits LSD $(G_1, G_2, \ldots, G_{3t+1})$.

Suppose $n \neq 3t + 1$. Then $n = 3t + 3$ or $n = 3t + 2$ and $n \leq 3$.

Case (i): $n = 3t + 3$. Then $y = \frac{\sum_{i=0}^{3t+2} l_i - 6}{2}$. Choose a vertex $u_1 \in V(K_4)$. Therefore three edges incident to u_1. Then $S_3 = G_3$. Now one pair of vertex is degree 2 (say u_2 and u_3) in $K_4 - G_3$. Let $G_2 = \{u_2u_3\}$. Choose a vertex u_4 in $K_4 \cup (G_2 \cup G_3)$. Then the edges of a star $K_{1, \frac{\sum_{i=0}^{3t+2} l_i - 2}{2}}$ is incident to u_4. Therefore, $K_{1, \frac{\sum_{i=0}^{3t+2} l_i - 2}{2}}$ is decomposed into G_1, $(G_4, G_7, \ldots, G_{3t+1})$, $(G_5, G_8, \ldots, G_{3t+2})$. Now, $H_1 = G - [(G_2 \cup G_3) \cup$

$K_{1,\frac{\sum_{i=0}^{3t+2} l_i - 2}{2}}]$. Then H_1 is a star $K_{1,\frac{\sum_{i=0}^{3t+2} l_i - 6}{2}}$ which is decomposed into $(G_6, G_9, \ldots, G_{3t+3})$. Hence G admits LSD$(G_1, G_2, \ldots, G_{3t+3})$.

Case (ii): $n = 3t+2$ and $n \leq 3$. Then y is a fraction. Thus G does not admit LSD. Hence G admits LSD$(G_1, G_2, \ldots, G_{3t+1})$ and $(G_1, G_2, \ldots, G_{3t+3})$.

(iii) Let $j = 5$. Then $q(G) = 2y + 10$. Therfore, $y = \frac{\sum_{i=0}^{n-1} l_i - 10}{2}$.

Case (i): $n = 3t+3$. Then $y = \frac{\sum_{i=0}^{3t+2} l_i - 10}{2}$. Choose a vertex $u \in V(K_5)$. Therefore four edges of the stars S_3 and S_1 is incident to a vertex u. Then $S_3 = G_3$ and $S_1 = G_2$. Choose a vertex $v \in V(K_5 - (G_2 \cup G_3))$ with $deg(v) = 3$. Therefore all edges of a star S_2 is incident to u. Then $S_2 = G_1$. Now, remaining one edge incident to u. Then $G_1 = S_2$. Let $H = G - (G_1 \cup G_2 \cup G_3)$. Then H contains two stars $K_{1,\frac{\sum_{i=0}^{3t+2} l_i - 6}{2}}$ and $K_{1,\frac{\sum_{i=0}^{3t+2} l_i - 6}{2}}$. Now, $K_{1,\frac{\sum_{i=0}^{3t+2} l_i - 6}{2}} = G_6 \cup G_9 \cup \ldots \cup G_{3t+3}$ and $K_{1,\frac{\sum_{i=0}^{3t+2} l_i - 6}{2}} = (G_4 \cup G_7 \cup \ldots \cup G_{3t+1}) \cup (G_5 \cup G_8 \cup \ldots \cup G_{3t+2})$. Thus G is decomposed into G_1, G_2, G_3, $(G_4, G_7, \ldots, G_{3t+1})$, $(G_5 \cup G_8 \cup \ldots \cup G_{3t+2})$ and $(G_6 \cup G_9 \cup \ldots \cup G_{3t+3})$. Hence G admits LSD$(G_1, G_2, \ldots, G_{3t+3})$.

Case (ii): $n \neq 3t+3$. Then $n = 3t+4$ or $n = 3t+5$ and $n \leq 5$.

Subcase (i): $n = 3t+4$. Then $y = \frac{\sum_{i=0}^{3t+3} l_i - 10}{2}$. Choose a vertex $u \in V(K_5)$. Then four edges incident to u is a star S_4. Then $S_4 = G_4$. Choose a vertex $v \in V(K_5 - G_4)$ with three edges incident to v. Therefore, $S_3 = G_3$. Choose a vertex $w \in V(K_5 - (G_4 \cup G_3))$ with three edges incident to w. Then S_2 is a star all edges incident to w. Therefore, $S_2 = G_1$ and the remaining edge in $K_5 - (G_4 \cup G_3 \cup G_1) = G_2$. Let $H = G - (G_1 \cup G_2 \cup G_3 \cup G_4)$. Then H is a star $K_{1,\frac{\sum_{i=0}^{3t+3} l_i - 10}{2}}$ and $K_{1,\frac{\sum_{i=0}^{3t+3} l_i - 10}{2}}$. Then the edges of a star $K_{1,\frac{\sum_{i=0}^{3t+3} l_i - 10}{2}}$ is decomposed into $(G_7, G_{10}, \ldots, G_{3t+4})$, $(G_5, G_8, \ldots, G_{3t+2})$ and $(G_6, G_9, \ldots, G_{3t+3})$. Hence G admits LSD$(G_1, G_2, \ldots, G_{3t+4})$.

Subcase (ii): $n = 3t+5$ and $n \leq 5$. Then y is a fraction. Thus G does not admit LSD. Hence G admits LSD$(G_1, G_2, \ldots, G_{3t+3})$ and $(G_1, G_2, \ldots, G_{3t+4})$.

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