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INTERVAL INTEGRO DYNAMIC EQUATIONS ON TIME SCALES UNDER GENERALIZED HUKUHARA DELTA DERIVATIVE

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ABSTRACT. This paper is devoted to study the local existence and uniqueness results for interval valued integro-dynamic equations on time scales (IIDETs)using Banach contraction principle under generalized delta derivative(Δ_g-derivative).

1. INTRODUCTION

Interval analysis was introduced to handle interval uncertainty that occurs in many mathematical or computer models of real world phenomena. The interval valued differential and integro differential equations can be used to model dynamical systems with uncertainties by several authors. Stefanini [3] defined the strongly generalized derivative of interval valued functions which was the starting point of studying the interval differential equations whose solutions have decreasing length of their values. The theory of time scales attracted many researchers for its ability to model various real-world phenomena as the dynamical processes include both discrete and continuous variables simultaneously [2]. By knowing the importance of interval analysis and time scales, we incorporate the uncertainty in terms of intervals into dynamic equations on time scales which leads to IIDETs. In [10], Vasavi et. al., introduced the concept of generalized

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delta derivative (Δ_g-derivative) and studied the fuzzy dynamic equations on time scales. For applications on fuzzy and time scales, we refer [1]- [12].

2. PRELIMINARIES

Let k^1 be the non-empty set of compact intervals of $\mathbb{R}$. For $a, b \in k^1$, $a = [\underline{a}, \overline{a}], b = [\underline{b}, \overline{b}]$, $\underline{a} \leq \overline{a}$. The sum and scalar multiplication in k^1 is defined as $a + b = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$. For any $\xi \in \mathbb{R}$,

$$\xi A = \begin{cases} [\xi \underline{a}, \xi \overline{a}], \xi > 0 \\ [\xi \overline{a}, \xi \underline{a}], \xi < 0 \end{cases} .$$

For $a, b \in k^1$, the Hausdorff metric is defined as:

$$D_H(a, b) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}. \tag{2.1}$$

Clearly, (k^1, D_H) is a complete metric space. For $a \in k^1, D(a, 0) = \max\{|\underline{a}|, |\overline{a}|\}$, $len(a) = |\overline{a} - \underline{a}|$.

The generalized Hukuhara difference of a, b is defined as:

$$a \ominus_{gH} b = \begin{cases} [\underline{a} - \underline{b}, \overline{a} - \overline{b}], \ \ if \ \ len(a) \geq len(b) \\ [\overline{a} - \overline{b}, \underline{a} - \underline{b}], \ \ if \ \ len(a) < len(b). \end{cases}$$

For any $a, b, c \in k^1$, then

$$a \ominus_{gH} b = c \Leftrightarrow \begin{cases} a = b + c, \ \ if \ \ len(a) \geq len(b) \\ b = a + (-1)c, \ \ if \ \ len(a) < len(b). \end{cases}$$

An interval valued function $\mathcal{F} : [a, b]_{\mathbb{T}} \to k^1$ is continuous at $\tau_0 \in \mathbb{T}$, if for every $\xi > 0$, $\exists\ \delta_\xi > 0$, $D_H(\mathcal{F}(\tau), \mathcal{F}(\tau_0)) \leq \xi$, $\forall \tau \in [a, b]$, with $|\tau - \tau_0| < \delta_\xi$.

Definition 2.1. *Let* $\mathcal{F} : [a, b]_{\mathbb{T}} \to k^1$ *be such that* $\mathcal{F}(t) = [\underline{f}(t), \overline{f}(t)]$, $t \in \mathbb{T}$, *where* $\underline{f}, \overline{f}$ *are* Δ*-differentiable at* $t_0 \in \mathbb{T}^k$*, then* $\mathcal{F}$ *is said to be* $\Delta_g - differentiable$ *at* $t_0 \in \mathbb{T}^k$*,*

$$\mathcal{F}^{\Delta_g}(t) = [\min\{(\underline{f})^\Delta(t_0), (\overline{f})^\Delta(t_0)\}, \max\{(\underline{f})^\Delta(t_0), (\overline{f})^\Delta(t_0)\}].$$

We say that $\mathcal{F}$ is non-decreasing (non-increasing) if $t \to len(\mathcal{F}(t))$ is non-decreasing (non-increasing) for $t \in [a, b)_{\mathbb{T}}$.

$$\mathcal{F}^{\Delta_g}(t) = \begin{cases} (i)[(\underline{f})^{\Delta}(t), (\overline{f})^{\Delta}(t)], & (\overline{f})^{\Delta}(t) - (\underline{f})^{\Delta}(t) \geq 0 \\ (ii)[(\overline{f})^{\Delta}(t), (\underline{f})^{\Delta}(t)], & (\overline{f})^{\Delta}(t) - (\underline{f})^{\Delta}(t) \leq 0 \end{cases}.$$

We say that $\mathcal{F}$ is $\Delta_{1,g}$-differentiable, if $\mathcal{F}$ is differentiable as in (i), $\Delta_{2,g}$-differentiable, if $\mathcal{F}$ is differentiable as in (ii).

Definition 2.2. *Let $\mathcal{F} : [a, b]_{\mathbb{T}} \to k^1$ be such that $\mathcal{F}(t) = [\underline{f}(t), \overline{f}(t)]$, $t \in \mathbb{T}$. Then $\mathcal{F}$ is Riemann Δ_g-integrable on $[a, b)_{\mathbb{T}}$ if and only if $\underline{f}, \overline{f}$ are Riemann Δ_g-integrable on $[a, b)_{\mathbb{T}}$. Moreover,*

$$\int_a^b \mathcal{F}(t)\Delta t = \left[\int_a^b \underline{f}(t)\Delta t, \int_a^b \overline{f}(t)\Delta t \right].$$

Remark 2.1. *Let $\mathcal{F} : [a, b]_{\mathbb{T}} \to k^1$ be continuous on $[a, b]_{\mathbb{T}}$. If $\mathcal{F}$ is Riemann Δ_g-differentiable on $[a, b)_{\mathbb{T}}$ such that $\mathcal{F}^{\Delta}$ is continuous on $[a, b)_{\mathbb{T}}$, then*

$$\int_a^b \mathcal{F}^{\Delta_g}(t)\Delta t = \mathcal{F}(b) \ominus_{gH} \mathcal{F}(a). \tag{2.2}$$

Remark 2.2. *If $\mathcal{F}$ is non-decreasing on $[a, b]_{\mathbb{T}}$, then* (2.2) *is equivalent with*

$$\mathcal{F}(b) = \mathcal{F}(a) + \int_a^b \mathcal{F}^{\Delta_g}(t),$$

and if $\mathcal{F}$ is non-increasing, then (2.2) *is equivalent with*

$$\mathcal{F}(b) = \mathcal{F}(a) \ominus_{gH} \int_a^b \mathcal{F}^{\Delta_g}(t).$$

Now, we recall some results related to time scale calculus [2]. Let $\mathbb{T}$ be a time scale, i.e., an arbitrary non-empty closed subset of real numbers. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ defined by
$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ defined by
$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ is defined by
$\mu(t) = \sigma(t) - t$, for $t \in \mathbb{T}$.
Denote $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

3. Interval Integro Dynamic Equations on Time Scales

The main aim of this study is to develop the existence as well as uniqueness results of solutions for IIDETs of the form:

$$y^{\Delta_g}(\tau) = \mathcal{F}(\tau, y(\tau)) + \int_{\tau_0}^{\tau} G(\tau, s, y(s))\Delta s, \quad \tau_0, s \in [\tau_0, \tau_0 + a]_{\mathbb{T}}, \tag{3.1}$$

subject to the condition

$$y(\tau_0) = y_0, \tag{3.2}$$

where $\mathcal{F} : [\tau_0, \tau_0+a]_{\mathbb{T}} \times K' \to K'$ and $G : [\tau_0, \tau_0+a]^2_{\mathbb{T}} \times K' \to K'$ are rd-continuous interval valued functions, y^{Δ_g} is the Δ_g-derivative of y, $\tau \in \mathbb{T}$, $y_0 \in K'$. By a solution to (3.1), (3.2), we mean an interval valued mapping $y \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K')$ that satisfies (3.1), (3.2) for $\tau_0 \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$.

Lemma 3.1. *Let $\mathcal{F} \in [\tau_0, \tau_0 + a]_{\mathbb{T}} \times K' \to K'$ and $G : [\tau_0, \tau_0 + a]^2_{\mathbb{T}} \times K' \to K'$ and $y \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K')$. Then the interval valued mapping $\tau \to \mathcal{F}(\tau, y(\tau)) + \int_{\tau_0}^{\tau} G(\tau, s, y(s))\Delta s \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K')$.*

Remark 3.1. *The mapping $\tau \to \mathcal{F}(\tau, y(\tau)) + \int_{\tau_0}^{\tau} G(\tau, s, y(s))\Delta s$ is integrable and hence bounded on the interval $[\tau_0, \tau_0 + a]_{\mathbb{T}}$.*

Lemma 3.2. *Assume that $\mathcal{F} \in [\tau_0, \tau_0 + a]_{\mathbb{T}} \times K' \to K'$ and $G : [\tau_0, \tau_0 + a]^2_{\mathbb{T}} \times K' \to K'$ and $y \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K^1)$ is called a solution to* (3.1), (3.2) *on $[\tau_0, \tau_0 + a]_{\mathbb{T}}$ if and only if y is a rd-continuous interval valued mapping and it satisfies one of the following interval valued integral equations*

$$y(\tau) = y_0 + \int_{\tau_0}^{\tau} \mathcal{F}(s, y(s))\Delta s + \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^{s} G(s, \tau, y(\tau))\Delta\tau \right) \Delta s,$$

if y is $\Delta_{1,g}$-differentiable,

$$y_0 = y(\tau) + (-1) \int_{\tau_0}^{\tau} F(s, y(s))\Delta s + (-1) \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^{s} G(s, \tau, y(\tau))\Delta\tau \right) \Delta s, \tag{3.3}$$

if y is $\Delta_{2,g}$-differentiable, provided the H-difference

$$y_0 \ominus (-1) \int_{\tau_0}^{\tau} F(s, y(s))\Delta s + (-1) \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^{s} G(s, \tau, y(\tau))\Delta\tau \right) \Delta s$$

exists.

Proof. We prove the case of $\Delta_{2,g}$-differentiability, the proof of the above case is similar. Let $y \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, k^1)$ be a solution to (3.1), (3.2). Therefore, y is $\Delta_{2,g}$-differentiable on $[\tau_0, \tau_0 + a]_{\mathbb{T}}$ and y_g^{Δ} is integrable as a rd-continuous function. Hence, we obtain

$$y(\tau_0) = y(\tau) + (-1) \int_{\tau_0}^{\tau} y^{\Delta_g}(s) \Delta s,$$

for every $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$. Since $y(\tau_0) = y_0$ and

$$y^{\Delta_g}(s) = \mathcal{F}(s, y(s)) + \int_{\tau_0}^{s} G(s, \tau, y(\tau)) \Delta\tau \text{for } s \in [\tau_0, \tau_0 + a]_{\mathbb{T}},$$

we obtain

$$y_0 = y(\tau) + (-1) \int_{\tau_0}^{\tau} F(s, y(s)) \Delta s + (-1) \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^{s} G(s, \tau, y(\tau)) \Delta\tau \right) \Delta s.$$

To prove the converse part, let us assume that $y : [\tau_0, \tau_0 + a]_{\mathbb{T}} \to K'$ is a rd-continuous interval valued mapping and it satisfies (3.3), which allows that $y(\tau_0) = y_0$ and there exists the H-difference

$$y(\tau_0) \ominus (-1) \int_{\tau_0}^{\tau} F(s, y(s)) \Delta s + (-1) \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^{s} G(s, \tau, y(\tau)) \Delta\tau \right) \Delta s,$$

for every $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$.

(i) If $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$ is right-dense such that $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$. Then

$$y(\tau) \ominus y(\tau + h) = (-1) \int_{\tau}^{\tau + h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^{s} G(s, \tau, y(\tau)) \Delta\tau \right) \Delta s.$$

For $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$, consider

$$\begin{aligned} & y(\tau + h) + (-1) \int_{\tau}^{\tau + h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^{s} G(s, \tau, y(\tau) \Delta\tau \right) \Delta s \\ =& y(\tau_0) \ominus (-1) \int_{\tau_0}^{\tau + h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^{s} G(s, \tau, y(\tau)) \Delta\tau \right) \Delta s \\ & + (-1) \int_{\tau}^{\tau + h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^{s} G(s, \tau, y(\tau) \Delta\tau \right) \Delta s \\ =& y(\tau_0) \ominus (-1) \int_{\tau_0}^{\tau + h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^{s} G(s, \tau, y(\tau) \Delta\tau \right) \Delta s \end{aligned}$$

$$+\,(-1)\int_{\tau_0}^{\tau+h}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau)\Delta\tau\right)\Delta s$$
$$\ominus\,(-1)\int_{\tau_0}^{\tau}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau)\Delta\tau\right)\Delta s=y(\tau).$$

for every $\tau\in[\tau_0,\tau_0+a]_{\mathbb{T}}$. Dividing by $-h$ and passing to limit $h\to 0$,

$$\lim_{h\to 0^+}\frac{y(t)\ominus y(t+h)}{-h}$$
$$=\lim_{h\to 0^+}\frac{1}{h}\int_{t}^{t+h}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau))\Delta\tau\right)\Delta s.$$

Since $\mathcal{F},G$ are rd-continuous, hence

$$\lim_{h\to 0^+}\frac{y(t)\ominus y(t+h)}{-h}=\mathcal{F}(t,y_t)+\int_{\tau_0}^{t}G(t,s,y(s))\Delta s.$$

Hence y is $\Delta_{2,g}$-differentiable.

(ii) If t is right-scattered, then

$$y(t)\ominus y(\sigma(t))=(-1)\int_{t}^{\sigma(t)}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau))\Delta\tau\right)\Delta s.$$

Consider

$$y(\sigma(t))+(-1)\int_{t}^{\sigma(t)}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau)\Delta\tau\right)\Delta s$$
$$=y(\tau_0)\ominus(-1)\int_{\tau_0}^{\sigma(t)}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau))\Delta\tau\right)\Delta s$$
$$+\,(-1)\int_{\tau_0}^{\sigma(t)}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau)\Delta\tau\right)\Delta s$$
$$=y(\tau_0)\ominus(-1)\int_{\tau_0}^{\sigma(t)}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau)\Delta\tau\right)\Delta s$$
$$+\,(-1)\int_{\tau_0}^{\sigma(t)}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau)\Delta\tau\right)\Delta s$$
$$\ominus\,(-1)\int_{\tau_0}^{t}\left(\mathcal{F}(s,y_s)+\int_{\tau_0}^{s}G(s,\tau,y(\tau)\Delta\tau\right)\Delta s=y(t).$$

Dividing by $-\mu(t)$ and passing to limit $\mu(t) \to 0$,

$$\int_t^{\sigma(t)} \mathcal{F}(s, y_s)\Delta s = \mu(t)\mathcal{F}(t, y_t).$$

Hence y is $\Delta_{2,g}$-differentiable.

$$y^{\Delta_g}(t) = \mathcal{F}(t, y(t) + \int_{\tau_0}^t G(t, s, y(s))\Delta s, \text{ for every } \tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}.$$

□

Definition 3.1. *Let D_H be the Hausdorff metric on K'. We define the space of all rd-continuous functions on time scales $C_{rd}[\tau_0, \tau_0 + a]_{\mathbb{T}}, K'$, with β metric by*

$$d_\beta(u, v) = \sup_{\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}} \frac{D(u(\tau), v(\tau))}{e_\beta(\tau, \tau_0)}, \text{ for each } u, v \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K').$$

It is clear that $C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K')$ is a complete metric space.

Theorem 3.1. *Let $\mathcal{F} : [\tau_0, \tau_0 + a]_{\mathbb{T}}^k \times K' \to k^1$, and $G : [\tau_0, \tau_0 + a]_{\mathbb{T}}^2 \times K' \to k^1$ be rd-continuous. If there exists constants $L > 0, M > 0, \beta > 0, \gamma > 1$ and $\beta = L\gamma$ such that $\mathcal{F}, G$ are rd-continuous and satisfy*

$$D_H(\mathcal{F}(\tau, u), \mathcal{F}(\tau, v)) \leq MD_H(u, v), \ \forall(\tau, u), (\tau, v) \in [\tau_0, \tau_0 + a]_{\mathbb{T}}^k \times K',$$
$$D_H(\mathcal{G}(\tau, s, u), \mathcal{G}(\tau, s, v)) \leq LD_H(u, v), \ \forall(\tau, u), (\tau, v) \in [\tau_0, \tau_0 + a]_{\mathbb{T}}^2 \times K'.$$

If $\frac{M}{\beta}(1 + \frac{1}{\gamma}) < 1$, then (3.1)*,* (3.2) *has two unique solutions ($\Delta_{1,g}, \Delta_{2,g}$) $y_1, y_2 \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K')$.*

Proof. Let $L, M > 0$ be the constants. Let

$$T(y)(\tau) = y_0 + \int_{\tau_0}^{\tau} \mathcal{F}(s, y(s))\Delta s + \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^{s} G(s, \tau, y(\tau))\Delta\tau \right) \Delta s, \tag{3.4}$$

Now, we verify that operator T is a contraction map. (3.4) is well defined, as $\mathcal{F}, G$ is rd-continuous. Hence $T(y) \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K')$ for every $y \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K')$. Further, $T(y)\tau_0 = y_0 \in K'$. Therefore, $T : C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K') \to C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K^1)$. Now, we prove that there exists a unique rd-continuous function y such that $Ty = y$, i.e., the point of T will be the solution of (3.1), (3.2). Let $u, v \in C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K')$, we have

$$
\begin{aligned}
D_\beta(T(u), T(v)) &= \sup_{\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}} \frac{D_H(Tu(\tau), Tv(\tau))}{e_\beta(\tau, \tau_0)} \\
&= \sup_{\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}} \left[\frac{1}{e_\beta(\tau, \tau_0)} \left\{ D_H \left(y_0 + \int_{\tau_0}^{\tau} F(s, u(s)) \Delta s \right.\right.\right. \\
&\quad \left. + \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^{s} G(s, \tau, u(\tau)) \Delta \tau \right) \right) \Bigg] \Delta s, \\
&\quad \left(y_0 + \int_{\tau_0}^{t} F(s, v(s)) \Delta s + \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^{s} G(s, \tau, v(\tau)) \Delta \tau \right) \right) \Bigg\} \Bigg] \Delta s, \\
&\leq \sup_{\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}} \frac{1}{e_\beta(\tau, \tau_0)} M \left[\int_{\tau_0}^{\tau} D_H(u(s), v(s)) + L \int_{\tau_0}^{s} D_H(u(\tau), v(\tau)) \Delta \tau \right] \Delta s \\
&= \sup_{\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}} \frac{1}{e_\beta(\tau, \tau_0)} M \int_{\tau_0}^{\tau} \left[\frac{D_H(u(s), v(s))}{e_\beta(s, \tau_0)} e_\beta(s, \tau_0) \right. \\
&\quad \left. + L \int_{\tau_0}^{s} \frac{D_H(u(\tau), v(\tau))}{e_\beta(\tau, \tau_0)} e_\beta(\tau, \tau_0) \Delta \tau \right] \Delta s \\
&\leq \sup_{\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}} \frac{1}{e_\beta(\tau, \tau_0)} M \int_{\tau_0}^{\tau} \left[D_\beta(u, v) e_\beta(s, \tau_0) + L \int_{\tau_0}^{s} D_\beta(u, v) e_\beta(\tau, \tau_0) \Delta \tau \right] \Delta s \\
&= \sup_{\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}} \frac{1}{e_\beta(\tau, \tau_0)} M D_\beta(u, v) \int_{\tau_0}^{\tau} \left[e_\beta(s, \tau_0) + L \int_{\tau_0}^{s} e_\beta(\tau, \tau_0) \Delta \tau \right] \Delta s \\
&= \sup_{\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}} \frac{1}{e_\beta(\tau, \tau_0)} M D_\beta(u, v) \int_{\tau_0}^{\tau} \left[e_\beta(s, \tau_0) + L \left(\frac{e_\beta(s, \tau_0) - 1}{\beta} \right) \right] \Delta s \\
&= M D_\beta(u, v) \left(1 + \frac{L}{\beta} \right) \sup_{\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}} \frac{1}{e_\beta(\tau, \tau_0)} \left(\frac{e_\beta(\tau, \tau_0) - 1}{\beta} \right) \\
&= \frac{M}{\beta} (1 + \frac{1}{\gamma}) D_\beta(u, v).
\end{aligned}
$$

As $\dfrac{M}{\beta}\left(1 + \dfrac{1}{\gamma}\right) < 1$, from Banach fixed point theorem, T has a unique fixed point in $C_{rd}([\tau_0, \tau_0 + a]_{\mathbb{T}}, K')$, which is a solution of (3.1), (3.2). □

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