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APPROXIMATE ANALYTICAL SOLUTION OF KLEIN-GORDON EQUATIONS BY THE MODIFIED ADOMIAN DECOMPOSITION METHOD

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ABSTRACT. In this work, the authors presented a new modification of Adomian Decomposition Method (ADM) to resolve Klein-Gordon equation. The Klein-Gordon Equation describes a wide variety of physical phenomena such as in wave propagation, in continuum mechanics and in the theoretical description of spinless particles in relativistic quantum mechanics. The authors applied this new procedure to solve this equation. The results resulting from the application of this method were good. And many times the exact solutions were obtained.

1. INTRODUCTION

A Partial Differential Equation (PDE) is an equation involving an unknown function, its partial derivatives, and the independent variables. PDE's are clearly ubiquitous in science; the unknown function might represent such quantities as temperature, electrostatic potential, concentration of a material, velocity of a fluid, displacement of an elastic material, acoustic pressure, etc. These quantities may depend on many variables, and one would like to find how the unknown quantity depends on these variables.

The Klein-Gordon Equation is an important group of partial differential equations and is present in relativistic quantum mechanics and field theory. The

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nonlinear Klein-Gordon Equation model as many physical phenomena in plasma physics, particle physics, quantum field theory. The Klein-Gordon Equation provides a simple but rich model to describe a self-interacting scalar field. From the mathematical viewpoint, the equation relies in the category of dispersive equations [1]. In physics, the equation has been used to study various phenomena from ferromagnetism to DNA dynamics and black-hole theory [2]. Many authors were interested in studying this equation and presented several method to solve it, such as the Variational Iteration Method [3,4], the Finite Element Method [5], the Cubic B-Spline Collocation Method [6], the Finite Difference Method [7], the Decomposition Method [8], Exp-Function Method [9,10], the Homotopy Perturbation Method [11].

The beginning of the ADM was in 1980s by the scientist George Adomian [12,13]. The ADM considered an reliable and simple procedure for obtaining analytical solutions. The ADM used to solve ordinary [14,15,16,17], partial, integral, fractional differential equation. In this work the authors focused on solving the Klein-Gordon Equation by using a new differential operator.

2. Analysis of the method

We consider the partial Klein-Gordon Equation as following:

$$u_{tt}(x,t) + au_t(x,t) + cu(x,t) = z(u, u_t, u_{tt}), \tag{2.1}$$

with the following initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x),$$

where $z(u, u_x, u_t)$ is a given functions, and $f(x), g(x)$ is the source term. Under the transformation $a = 2n + m, \quad c = n(m+n)$.
Equation (2.1) transformed to:

$$u_{tt}(x,t) + (2n+m)u_t(x,t) + n(n+m)u(x,t) = z(u, u_t, u_{tt}). \tag{2.2}$$

We discuss the solution of the Klein-Gordon Equation using (MADM).

Equation (2.2) can be written in an operator form:

$$Lu(x,t) = z(u, u_t, u_{tt}), \tag{2.3}$$

where L is second order differential operators define by:

$$Lu(x,t) = e^{-(m+n)t}\frac{\partial}{\partial x}e^{mt}\frac{\partial}{\partial x}e^{nt}u(x,t), \tag{2.4}$$

and the inverse operators L_{tt}^{-1} take the formula

$$L_{tt}^{-1}(.) = e^{-nt}\int_0^x e^{-mt}\int_0^x e^{(m+n)}(.)\partial x\partial x. \tag{2.5}$$

Applying L_{tt}^{-1} of (2.5) to the terms

$$u_{tt}(x,t) + (2n+m)u_t(x,t) + n(n+m)u(x,t)$$

of equation (2.2), we have

$$\begin{aligned} &L_{tt}^{-1}(u_{tt}(x,t) + (2n+m)u_t(x,t) + n(n+m)u(x,t)) \\ =&e^{-nt}\int_0^x e^{-mt}\int_0^x e^{(m+n)}(u_{tt}(x,t) \\ &+(2n+m)u_t(x,t) + n(n+m)u(x,t))\partial x\partial x. \end{aligned}$$

Therefore

$$\begin{aligned} u(x,t) = e^{-nt}u(x,0) - \frac{e^{-(m+n)t}}{m}u_t(x,0) + \frac{e^{-nt}}{m}u_t(x,0) \\ - \frac{ne^{-(m+n)t}}{m}u(x,0) + \frac{ne^{-nt}}{m}u(x,0) + L^{-1}(z(u,u_t,u_tt)), \end{aligned}$$

and using the initial conditions $u(x,0) = f(x), u_x(x,0) = g(x)$, we get:

$$\begin{aligned} u(x,t) = e^{-nt}f(x) - \frac{e^{-(m+n)t}}{m}g(x) + \frac{e^{-nt}}{m}g(x) - \frac{ne^{-(m+n)t}}{m}f(x) \\ + \frac{ne^{-nt}}{m}f(x) + L^{-1}(z(u,u_t,u_tt)). \end{aligned} \tag{2.6}$$

We will define the solution $u(x,t)$ by series gives as follows:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{2.7}$$

the non-linear part of equation (2.2) represented by an infinite series, called Adomian's polynomials

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n, \tag{2.8}$$

where A_n define by:

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[N(\sum_{i=0}^{\infty}\lambda^i u_i)\right]_{\lambda=0}, n = 0, 1, 2, ... \tag{2.9}$$

We will put the decomposition series equation (2.8) and equation (2.7) into equation (2.6), we get:

$$\sum_{n=0}^{\infty} u_n(x,t) = \delta(x) + L^{-1}z(u, u_t, u_{tt}) + \sum_{n=0}^{\infty} A_n,$$

we get following recursive relation

$$u_0(x,t) = \delta(x),$$

$$u_{n+1}(x,t) = L^{-1}z(u, u_t, u_{tt}) + \sum_{n=0}^{\infty} A_n.$$

The n-term approximant

$$\nu_n(x,t) = \sum_{i=0}^{n-1} u_i,$$

with

$$u(x,t) = \lim_{n\to\infty} \nu_n(x,t).$$

3. Applications of MADM

In order to assess both the applicability and the accuracy of MADM, we apply MADM to several Klein-Gordon Equations as indicated in the following examples.

Problem 1: Consider the following homogeneous linear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) - u(x,t) = 0. \tag{3.1}$$

With the initial conditions

$$\begin{aligned} u(x,0) &= 0, \\ u_t(x,0) &= \sin x, \end{aligned}$$

using the MADM in operator form equation (3.1) becomes

$$L_{tt}u(x,t) = u_{xx}(x,t) + u(x,t). \tag{3.2}$$

Applying the operator $L_{tt}^{-1} = e^{-nt}\int_0^t e^{-mt}\int_0^t e^{(m+n)t}(\cdot)\partial t\partial t$, to both sides of equation (3.2) Where

$$a = 0, c = -1, n = 1, m = -2$$

and using the initial conditions yields

$$u(x,t) \;=\; \frac{1}{2}e^t \sin x - \frac{1}{2}e^{-t}\sin x + L_{tt}^{-1}\Big(u_{xx}(x,t)\Big).$$

Identifying the Zeroth component $u_0(x,t)$, by all terms that are not included under the inverse operator L_{tt}^{-1} and following the above discussion leads to the recursive relation

(3.3)
$$\begin{aligned}
u_0(x,t) &= \frac{1}{2}e^t \sin x - \frac{1}{2}e^{-t}\sin x,\\
u_1(x,t) &= L_{tt}^{-1}(u_{0xx}(x,t)),\\
u_1(x,t) &= L_{tt}^{-1}\Big(\frac{\partial^2}{\partial x^2}\Big(\frac{1}{2}(e^t - e^{-t})\Big)\sin x\Big),\\
u_1(x,t) &= L_{tt}^{-1}\Big(-\frac{1}{2}\Big(e^t - e^{-t}\Big)\sin x\Big),\\
u_1(x,t) &= -\frac{1}{6}t^3 \sin x.
\end{aligned}$$

Expanding equation (3.3) using Maclaurine series of order 4, we have

(3.4)
$$u_0(x,t) = t\sin x + \frac{1}{6}t^3\sin x.$$

Accordingly, the series solution is given by

$$u(x,t) \;=\; t\sin x.$$

However solving problem (1) using Adomian Decomposition Method (ADM) where

(3.5)
$$L_{tt}u(x,t) = u_{xx}(x,t) + u(x,t).$$

Applying the operator $L_{tt}^{-1} = \int_0^t\int_0^t(\cdot)dtdt$, to both sides of equation (3.5), and using the initial conditions yields

$$u(x,t) \;=\; t\sin x\,.$$

Problem 2 Consider the following nonlinear Klein-Gordon Equation

(3.6)
$$u_{tt} + 3u_t + 2u + u_x u + u_{xx} = -6t + 15t^2 + 22t^3 + 4t^4.$$

With the initial conditions

$$u(x,0)=0, u_t(x,0)=0,$$

the exact solution is $u(x,t)=-t^3+2t^4$.

Using the MADM in operator form equation (3.6), becomes

$$L_{tt}=-6t+15t^2+22t^3+4t^4-u_xu-u_{xx}, \tag{3.7}$$

by applying the operator $L_{tt}^{-1}=e^{-nt}\int_0^t e^{-mt}\int_0^t e^{(m+n)t}(\cdot)\partial t\partial t$, to both sides of equation (3.7) where

$$a=3, c=2, n=1, m=1,$$

and using the initial conditions yields

$$u(x,t)=-t^3+2t^4-L_{tt}^{-1}(u_xu)-L_{tt}^{-1}(u_{xx}).$$

Identifying the Zeroth component $u_0(x,t)$, by all terms that are not included under the inverse operator L_{tt}^{-1} and following the above discussion leads to the recursive relation

$$\begin{aligned} u_o &= -t^3, \\ u_{n+1} &= -L_{tt}^{-1}(A_n)-L_{tt}^{-1}(u_{nxx}), \quad n\geq 1. \end{aligned} \tag{3.8}$$

This will enable us to determine the components $u_n(x,t)$, recurrently. In view of the recursive relation (3.8) we obtain

$$\begin{aligned} u_1 &= 2t^4-L_{tt}^{-1}(A_0)-L_{tt}^{-1}(u_{0xx}), \\ u_2 &= -L_{tt}^{-1}(A_1)-L_{tt}^{-1}(u_{1xx})=0, \\ u_n(x,t) &= 0, \quad n\geq 3. \end{aligned}$$

Accordingly, the series solution is given by

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)=-t^3+2t^4.$$

However solving problem 2 is given by Adomian Decomposition Method (ADM). Here

$$L_{tt}=-6t+15t^2+22t^3+4t^4-3u_t-2u-u_xu-u_{xx}, \tag{3.9}$$

by applying $L_{tt}^{-1}(.)=\int_0^t\int_0^t(.)$ on both side of equation (3.9), we get

$$\begin{aligned} u(x,t) &= -t^3+\frac{5\,t^4}{4}+\frac{11\,t^5}{10}+\frac{2\,t^6}{15}-L_{tt}^{-1}(-3u_t-2u-u_xu-u_{xx}) \\ u_{n+1} &= -3L_{tt}^{-1}u_{nt}-2L_{tt}^{-1}u_n-L_{tt}^{-1}(A_n)-L_{tt}^{-1}(u_{nxx}), \quad n\geq 1. \end{aligned}$$

TABLE 1. Comparison of numerical errors

X	EXACT	ADM	MADM	ABS ERROR [ADM]	ABS ERROR [MADM]
0.0	0.00000000	0.00000000	0.000000000	0.00000000	0.00000000
0.1	-0.00080000	-0.00080061	-0.000800000	0.00000061	0.00000000
0.2	-0.00480000	-0.00483988	-0.004800000	0.00003988	0.00000000
0.3	-0.01080000	-0.01126271	-0.010800000	0.00046271	0.00000000
0.4	-0.01280000	-0.01544797	-0.012800000	0.00264797	0.00000000
0.5	0.00000000	-0.01028646	0.000000000	0.01028646	0.00000000
0.6	0.04320000	0.01192715	0.043200000	0.03127285	0.00000000
0.7	0.13720000	0.05692417	0.137200000	0.08027583	0.00000000
0.8	0.30720000	0.12514723	0.307200000	0.18205277	0.00000000
0.9	0.58320000	0.20762305	0.583200000	0.37557695	0.00000000
1.0	1.00000000	0.28095238	1.000000000	0.71904762	0.00000000

The first few components are

$$\begin{aligned} u_0 &= -t^3, \\ u_1 &= 2t^4 + \frac{6t^5}{5} + \frac{2t^6}{15}, \\ u_2 &= \frac{-6t^5}{5} - \frac{11t^6}{15} - \frac{4t^7}{35} - \frac{t^8}{210}. \end{aligned}$$

Accordingly, the series solution is given by

$$u(x,t) = -t^3 + 2\,t^4 - \frac{3\,t^6}{5} - \frac{4\,t^7}{35} - \frac{t^8}{210}.$$

Table 1 exhibits a comparison between the errors obtained by using the proposed Modify Adomian Decomposition Method (MADM) and the Standard Adomian Decomposition Method (SADM). Examining this table closely shows the improvements obtained by using the proposed scheme.

4. CONCLUSION

We have solved the Klein-Gordon equation using the MADM and SADM. As it is clear from examples in first example, the result was similar in both, in the second example the result in MADM was more accuracy of the SADM. The new method in this work is easy and effective in finding approximate solutions to the linear and nonlinear Klein-Gordon Equation.

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