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AN EXISTENCE THEOREM FOR A NONLINEAR INTEGRAL EQUATION OF URYSOHN TYPE IN $L^P(\mathbb{R}^N)$

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ABSTRACT. The aim of this paper is investigating and solvability of the nonlinear integral Equation due to Urysohn, in the space of p^{th} Lebesgue integrable functions on $\mathbb{R}^{\mathbb{N}}$, $(L^p(\mathbb{R}^{\mathbb{N}}))$. The Urysohn integral equations are enjoying interest among mathematicians, physicists and engineers. We try to assume the sufficient conditions under which the existence theorem of the given integral equation can be proved. The main tool is using Dabo fixed point theorem via a certain measure of noncompactness introduced by Aghajani et. Al [3] in the space $L^p(\mathbb{R}^{\mathbb{N}})$, as an application to prove the desired existence theorem of our Urysohn integral equation. At the end of this paper, we introduce an example that ensure the importance of the hypothesis that assumed in our existence theorem.

1. INTRODUCTION

Many applications in mathematics, physics,..., etc, depend on the class of integral equations, where the methods of integral equations are used in solving some physical problems. It worth mention that Urysohn integral equation is one of the most frequently studied equation in nonlinear analysis, which have many useful applications in describing problems in the real world. The technique of measure of noncompactness is the main tool for solvability of several types of integral equations, see, for example, [1,2,5-22,24], and the references cited therein. Besides, it has been frequently applied in several branches of nonlinear analysis. The first measure of noncompactness was

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introduced by Kuratowski (Kuratowski, 1934). The most important fixed point theorem of this measure was introduced by Darbo (Darbo, 1955). In the last years, authors have applied this result to establish the existence and uniqueness of solutions for integral equations in Banach spaces. The space $L^p(R^N)$ is one of the most important spaces where several integral equations have been solved in this apace. The additional advantage of this space depends on the fact that the functions of the space $L^p(R^N)$ are not necessary to be continuous.

In [4], the authors discussed the solvability of the Urysohn integral equation

$$x(t) = f(t) + \int_0^\infty u(t, s, x(s))ds,$$

in [3] the authors proved the existence theorem for the functional integral equation

$$u(t) = f(t, u(t)) + \int_{R^N} k(t, s)(Qu)(s)ds,$$

while the authors in [10] studied the existence of integral solutions of the following integral equation

$$x(t) = f_1\left(t, \int_0^t k(t, s)f_2(s, x(s)ds\right).$$

In [11], the authors studied the existence of solutions for the perturbed functional integral equations of convolution type

$$x(t) = f_1(t, x(t)) + f_2\left(t, \int_0^\infty k(t-s)Q(x)(s)ds\right),\ t \in \mathbb{R}_+$$

in the space$L^p(\mathbb{R}_+)$ (the space of lebesgue integrable functions on $\mathbb{R}_+$). The measures of noncompactness play major roles in fixed point theory .

In the present work, we will use special measure of noncompactness due to Aghajani et. al [3] to prove the existence theorem of the Urysohn integral equation

$$u(x) = f(x) + g(x, u(x)) + \int_{\mathbb{R}^N} \psi(x, y, (Qu)(y))dy, \tag{1.1}$$

by using Darbo fixed point theorem.

2. Notation, Definitions, and Auxiliary Facts

We will collect in this section some definitions and basic results which will be used further on throughout the paper.

First, we denote $L^p(U)$ $(U \in \mathbb{R}^N)$ the space of Lebesgue integrable functions on U with the standard norm $\| x \|_{L^p(U)} = \left(\int_U | x(t) |^p \, dt\right)^{\frac{1}{p}}$.

Theorem 2.1. *[3, 21, 22] Let F be a bounded set in $L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$. The closure of F in $L^p(\mathbb{R}^N)$ is compact if and only if*

$$\lim_{h \to 0} \| \tau_h f - f \|_{L^p(\mathbb{R}^N)} = 0 \; uniformly \; in \; f \in F,$$

where $\tau_h f(x) = f(x+h)$ for all $x, h \in \mathbb{R}^N$. Also for $\epsilon > 0$ there is a bounded and measurable subset $\Omega \subset (\mathbb{R}^N)$ such that

$$\| f \|_{(\mathbb{R}^N \setminus \Omega)} < \epsilon \; for \; all \; f \in F.$$

Next, we recall the concept of measure of noncompactness, let E be an infinite dimensional Banach space with norm $\|.\|$ and zero element θ. Denote by$\mathcal{M}_E$ the family of all nonempty and bounded subsets of E , $\mathcal{N}_E$ and $\mathcal{N}_E^W$ the family of all nonempty relatively compact and weakly relatively compact sets, respectively. The symbols $\bar{X}$ and ConvX stand for the closure and closed convex hull of a subset X of E, respectively.The symbol $\bar{X}^W$ stands for the weak closure of a set X while, we denote $B_r = B(\theta, r)$ the closed ball centered at θ and with radius r.

Definition 2.1. *(Measure of noncompactness) [23] A mapping $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:*

(1) *the family $\ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathcal{N}_E$, where $ker\mu$ is called the kernel of the measure μ.*
(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
(3) $\mu(ConvX) = \mu(X) = \mu(\overline{X})$.
(4) $\mu[\lambda X + (1-\lambda)Y] \leq \lambda\mu(X) + (1-\lambda)\mu(Y), \; \lambda \in [0,1]$.
(5) *If $X_n \in \mathcal{M}_E, \; X_n = \bar{X_n}$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if*

$$\lim_{n \to \infty} \mu(X_n) = 0, \; then \; X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \phi.$$

Theorem 2.2. *[3] Suppose $1 \leq p < \infty$ and X is a bounded subset of $(\mathbb{R}^N)$. For $x \in X$ and $\epsilon > 0$*

$$\begin{aligned}
w^T(x, \epsilon) &= \sup\{\| \tau_h x - x \|_{L^p(B_T)} : \|h\|_{\mathbb{R}^N} < \epsilon\}, \\
w^T(X, \epsilon) &= \sup\{w^T(x, \epsilon) : x \in X\}, \\
w^T(X) &= \lim_{\epsilon \to 0} w^T(X, \epsilon), \\
w(X) &= \lim_{T \to \infty} w^T(X), \\
d(X) &= \lim_{T \to \infty} \sup\{\|x\|_{L^p(\mathbb{R}^N \setminus B_T)} : x \in X\},
\end{aligned}$$

where $B_T = \{a \in R^N : \|a\|_{R^N} \leq 1\}$. Then

$$\mu(X) = w(X) + d(X)$$

is a measure of non compactness on $L^p(\mathbb{R}^N)$.

In the end of this section, we recall the fixed point of Darbo which enables us to prove the solvability of several integral equations considered in nonlinear analysis. To quote this theorem we need the following definitions.

Theorem 2.3. *(Darbo fixed point theorem) [25] Let* Ω *be a nonempty, bounded, closed and convex subset of E and let* $f : \Omega \to \Omega$ *be a continuous transformation which is a contraction with respect to the measure of noncompactness* μ*, i.e. there exists a constant* $k \in [0, 1)$ *such that*

$$\mu(fX) \leq k\mu(X),$$

for any nonempty subset X of Ω*. Then* f *has at least one fixed point in the set* Ω*.*

3. Main Results

In this section, we study the existence of solutions to,(1.1) in the space $L^p(\mathbb{R}^{\mathbb{N}})$. We consider the equation(1.1) under the following assumptions:

(i) $f \in L^p(\mathbb{R}^N)$,

(ii) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition and there exists a constant $l \in [0, 1)$ and $a_1 \in L^p(\mathbb{R}^N)$ such that

$$| \, g(x, u) - g(y, v) \, | \leq |a_1(x) - a_1(y) \, | + l \, |u - v|,$$

for any $u, v \in \mathbb{R}$ and almost all $x, y \in \mathbb{R}^N$,

(iii) $g(., 0) \in L^p(\mathbb{R}^N)$,

(iv) $\psi : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that

$$| \, \psi(x, y, u) \, | \leq k(x, y)[a_2(y) + b \mid u \mid], a_2 \in L^p(\mathbb{R}^N)$$

$b > 0$ where $k(x, y)$ satisfies Carathéodory condition $k : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ and there exist $g_1, g_2 \in L^p(\mathbb{R}^N)$ and $g^* \in L^q(\mathbb{R}^N)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ s.t $| \, k(x, y) \, | \leq g^*(y).g_1(x)$ for all $x, y \in \mathbb{R}^N$ and

$$| \, k(x_1, y) - k(x_2, y) \, | \leq g^*(y)|g_2(x_1) - g_2(x_2)|.$$

(v) The operator Q is bounded linear operator and maps continuously the space $L^p(\mathbb{R}^N)$ into itself. Moreover, there exists a nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\| \, Qu \, \|_{L^p(\mathbb{R}^N)} \leq h(\| \, u \, \|_{L^p(\mathbb{R}^N)})$$

for any $u \in L^p(\mathbb{R}^N)$.

(vi) There exists a positive constant r_0 to the inequality

$$\begin{aligned} lr_0 + & \parallel f \parallel_{L^p(\mathbb{R}^N)} + \parallel g(.,0) \parallel_{L^p(\mathbb{R}^N)} \\ + & \; b \parallel k \parallel_1 h(r_0) + \parallel k \parallel_1 \parallel a_2 \parallel_{L^p(\mathbb{R}^N)} \leq r_0, \end{aligned}$$

where

$$(Ku)(x) = \int_{\mathbb{R}^N} k(x,y)u(y)dy$$

and

$$\parallel K \parallel_1 = \{\sup \parallel Ku \parallel_{L^p(\mathbb{R}^N)} : \parallel u \parallel_{L^p(\mathbb{R}^N)} \leq r_0\}.$$

Now, we are in a position to state our min result.

Remark 3.1. *The linear fredholm integral operator $K : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is continuous operator and $\parallel K \parallel_1 \leq \infty$.*

Theorem 3.1. *Under the assumptions (i)-(vi) then the integral equation* (1.1) *has at least one solution in the space $L^p(\mathbb{R}^N)$.*

Proof. First of all, we define the operator $F : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, by

$$(Fu)(x) = f(x) + g(x,u(x)) + \int_{\mathbb{R}^N} \psi(x,y,(Qu)(y))dy.$$

It is clear that Fu is measurable for any $u \in L^p(\mathbb{R}^N)$. Next, we will prove that $Fu \in L^p(\mathbb{R}^N)$ for any $u \in L^p(\mathbb{R}^N)$. To establish this we use the above assumptions, then, we have the following inequality

$$\begin{aligned} & \mid (Fu)(x) \mid = \left| f(x) + g(x,u(x)) + \int_{\mathbb{R}^N} \psi(x,y,(Qu)(y))dy \right| \\ \leq & \mid f(x) \mid + \mid g(x,u(x)) \mid + \int_{\mathbb{R}^N} \mid \psi(x,y,(Qu)(y)) \mid dy \\ \leq & \mid f(x) \mid + \mid g(x,u(x)) - g(x,0) \mid + \mid g(x,0) \mid + \int_{\mathbb{R}^N} \mid \psi(x,y,(Qu)(y)) \mid dy \\ \leq & \mid f(x) \mid + l \mid u(x) \mid + \mid g(x,0) \mid + \int_{\mathbb{R}^N} k(x,y)(a_2(y) + b \mid (Qu)(y) \mid)dy \\ \leq & \mid f(x) \mid + l \mid u(x) \mid + \mid g(x,0) \mid + \int_{\mathbb{R}^N} k(x,y)a_2(y)dy + b \int_{\mathbb{R}^N} k(x,y) \mid (Qu)(y) \mid dy. \\ \parallel & Fu \parallel_{L^p(\mathbb{R}^N)} \leq \parallel f \parallel_{L^p(\mathbb{R}^N)} + l \parallel u \parallel_{L^p(\mathbb{R}^N)} + \parallel g(.,0) \parallel_{L^p(\mathbb{R}^N)} + \parallel K \parallel_{L^p(\mathbb{R}^N)} \\ & (\parallel a_2 \parallel_{L^p(\mathbb{R}^N)} + b \parallel Qu \parallel_{L^p(\mathbb{R}^N)}) \\ \leq & \parallel f \parallel_{L^p(\mathbb{R}^N)} + l \parallel u \parallel_{L^p(\mathbb{R}^N)} + \parallel g(.,0) \parallel_{L^p(\mathbb{R}^N)} + \parallel K \parallel_1 \\ \times & \parallel a_2 \parallel_{L^p(\mathbb{R}^N)} + b \parallel K \parallel_1 h(\parallel u \parallel) \\ < & \infty. \end{aligned}$$

Hence, $F(u) \in L^p(\mathbb{R}^N)$ and F is will defined, also we see that F is continuous in $L^p(\mathbb{R}^N)$, because $g(x,.)$, k and Q are continuous for a.e. $x \in \mathbb{R}^N$.

Next, we show that $F : B_{r_0} \to B_{r_0}$, let $u \in B_{r_0}$ where $(\| u \| \leq r_0)$

$$\begin{aligned} \| Fu \|_{L^p(\mathbb{R}^N)} &\leq \| f \|_{L^p(\mathbb{R}^N)} + lr_0 + \| g(.,0) \|_{L^p(\mathbb{R}^N)} \\ &+ \| K \|_1 (\| a_2 \|_{L^p(\mathbb{R}^N)} + bh(r_0)) \\ &\leq r_0. \end{aligned}$$

For any nonempty set $X \subset B_{r0}$ we have $w_0(FX) \leq lw_0(X)$ to do this, we fix arbitrary $T > 0$ and $\epsilon > 0$, let us choose $u \in X$ and for $x, h \in B_T$ with $\| h \|_{\mathbb{R}^N} \leq \epsilon$. Then, we have

$$\begin{aligned} &|(Fu)(x+h) - (Fu)(x)| \leq |f(x+h) - f(x)| + |g(x+h, u(x+h)) - g(x, u(x))| \\ + &\int_{\mathbb{R}^N} |\psi(x+h, y, (Qu)(y)) - \psi(x, y, (Qu)(y))| dy \\ \leq &|f(x+h) - f(x)| + |g(x+h, u(x+h)) - g(x+h, u(x))| \\ + &|g(x+h, u(x)) - g(x, u(x))| + \int_{\mathbb{R}^N} (|k(x+h, y) - k(x, y)| [a_2(y) + b|(Qu)(y)|]) dy \\ \leq &|f(x+h) - f(x)| + |a_1(x+h) - a_1(x+h)| + |a_1(x+h) - a_1(x)| \\ + &l|u(x+h) - u(x)| + \int_{\mathbb{R}^N} |k(x+h, y) - k(x, y)| a_2(y) dy \\ + &b \int_{\mathbb{R}^N} (|k(x+h, y) - k(x, y)|) |(Qu)(y)| dy \\ \leq &|f(x+h) - f(x)| + l|u(x+h) - u(x)| + |a_1(x+h) - a_1(x)| \\ + &\int_{\mathbb{R}^N} g^*(y) |g_2(x+h) - g_2(x)| a_2(y) dy + b \int_{\mathbb{R}^N} g^*(y) |g_2(x+h) - g_2(x)| |(Qu)(y)| dy. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\|\tau_h Fu - F\|_{L^p} = \left(\int_{B_T} |(Fu)(x+h) - (Fu)(x)|^p dx \right)^{\frac{1}{p}} \\ \leq &\left(\int_{B_T} |f(x+h) - f(x)|^p dx \right)^{\frac{1}{p}} + l \left(\int_{B_T} |u(x+h) - u(x)|^p dx \right)^{\frac{1}{p}} \\ + &\left(\int_{B_T} |a_1(x+h) - a_1(x)|^p dx \right)^{\frac{1}{p}} \\ + &\left(\int_{B_T} (\int_{\mathbb{R}^N} |g^*(y)|^q |g_2(x+h) - g_2(x)|^q |a_2(y)|^q dy)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\ + &b \left(\int_{B_T} (\int_{\mathbb{R}^N} |g^*(y)|^q |g_2(x+h) - g_2(x)|^q |(Qu)(y)|^q dy)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}. \end{aligned}$$

$$
\begin{aligned}
& \|\tau_h Fu - Fu\|_{L^p} \leq \|\tau_h f - f\|_{L^p(B_T)} + l\|\tau_h u - u\|_{L^p(B_T)} + |\tau_h a_1 - a_1\|_{L^p(B_T)} \\
+ \quad & \|g^*\|_{L^q(\mathbb{R}^N)}\|\tau_h g_2 - g_2\|_{L^p(B_T)}\|a_2\|_{L^p(\mathbb{R}^N)} + b\|g^*\|_{L^q\mathbb{R}^N)}\|\tau_h g_2 - g_2\|_{L^p(B_T)}\|Qu\|_{L^p\mathbb{R}^N)} \\
\leq \quad & w^T(f,\epsilon) + lw^T(u,\epsilon) + w^T(a_1,\epsilon) + w^T(g_2,\epsilon)\ \|g^*\|_{L^q(\mathbb{R}^N)}\|a_2\|_{L^p(\mathbb{R}^N)} \\
+ \quad & b\|g^*\|_{L^q(\mathbb{R}^N)}w^T(g_2,\epsilon)\ h(\|u\|_{L^p(\mathbb{R}^N)}).
\end{aligned}
$$

Then, we have,

$$
\begin{aligned}
w^T(Fx,\epsilon) \quad \leq \quad & w^T(f,\epsilon) + lw^T(X,\epsilon) + w^T(a_1,\epsilon) + \|g^*\|_{L^q(\mathbb{R}^N)}\|a_2\|_{L^p(\mathbb{R}^N)}w^T(g_2,\epsilon) \\
+ \quad & b\|g^*\|_{L^q(\mathbb{R}^N)}w^T(g_2,\epsilon)h(r_0).
\end{aligned}
$$

$w^T(g_2,\epsilon)$,$w^T(f,\epsilon)$, and $w^T(a_1,\epsilon) \to 0$ as $\epsilon \to 0.$

Then, we obtain

$$
w(FX) \leq\ lw(X),\ l \in [0,1). \tag{3.1}
$$

In the following, we fix an arbitrary number $T > 0$. Then, taking into account our assumptions, for an arbitrary function $u \in X$, we obtain

$$
\begin{aligned}
\left(\int_{\mathbb{R}^N\ \backslash B^T}|(Fu)(x)|^p dx\right)^{\frac{1}{p}} \quad \leq \quad & \left(\int_{\mathbb{R}^N\backslash B^T}|f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N\backslash B^T}|g(x,u(x))|^p dx\right)^{\frac{1}{p}} \\
+ \quad & \left(\int_{\mathbb{R}^N\backslash B^T}|\psi(x,y,(Qu)(y))|^p dy\right)^{\frac{1}{p}} \\
\leq \|f\|_{L_p(\mathbb{R}^N\backslash B^T)} + \left(\int_{\mathbb{R}^N\backslash B^T}|g(x,0)|^p dx\right)^{\frac{1}{p}} & + \left(\int_{R^N\backslash B^T}|g(x,u(x)) - g(x,0)|^p dx\right)^{\frac{1}{p}} \\
+ \quad & \left(\int_{\mathbb{R}^N\backslash B^T}|(\int_{\mathbb{R}^N}|k(x,y)| \times [a_2(y) + b|(Qu)(y)|]dy)|^p dx\right)^{\frac{1}{p}} \\
\leq \|f\|_{L^p(\mathbb{R}^N\backslash B^T)} + l\left(\int_{\mathbb{R}^N\backslash B^T}|u(x)|^p dx\right)^{\frac{1}{p}} & + \left(\int_{\mathbb{R}^N\backslash B^T}|g(x,0)|^p dx\right)^{\frac{1}{p}} \\
+ \quad & \left(\int_{\mathbb{R}^N\backslash B^T}(\int_{\mathbb{R}^N}|k(x,y)|^q|a_2(y)|^q dy)^{\frac{p}{q}}dx\right)^{\frac{1}{p}} \\
+ \quad & b\left(\int_{\mathbb{R}^N\backslash B^T}(\int_{\mathbb{R}^N}|k(x,y)|^q dy)^{\frac{p}{q}}dx\right)^{\frac{1}{p}}\|(Qu)(y)\|_{L^p}
\end{aligned}
$$

$$\begin{aligned}
&\leq \|f\|_{L^p(\mathbb{R}^N\backslash B^T)} + l \parallel u \parallel_{L^p(\mathbb{R}^N\backslash B^T)} \\
&+ \parallel g(.,0) \parallel_{L^p(\mathbb{R}^N\backslash B^T)} + \parallel g^* \parallel_{L^q(\mathbb{R}^N)} \times \parallel g_1 \parallel_{L^p(\mathbb{R}^N\backslash B^T)} \\
&\times \left(\parallel a_2 \parallel_{L^p(\mathbb{R}^N\backslash B^T)} + bh(\|u\|_{L^p(\mathbb{R}^N)})\right).
\end{aligned}$$

Also we have $\|f\|_{L^p(\mathbb{R}^N\backslash B^T)}, \parallel g(.,0) \parallel_{L^p(\mathbb{R}^N\backslash B^T)}, \parallel g_1 \parallel_{L^p(\mathbb{R}^N\backslash B^T)} \to 0$ as $T \to \infty$ and hence we obtain that

$$d(FX) \leq ld(X). \tag{3.2}$$

From (3.1)and (3.2), we get

$$\mu(FX) \leq l\ \mu(X). \tag{3.3}$$

□

From the above inequality(3.3) and the Theorem (2.3) we obtain that the operator F satisfies all conditions of Darbo fixed point theorem, which enables us to deduce that F has a fixed point u in B_{r_0} and thus the integral equation (1.1) has at least one solution in $L^p(\mathbb{R}^N)$. Thus the proof is finished. Next, we will need the following theorem that help us in a coming example.

Theorem 3.2. *[13] Let $\Omega \subseteq \mathbb{R}^N$ be a measure space and suppose $k : \Omega \times \Omega \to \mathbb{R}$ is a measurable function for which there is constant $C > 0$ such that*

$$\int_I |k(x,y)|dx \leq C \ a.e. \ y \in \Omega$$

and

$$\int_I |k(x,y)|dy \leq C \ a.e. \ x \in \Omega.$$

If $K : L^p(\Omega) \to L^p(\Omega)$ is defined by

$$(Kf)(x) = \int_\Omega f(y)\, dy,$$

then K is a bounded and continuous operator and $\|K\|_1 \leq C$.

Example 1. *Consider the integral equation*

$$\begin{aligned}
u(x) &= e^{-\|x\|} + \frac{\sin u}{\parallel x \parallel + 3} \\
&+ \int_{\mathbb{R}^2} \left(\frac{e^{-(|x_1|+|y_2|+1)}}{(|x_2|+4)^2(1+|y_1|^2)}\right)\left(\sqrt[p]{\frac{x_1}{1+|y_2|^2}} + \frac{1}{4}e^{-|u|}u(y)\right) dy,
\end{aligned} \tag{3.4}$$

this equation is a special case of (1.1) where $x = (x_1, x_2) \in \mathbb{R}^2$ *and* $\|x\|$ *is the Euclidean norm, with* $f(x) = e^{-\|x\|}$, $g(x, u(x)) = \frac{\sin u}{\|x\|+3}$,

$$\psi(x, y, (Qu)(y)) = \left(\frac{e^{-(|x_1|+|y_2|+1)}}{(|x_2|+4)^2(1+|y_1|^2)}\right)\left(\sqrt[p]{\frac{x_1}{1+|y_2|^2}} + \frac{1}{4}e^{-|u|}u(y)\right).$$

First, note that $f(x) \in L^p(\mathbb{R}^N)$ *i.e the assumption (i) is satisfied with*

$$\|f\|^p_{L^p(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |e^{-\|x\|}|^P dx = \frac{2\pi}{p^2}.$$

Next the function $g(x, u(x))$ *satisfies the assumption (ii) with*

$$a_1(x) = \frac{1}{\| x \| + 3}$$

and $l = \frac{1}{3}$, *indeed by using Mean value Theorem, we have*

$$\begin{aligned} |g(x,u) - g(y,v)| &= \left| \frac{\sin u}{\| x \| + 3} - \frac{\sin v}{\| y \| + 3} \right| \\ &\leq \left| \frac{1}{\| x \| + 3} - \frac{1}{\| y \| + 3} \right| \mid \sin u \mid + \frac{1}{\| y \| + 3} \mid \sin u - \sin v \mid \\ &\leq \left| \frac{1}{\| x \| + 3} - \frac{1}{\| y \| + 3} \right| + \frac{1}{3} \mid u - v \mid = \mid a_1(x) - a_1(y) \mid + l \mid u - v \mid, \end{aligned}$$

where $a_1(x) \in L^p(\mathbb{R}^2)$ *as*

$$\begin{aligned} \|a_1\|^p_{L^p(\mathbb{R}^2)} &= \int_{\mathbb{R}_2} \left|\frac{1}{\|x\|+3}\right|^P dx = \int_0^{2\pi} \int_0^\infty \frac{r}{(r+3)^p} dr\, d\theta \\ &= 2\pi \left(\frac{1}{3^{p-2}(p-2)} + \frac{3}{3^{p-1}(1-p)} \right), \end{aligned}$$

for all $p > 2$. *Also it is easily seen that* $g(.,0)$ *satisfies the assumption (iii) with* $\|g(.,0)\| = 0$. *In the sequel* $\psi(x, y, (Qu)(y))$ *satisfies the assumption (iv) where*

$$|k(x,y)| = \left|\frac{e^{-(|x_1|+|y_2|+1)}}{(|x_2|+4)^2(1+|y_1|^2)}\right|,$$

$b = \frac{1}{4}$, *and* $a_2(x) = \sqrt[p]{\frac{x_1}{1+|y_2|^2}}$, $\|a_2\|_{L^p(\mathbb{R}^2)} = 0$ *we get* $g_1(x) = g_2(x) = \frac{e^{-|x_1|}}{(|x_2|+4)^2}$ *and* $g^*(y) = \frac{e^{-|y_2|}}{(1+|y_1|^2)}$, *we see that* g_1, g_2, $g^* \in L^p(\mathbb{R}^2)$ *for all* $1 \leq p < \infty$. *Also we have*

$$\begin{aligned} \int_{\mathbb{R}^2} |k(x,y)| dx &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-(|x_1|+|y_2|+1)}}{(|x_2|+4)^2(1+|y_1|^2)} dx_1 dx_2 \leq \frac{1}{e}, \\ \int_{\mathbb{R}^2} |k(x,y)| dy &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-(|x_1|+|y_2|+1)}}{(|x_2|+4)^2(1+|y_1|^2)} dy_1 dy_2 \leq \frac{\pi}{8e}. \end{aligned}$$

Thus from Theorem (3.1), $\|K\|_1 \leq \frac{1}{e}$, *also* $Q(u)(x) = e^{-|u|}u(x)$ *satisfies the assumption (v) with* $h(t) = t$.

Finally, the inequality from assumption (vi), has the form

$$\begin{aligned} lr_0 + \| f \|_{L^p(\mathbb{R}^2)} + \| g(.,0) \|_{L^p(\mathbb{R}^2)} + b \| k \|_1 \, h(r_0) + \| k \|_1 \| a_2 \|_{L^p(\mathbb{R}^2)} \\ = (\frac{2\pi}{p^2})^{\frac{1}{p}} + \frac{1}{4e} r_0 + \frac{r_0}{3} = (\frac{2\pi}{p^2})^{\frac{1}{p}} + (\frac{4e+3}{12e}) r_0 \leq r_0. \end{aligned}$$

Thus, for the number r_0 we can take $r_0 = (\frac{2\pi}{p^2})^{\frac{1}{p}} \times \frac{12e}{(8e-3)}$. Hence, all the assumptions of Theorem (3.1) *are satisfied and so,* (3.4) *has at least one solution in the space $L^p(\mathbb{R}^2)$ if $p > 2$.*

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