Advances in Mathematics: Scientific Journal **9** (2020), no.11, 10045–10052
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.11.112

QUASI IDEALS IN TERNARY PARTIAL SEMIRINGS

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ABSTRACT. This paper attempts to present the notions of quasi ideal, principal quasi ideal and minimal quasi ideal in a ternary partial semiring. The most important aspect of this paper is to study the properties of these three ideals. The notions of minimal left ideal, minimal lateral ideal and minimal right ideal in ternary partial semiring are also introduced by characterizing regular ternary partial semiring in terms of these ideals in ternary partial semiring.

1. INTRODUCTION

In 1932, D.H. Lehmar [5] introduced a note on a ternary analogue of abelian groups. In 1971, ternary rings are introduced by W.G. Lister [6]. In 2003, T.K. Dutta and S. Kar [1] are introduced the notion of ternary semiring is their note on regular ternary semirings. In 1980, Higgs [4] studied Σ- structures. In 1985, Streenstrup [3] studied partial semirings. In 2016, K. Sivaprasad and Y. Prabhakararao [2] have introduced the notion of ternary partial semiring and studied their congruence relations. In this paper, the notions of quasi ideal, principal quasi ideal and minimal quasi ideal in a ternary partial semiring are introduced. Also introduce the notions of minimal left ideal, minimal lateral ideal and minimal right ideal in ternary partial semiring are developed. It is shown that every minimal quasi ideal of a ternary partial semiring is the intersection of a minimal left ideal, a minimal lateral ideal and a minimal right ideal. Also it is observed that if Q is

2020 *Mathematics Subject Classification.* 16Y30, 16Y60.
Key words and phrases. quasi ideal, principal quasi ideal, minimal quasi ideal, minimal left ideal, minimal lateral ideal, minimal right ideal, regular ternary partial semiring.

quasi ideal of a ternary partial semiring and $Q^3 = Q$ then S is a regular ternary partial semiring.

2. Preliminaries

Definition 2.1. *A nonempty set S together with a binary operation called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup fulfilling the following axioms:*

(i) *(xyz)uv=x(yzu)v=xy(zuv)*
(ii) *(x+y)uv=xuv+yuv*
(iii) *x(y+z)u=xyu+xzu*
(iv) *xy(z+u)=xyz+xyu; for all x,y,z,u,v∈ S.*

Definition 2.2. *A nonempty set S together with a partial addition Σ defined on some (not necessarily all) families $(x_i : i \in I)$ in S and a ternary multiplication, denoted by juxtaposition, is said to be a ternary partial semiring if it satisfies the following axioms:*

(i) (S, Σ) *is a partial monoid*
(ii) *(xyz)uv=x(yzu)v=xy(zuv)*
(iii) *A family*$(x_i : i \in I)$ *is summable in* S *implies* $(x_i xy : i \in I)$ *is summable in* S *and* $(\Sigma_{i\in I} x_i)xy = \Sigma_{i\in I}(x_i xy)$,
(iv) *A family*$(x_i : i \in I)$ *is summable in* S *implies* $(x x_i y : i \in I)$ *is summable in* S *and* $a(\Sigma_{i\in I} x_i)y = \Sigma_{i\in I}(x x_i y)$,
(v) *A family*$(x_i : i \in I)$ *is summable in* S *implies* $(xy x_i : i \in I)$ *is summable in* S *and* $xy(\Sigma_{i\in I} x_i) = \Sigma_{i\in I}(xy x_i)$,
(vi) $0xy = x0y = xy0 = 0$*,where* $0 = \Sigma(x_i : i \in \emptyset)$*(empty family), acts as an additive zero for binary sums in* S*; for all* $x, y, z, u, v \in S$.

Definition 2.3. *A ternary partial semiring S is considered to be complete if all families of finite support in it are summable.*

Definition 2.4. *A nonempty subset H of a ternary partial semiring S is said to be a left (right) ideal of S if it satisfies the following axioms:*

(i) *Whenever* $(x_i : i \in I)$ *is a summable family in S and* $x_i \in H$ *for every* $i \in I$*, then* $\sum_{i\in I} x_i \in H$.
(ii) $s_1 s_2 h \in H (h s_1 s_2 \in H)$, *for all* $s_1, s_2 \in S$ *and* $h \in H$.

Definition 2.5. *A nonempty subset H of a ternary partial semiring S is said to be a lateral ideal of S if it satisfies the following axioms:*

(i) *Whenever $(x_i : i \in I)$ is a summable family in S and $x_i \in H$ for every $i \in I$, then $\sum_{i \in I} x_i \in H$.*
(ii) *$s_1 h s_2 \in H$ for all $s_1, s_2 \in S$ and $h \in H$.*

Definition 2.6. *Let S be a ternary partial semiring. A nonempty subset H of a S is said to be an ideal of S if H is a left, a right as well as a lateral ideal of S. If H$\neq$ S, then H is called a proper ideal of S.*

Definition 2.7. *Let S be a ternary partial semiring and an element u of S is called regular when there is an element v in S such that uvu = u.*

Definition 2.8. *If all elements of a ternary partial semiring S are regular then S is called regular ternary partial semiring.*

3. Quasi Ideals

Definition 3.1. *A nonempty subset H of a ternary partial semiring S is said to be a lateral ideal of S if it satisfies the following axioms:*

(i) *Whenever $(x_i : i \in I)$ is a summable family in S and $x_i \in H$ for every $i \in I$, then $\sum_{i \in I} x_i \in H$.*
(ii) *$SSQ \cap SQS \cap QSS \subseteq Q$.*

Remark 3.1. *Let S be a ternary partial semiring. Then every left ideal, right ideal and lateral ideal of S is a quasi ideal of S. But not conversely true.*

Example 1. *If M is the set of all 2×2 square matrices containing non positive integers, it becomes a ternary partial semiring with normal addition over families of finite support and ternary multiplication. Here*

$$Q = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \setminus x \in \mathbb{Z}^- \cup \{0\} \right\}$$

is a quasi ideal of M. but Q is not a right ideal, a lateral ideal or a left ideal of M.

Definition 3.2. *Let S be a ternary partial semiring and $x \in S$. Then the smallest quasi ideal of S that contains x is called the principal quasi ideal of S generated by x and is denoted by $\langle x \rangle_q$.*

Theorem 3.1. *Let S be a complete ternary partial semiring. If $x \in S$, then the principal quasi ideal generated by x is the set $\langle x \rangle_q = \{a \in S/a = \Sigma_i a_i b_i x + \Sigma_j x c_j d_j + \Sigma_k u_k x v_k + nx$ where $a_i, b_i, c_j, d_j, u_k, v_k \in S$ and $n \in \mathbb{Z}^+ \cup \{0\}\}$. where nx denotes the sum of n copies of x for $n \in \mathbb{Z}^+$ and $n = 0$ implies $nx = 0$.*

Proof. Let $T = \{a \in S/a = \Sigma_i a_i b_i x + \Sigma_j x c_j d_j + \Sigma_k u_k x v_k + nx$ where $a_i, b_i, c_j, d_j, u_k, v_k \in S$ and $n \in \mathbb{Z}^+ \cup \{0\}\}$. We first prove that T is an ideal of S. Let $(a_i : i \in I)$ be a family of finite support in S and $a_i \in T, i \in I$. Then there are only a finite number of nonzero terms in $(a_i : i \in I)$, namely $a_1, a_2, ..., a_p$. Since each $a_m \in T, 1 \leq m \leq p, a_m = \Sigma_{m_i} a_{m_i} b_{m_i} x + \Sigma_{m_j} x c_{m_j} d_{m_j} + \Sigma_{m_k} u_{m_k} x v_{m_k} + nx$, where $a_{m_i}, b_{m_i}, c_{m_j}, d_{m_j}, u_{m_k}, v_{m_k} \in S$ and $n \in \mathbb{Z}^+ \cup \{0\}$. Since S is complete, $\sum\limits_{m=1}^{p} a_m$ is in S and $\sum\limits_{m=1}^{p} a_m = \Sigma_m \Sigma_{m_i} a_{m_i} b_{m_i} x + \Sigma_m \Sigma_{m_j} x c_{m_j} d_{m_j} + \Sigma_m \Sigma_{m_k} u_{m_k} x v_{m_k} + pnx \in S$, so that $\sum\limits_{m=1}^{p} a_m \in T$ and $\Sigma_{i \in I} a_i \in T$. Let $a \in T$ and $s, s' \in S$. Now $a \in T \Rightarrow a \in S$ and $a = \Sigma_i a_i b_i x + \Sigma_j x c_j d_j + \Sigma_k u_k x v_k + nx$ where$a_i, b_i, c_j, d_j, u_k, v_k \in S$ and $n \in \mathbb{Z}^+ \cup \{0\}$. Consider $ss'a = ss'(\Sigma_i a_i b_i x + \Sigma_j x c_j d_j + \Sigma_k u_k x v_k + nx) = ss'(\Sigma_i a_i b_i x) + ss'(\Sigma_j x c_j d_j) + ss'(\Sigma_k u_k x v_k) + ss'(nx) = \Sigma_i (ss'a_i) b_i x + \Sigma_j (ss' x c_j d_j) + \Sigma_k (ss' u_k) x v_k + n(ss'x)$. So, $ss'a \in T$. Similarly, we can prove that $sas' \in T$ and $ass' \in T$. Therefore, T is an ideal of S. Since $x = 00x + x00 + 0x0 + 00x00 + nx$, where $n = 1 \in \mathbb{Z}^+ \cup \{0\}$, it follows that $x \in T$. Let U be an ideal of S such that $x \in U$. let $a \in T$. Then $a \in S$ and $a = \Sigma_i a_i b_i x + \Sigma_j x c_j d_j + \Sigma_k u_k x v_k + nx$, where $a_i, b_i, c_j, d_j, u_k, v_k \in S$ and $n \in \mathbb{Z}^+ \cup \{0\}$. Since U is an ideal of S, $a \in S$ and $x \in U$, it follows that $a = \Sigma_i a_i b_i x + \Sigma_j x c_j d_j + \Sigma_k u_k x v_k + nx \in U$ and that $a \in U$. Therefore, $T \subseteq U$. Hence, T is the smallest ideal of S containing x. Thus $\langle x \rangle = T$. □

The following theorems can be derived by using the above theorem.

Theorem 3.2. *Let S be a complete ternary partial semiring. If $x \in S$, then the principal left ideal generated by x is the set $\langle x \rangle_l = \{a \in S/a = \Sigma_i a_i b_i x + nx$, where $a_i, b_i \in S$ and $n \in \mathbb{Z}^+ \cup \{0\}\}$.*

Theorem 3.3. *Let S be a complete ternary partial semiring. If $x \in S$, then the principal right ideal generated by x is the set $\langle x \rangle_r = \{a \in S/a = \Sigma_i x a_i b_i + nx$, where $a_i, b_i \in S$ and $n \in \mathbb{Z}^+ \cup \{0\}\}$.*

Theorem 3.4. *Let S be a complete ternary partial semiring. If $x \in S$, then the principal two-sided ideal generated by x is the set* $\langle x \rangle_t = \{a \in S/a = \Sigma_i a_i b_i x + \Sigma_j x c_j d_j + \Sigma_k a'_k b'_k x c'_k d'_k + nx$, *where* $a_i, b_i, c_j, d_j, a'_k, b'_k, c'_k, d'_k \in S$ *and* $n \in \mathbb{Z}^+ \cup \{0\}\}$.

Theorem 3.5. *The intersection of left ideal H, a right ideal K and a lateral ideal L of a ternary partial semiring S is a quasi ideal of S.*

Proof. Let H be a left ideal and K be a right ideal and L be a lateral ideal of a ternary partial semiring S. Let $(x_i : i \in I)$ is a summable family in S and $x_i \in H \cap L \cap K$ for every $i \in I$. Then $\sum\limits_{i \in I} x_i \in H$ as H is a left ideal,$\sum\limits_{i \in I} x_i \in L$ as L is a lateral ideal and $\sum\limits_{i \in I} x_i \in K$ as K is a right ideal of S. We have $\sum\limits_{i \in I} \in H \cap L \cap K$.Further we have $SS(H \cap L \cap K) \cap S(H \cap L \cap K)S \cap (H \cap L \cap K)SS \subseteq SSH \cap SLS \cap KSS \subseteq H \cap L \cap K$. Hence $H \cap L \cap K$ is a quasi ideal of S. □

Theorem 3.6. *The intersection of arbitrary collection of quasi ideals of a ternary partial semiring S is either an empty set or a quasi ideal of S.*

Proof. Let $Q = \cap_{m \in \triangle} \{Q_m \backslash Q_m$ is a quasi ideal of S$\}$ where $\triangle$ denotes any indexing set, be a non empty set. Let $(x_i : i \in I)$ be a summable family in S and $x_i \in Q$ for every $i \in I$, then $x_i \in Q_m$ for each $m \in \triangle$. Then $\sum\limits_{i \in I} x_i \in Q_m$ as Q_m is a quasi ideal of S for each $m \in \triangle$. Hence $\sum\limits_{i \in I} x_i \in \cap_{m \in \triangle} Q_m = Q$. Further we have $SSQ \cap SQS \cap QSS = SS(\cap_{m \in \triangle}) \cap S(\cap_{m \in \triangle})S \cap (\cap_{m \in \triangle})SS \subseteq SSQ_m \cap SQ_mS \cap Q_mSS \subseteq Q_m$ for all $m \in \triangle$ as Q_m is a quasi ideal. Thus $SSQ \cap SQS \cap QSS \subseteq \cap_{m \in \triangle} = Q$. Hence Q is a quasi ideal of S. □

Lemma 3.1. *Let S be a ternary partial semiring and Q is quasi ideal of S then* $Q^3 \subseteq Q$.

Proof. Let Q be a quasi ideal of a ternary partial semiring S. We have $SSQ \cap SQS \cap QSS \subseteq Q$. We have $Q^3 = QQQ \subseteq SSQ$ and $Q^3 = QQQ \subseteq SQS$ and $Q^3 = QQQ \subseteq QSS$. Hence $Q^3 = QQQ \subseteq SSQ \cap SQS \cap QSS \subseteq Q$. □

4. Minimal Quasi Ideal

Definition 4.1. *Let S be a ternary partial semiring. A quasi ideal Q of S which does not contain any other proper quasi ideals of S is said to be a minimal quasi ideal of S.*

Theorem 4.1. *If H is a minimal left ideal, L is a minimal lateral ideal and K is a minimal right ideal of a ternary partial semiring S then $H \cap L \cap K$ is a minimal quasi ideal of S.*

Proof. let $Q = H \cap L \cap K$.From theorem 3.5, Q is a quasi ideal of S. let Q_1 be another quasi ideal of S such that $Q_1 \subseteq Q$. Then SSQ_1 is a left ideal, SQ_1S is a lateral ideal and Q_1SS is a right ideal of S. Also $Q_1 \subseteq H \Rightarrow SSQ_1 \subseteq SSH \subseteq H, Q_1 \subseteq L \Rightarrow SQ_1S \subseteq SLS \subseteq L, Q_1 \subseteq K \Rightarrow Q_1SS \subseteq KSS \subseteq K$.By minimality of H,L and K, we have $SSQ_1 = H, SQ_1S = L, Q_1SS = K$. Therefore $Q = H \cap L \cap K = SSQ_1 \cap SQ_1S \cap Q_1SS \subseteq Q_1$. Hence $Q = Q_1$ which shows that Q_1 is a minimal quasi ideal of S. □

Theorem 4.2. *If Q is a minimal quasi ideal of a ternary partial semiring S. A left(right) ideal of S generated by two non zero elements of Q is same.*

Proof. Let Q be a minimal quasi ideal of a ternary partial semiring S. For a non zero element $x \in Q, \langle x \rangle_l$ is the left ideal generated by x which is a quasi ideal of S. Then $\langle x \rangle_l \cap Q$ is a quasi ideal of S. But $\langle x \rangle_l \cap Q \subseteq Q$ and Q is a minimal quasi ideal of S. Then we have $\langle x \rangle_l \cap Q = Q$. From this $Q \subseteq \langle x \rangle_l$. Now,for another non zero element $y \in Q$, we get $y \in \langle x \rangle_l$. Hence $\langle y \rangle_l \subseteq \langle x \rangle_l$...(1). Similarly, we can show that $\langle x \rangle_l \subseteq \langle y \rangle_l$...(2). From (1)and(2),we have $\langle x \rangle_l = \langle y \rangle_l$. □

Theorem 4.3. *If Q is a minimal quasi ideal of a ternary partial semiring S then Q is the intersection of a minimal left ideal, a minimal lateral ideal and a minimal right ideal of S.*

Proof. Let Q be a minimal quasi ideal of a ternary partial semiring S and let $x \in Q$. Then SSx, SxS, xSS be a left ideal, a lateral ideal, a right ideal of S. Then from theorem 3.5,$SSx \cap SxS \cap xSS$ is a quasi ideal of S. But $SSx \cap SxS \cap xSS \subseteq SSQ \cap SQS \cap QSS \subseteq Q$ as Q is a minimal quasi ideal. Therefore $SSx \cap SxS \cap xSS = Q$ as Q is minimal quasi ideal. Now, we show that SSx is a minimal left ideal. Let H be a left ideal of S such that $H \subseteq SSx$...(1). Then $SSH \subseteq H \subseteq SSx$. Now, $SSH \cap SxS \cap xSS \subseteq SSx \cap SxS \cap xSS \subseteq Q$. Then $SSx \cap SxS \cap xSS$ is quasi ideal of S. Thus by minimality of Q, we have $Q = SSH \cap SxS \cap xSS \subseteq SSH$. Now $SSx \subseteq SSQ \subseteq SS(SSH) = (SSS)SH \subseteq SSH \subseteq H$...(2). From (1)and(2) we have $H = SSX$. Consequently SSx is a minimal left ideal of S. Similarly, we prove that SxS a minimal lateral ideal and xSS a minimal right ideal of S. □

Theorem 4.4. *If Q is a quasi ideal of a ternary partial semiring S and $Q^3 = Q$ then S is a regular ternary partial semiring.*

Proof. Let H be a minimal left ideal, L be a minimal lateral ideal and K be a minimal right ideal of a ternary partial semiring S. Then from theorem 3.5,$K \cap L \cap H$ is a quasi ideal of S. From the given hypothesis, we have

$$K \cap L \cap H = (K \cap L \cap H)^3 = (K \cap L \cap H)(K \cap L \cap H)(K \cap L \cap H) \subseteq KLH. \tag{4.1}$$

Again, we have $KLH \subseteq KSS \subseteq K, KLH \subseteq SLS \subseteq L, KLH \subseteq SSH \subseteq H$. Then

$$KLH \subseteq K \cap L \cap H. \tag{4.2}$$

From (4.1) and (4.2), we have $KLH = K \cap L \cap H$. Let $x \in S$. Now $x \in \langle x\rangle_r, x \in \langle x\rangle_t, x \in \langle x\rangle_l$, and that $x \in \langle x\rangle_r \cap \langle x\rangle_t \cap \langle x\rangle_l = \langle x\rangle_r\langle x\rangle_t\langle x\rangle_l$.This implies that x is a finite sum of terms of the form yzw, where $y \in \langle x\rangle_r, z \in \langle x\rangle_t, w \in \langle x\rangle_l$. Also, $y \in \langle x\rangle_r \Rightarrow y = \sum\limits_i xa_ib_i + nx$ as a finite sum, $z \in \langle x\rangle_t \Rightarrow z = \sum\limits_j u_jxv_j + \sum\limits_k p_kq_kxr_ks_k + n'x$ as a finite sum,$w \in \langle x\rangle_l \Rightarrow w = \sum\limits_m c_md_m + n''x$ as a finite sum for some $a_i, b_i, u_j, v_j, p_k, q_k, r_k, s_k, c_m, d_m \in S$ and some $n, n', n'' \in \mathbb{Z}^+ \cup \{0\}$. Therefore

$$\begin{aligned}
yzw &= (\sum_i xa_ib_i + nx)(\sum_j u_jxv_j + \sum_k p_kq_kxr_ks_k + n'x)(\sum_m c_md_m + n''x)\\
&= x\{\sum_i\sum_j\sum_m(a_ib_iu_jxv_jc_md_m) + \sum_i\sum_j(a_ib_iu_jxv_j)n''\\
&+ \sum_i\sum_k\sum_m(a_ib_ip_kq_kxr_ks_kc_md_m) + \sum_i\sum_k(a_ib_ip_kq_kxr_ks_k)n''\\
&+ \sum_i\sum_m(a_ib_ixc_md_m)n' + \sum_i(a_ib_ix)n'n''\\
&+ \sum_j\sum_m(u_jxv_jc_md_m)n + \sum_j(u_jxv_j)n'n''\\
&+ \sum_k\sum_{im}(p_kq_kxr_ks_kc_md_m)n + \sum_k(p_kq_kxr_ks_k)n'n'' + \sum_m(xc_md_m)n\\
&+ (nn'n'')x\}x \in xSx.
\end{aligned}$$

Hence $x \in xSx$. Thus there exists $t \in S$ such that x=xtx. This shows that x is regular. Hence S is a regular ternary partial semiring. □

Theorem 4.5. *Every minimal lateral ideal of a ternary partial semiring S is a minimal ideal of a ternary partial semiring S.*

Proof. Let L be a minimal lateral ideal of a ternary partial semiring S. Then SLS is a lateral ideal of S and $SLS \subseteq L$. Since L is minimal, we have SLS = L. Now $LSS = (SLS)SS = SLSSS \subseteq SLS \subseteq L$ and $SSL = SS(SLS) = SSSLS \subseteq SLS \subseteq L$. Thus L is a right ideal and also a left ideal of S. Also L is a lateral ideal of S. Hence L is an ideal of S. Now we show that L is minimal ideal of S. If possible, let M be an ideal of S such that $M \subseteq L$. Since M is an ideal of S, it is a lateral ideal of S. But from the given hypothesis, we have M = L. Hence L is minimal ideal of S. □

Corollary 4.1. *A minimal ideal of a ternary partial semiring S contains any minimal quasi ideal of S.*

Proof. Let S be a ternary partial semiring and Let Q be its minimal quasi ideal. Then $Q = H \cap L \cap K$ where H is a minimal left ideal, L is a minimal lateral ideal and K is a minimal right ideal of a ternary partial semiring S. From this $Q \subseteq L$. From theorem 4.4, L is minimal ideal of S. Hence Q is contained in a minimal ideal of S. □

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