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BOUNDS ON THE MULTIPLE DOMINATION NUMBER OF A SEMIGRAPH

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ABSTRACT. The notion of $k-$domination in graphs was introduced by Fink and Jacobson [1]. S.S.Kamath and R.S.Bhat [2] introduced the concept of adjacency domination in semigraphs. They inspire us to define multiple domination number of semigraphs. Let $G = (V, X)$ be a semigraph and let k be a positive integer. A set $D \subseteq V$ is called adjacent $k-$dominating set if every vertex $v \in V - D$ is adjacent to at least k vertices of D. The adjacency $k-$domination number γ_k^a is the minimum cardinality among the adjacent $k-$dominating sets of G. Also the end vertex adjacency $k-$domination number $\gamma_k^{ea}(G)$ is defined in the natural way. In this paper, the above multiple domination parameters are determined for various semigraphs, necessary and sufficient conditions and few bounds are obtained.

1. INTRODUCTION

Semigraphs introduced by E.Sampathkumar [3] are a new type of generalization of the concept of graph. A semigraph is a pair (V, X), where V is a nonempty set of vertices of G and X is a set of $n-$tuples, called edges of G of distinct vertices for $n \geq 2$ satisfying the following conditions:

(SG1) Any two edges have at most one vertex in common.

(SG2) Two edges $(u_1, u_2, \dots, u_n)$ and $(v_1, v_2, \dots, v_m)$ are considered to be equal if, and only if, (i) $m = n$ and (ii) either $u_i = v_i$ for all $i, 1 \leq i \leq n$ or

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$u_i = v_{n-i+1}$ for all $i, 1 \le i \le n$. Thus the edge $(u_1, u_2, \ldots, u_n)$ is the same as $(u_n, u_{n-1}, \ldots, u_1)$.

The vertices in a semigraph are divided into four types namely end vertices, middle vertices, middle-end vertices and isolated vertices.

Example 1. *[3] Let $G = (V, X)$ be a semigraph where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $X = \{(v_1, v_2, v_3, v_4), (v_4, v_5, v_6, v_7)\}$. An edge E is represented by an open Jordan curve whose end points are the end vertices of E. In G, v_1 and v_4 are end vertices (which are represented by thick dots), v_2 and v_3 are middle vertices (which are specially represented by small circle) of the edge (v_1, v_2, v_3, v_4).*

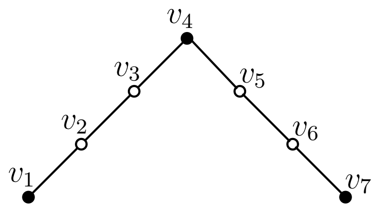

Figure 1: $4-$Uniform Semigraph G.

Definition 1.1. ***Adjacency of two vertices in a semigraph*** *[3] There are different types of adjacency of two vertices in a semigraph. Let $G = (V, X)$ be a semigraph.*

(1) *Two vertices u and v in a semigraph are said to be adjacent if they belong to the same edge.*
(2) *Two vertices u and v in a semigraph are said to be consecutively adjacent if in addition they are consecutive in order as well.*
(3) *Two vertices u and v are said to be $e-$adjacent, if they are the end vertices of an edge in G.*

Definition 1.2. ***Graphs associated with given semigraph*** *[3] Let $G = (V, X)$ be a semigraph. The following are three different graphs associated with G, each having the same vertex set V as that of G.*

End vertex graph G_e *: Two vertices in G_e are adjacent if and only if they are the end vertices of an edge in G.*

The adjacency graph G_a: *Two vertices in G_a are adjacent if and only if they are adjacent in G.*

The consecutive adjacency graph G_{ca}: *Two vertices in G_{ca} are adjacent if and only if they are consecutive adjacent vertices in G.*

Definition 1.3. *A semigraph G is complete if any two vertices in G are adjacent.*

Definition 1.4. *[4] A semigraph is $k-$ uniform if every edge contains exactly k vertices.*

2. Dominating Sets in Semigraphs

Various types of domination in semigraphs introduced by S.S. Kamath and R.S. Bhat [2], we give below definitions of some dominating parameters in semigraph.

Definition 2.1. *[5] Let $G = (V, X)$ be a semigraph and V_e be the set of all end vertices of G. A set $D \subseteq V$ is called adjacent dominating set(ad-set) if for every $v \in V - D$ there exists a $u \in D$ such that u is adjacent to v in G. The adjacency domination number $\gamma_a = \gamma_a(G)$ is the minimum cardinality of an $ad-$set of G. A set $D \subseteq V_e$ is called end vertex adjacency dominating set $ead-$set if (i) D is an $ad-$set and (ii) Every end vertex $v \in V - D$ is $e-$adjacent to some end vertex $u \in D$ in G. The end vertex adjacency domination number $\gamma_{ea} = \gamma_{ea}(G)$ is the minimum cardinality of an $ead-$set of G.*

Remark 2.1. *[2] For any semigraph G, (i) $\gamma_a(G) = \gamma(G_a)$ (ii) $\gamma_a(G) \leq \gamma_{ea}(G)$.*

Definition 2.2. *[5] Let $G = (V, X)$ be a semigraph. For any vertex $v \in V$, $N_a(v) = \{x \in V/x$ is adjacent to $v\}$ and $N_a[v] = N_a(v) \cup \{v\}$.*

3. Multiple Domination Number

Let V_e be the set of all end vertices of G and let V_m be the set of all middle vertices of a semigraph G.

Definition 3.1. *Let $G = (V, X)$ be a semigraph and let k be a positive integer. A subset $D \subseteq V$ is an adjacent $k-$dominating set (adk-set) if $|N_a(v) \cap D| \geq k$ for every $v \in V - D$. The adjacency k domination number $\gamma_k^a(G)$ is the minimum cardinality among the adjacent $k-$dominating set of G. Note that the adjacency $1-$domination number γ_1^a is the usual adjacency domination number $\gamma_a(G)$.*

Remark 3.1. *For $1 \leq j \leq k$, every adjacent $k-$dominating set is an adjacent $j-$dominating set. Therefore $\gamma_j^a(G) \leq \gamma_k^a(G)$.*

Definition 3.2. *A Set $D \subseteq V$ of semigraph G is called end vertex adjacency dominating set if (i) D is an adjacent $k-$ dominating set. (ii) For every end vertex $v \in V - D$, there exists an end vertex $u \in D$ such that u and v are $e-$adjacent. The end vertex adjacency $k-$domination number $\gamma_k^{ea}(G)$ is the minimum cardinality of an end vertex adjacency $k-$dominating set of G.*

Theorem 3.1. *[6]. An adjacent $k-$dominating set D of a semigraph $S = (V, X)$ is minimal if and only if for every $u \in D$ one of the following holds:*

(i) $|N_a(u) \cap D| < k$.

(ii) there exists a vertex v in V such that $|N_a(v) \cap D| = k$.

Proof. Suppose D is a minimal adjacent $k-$dominating set of a semigraph $S = (V, X)$. Let $u \in D$ and let $|N_a(u) \cap D| \geq k$. Suppose for every $v \in V - D$, such that $u \in N(v)$, there exists another vertex $w \in D$ such that $w \in N(v)$. Since u is adjacent to at least k vertices of $D - \{u\}$, $\therefore$ $D - \{u\}$ is adjacent $k-$dominating set of S. a contradiction, $\therefore$ u satisfies one of the conditions (i) and (ii).

Conversely suppose D is an adjacent $k-$dominating set of S such that every point $u \in D$ satisfies one of the conditions (i) and (ii). Since adjacency $k-$domination is a super hereditary property, it is enough to prove that D is minimal. Let $u \in D$. Consider $D - \{u\}$. If u satisfies (i), then $D - \{u\}$ cannot adjacent $k-$dominate u.If u satisfies (ii), then $D - \{u\}$ cannot adjacent $k-$dominate v. $\therefore$ D is minimal adjacent $k-$dominating set. ☐

Corollary 3.1. *For any semigraph G,*

(i) $\gamma_k^a(G) = \gamma_k(G_a)$.

(ii) If G has no vertices of degree zero, then $\gamma_k^{ea}(G) = \gamma_k(G_e)$.

Example 2. *The adjacency graph G_a associated with the semigraph G given in Example 1.1 is given below*

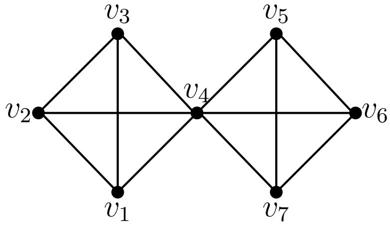

Figure 2: Graph G_a.

$\{v_2, v_4, v_6\}$ is a minimal $2-$dominating set. $\gamma_2(G_a) = 3$. For the semigraph G in Figure 1, $V_e = \{v_1, v_4, v_7\}$ is a minimal adjacent $2-$dominating set. $\gamma_2^a(G) = \gamma_2^{ea}(G) = 3$. Hence $\gamma_2^a(G) = \gamma_2(G_a) = 3$.

Corollary 3.2. *Let G be a semigraph. Then $\gamma_k^a(G) \leq \gamma_k^{ea}(G)$*

Theorem 3.2. *If G be a semigraph with no middle end vertices such that $\delta_a(G) \geq 2$, then the set of all vertices V_e of G is a minimal adjacent 2-dominating set.*

Proof. Since for each vertex in G the minimal adjacent degree $\delta_a(G) \geq 2$. Every middle vertex $u \in V(G) - V_e$ is adjacent to at least two end vertices of V_e. Therefore V_e is an adjavent 2- dominating set in G. Futhermore for every vertex $v \in V_e, V_e - \{v\}$ is not an adjacent 2-dominating set in G. Hence V_e is minimal adjacent 2-dominating set. $\therefore \gamma_2^a(G) = |V_e|$. □

Theorem 3.3. *Let G be a semigraph with no middle end vertices. If each edge of G has at least k middle vertices such that $\delta_a(G) \geq k + 1$, then the set of all middle vertices V_m contains a minimal adjacent $k-$ dominating set and $\gamma_k^a(G) = qk$.*

Proof. Since for each vertex in G the minimum adjacent degree $\delta_a(G) \geq k+1$. An end vertex $v \in V - V_m$ is adjacent to at least k middle vertices of V_m. Therefore V_m is an adjacent $k-$ dominating set in G. If D is a γ_k^a- set in G, then D consisting of k middle vertices from each edge of G. $\therefore |D| = qk = \gamma_k^a(G)$. □

Theorem 3.4. *Let G be a semigraph such that $2 < k \leq \delta_a - 1$, then $\gamma_2^a \leq p - \gamma_k^a$.*

Proof. Let D be a γ_k^a- set. Then $V - D$ contains a minimal adjacent 2-dominating set of G. $\gamma_2^a \leq |V - D| = |V| - |D|$. Since $\gamma_k^a = |D|$, $\gamma_2^a \leq p - \gamma_k^a$. □

Remark 3.2. $\gamma_a \leq \gamma_2^a \leq p - \gamma_k^a$.

We can characterise the class of semigraphs that satisfy the equality in Theorem 3.4.

Example 3. *Let S be a 5-uniform semigraph with no middle end vertices.*

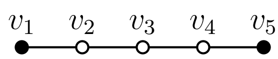

Figure 3: $5-$Uniform Semigraph S

Then $\gamma_2^a = p - \gamma_k^a = 2$. Thus the inequality in Theorem 3.4 is sharp.

Theorem 3.5. *Let G be a semigraph such that $\delta_a(G) \geq 2$, then $\gamma_a \leq p - \gamma_2^a$.*

Proof. Let V_e be a γ_2^a- set. Then $V - V_e$ contains a minimal adjacent dominating set. If D is a minimal adjacent dominating set of G. Thus we obtain $|D| \leq |V - V_e| = |V| - |V_e|$. $\therefore \gamma_a \leq p - \gamma_2^a$. $\square$

Theorem 3.6. *Let G be a semigraph with no middle end vertices and let Δ_a denotes the maximum adjacent degree of G. If each edge of G has at least k middle vertices, then $\dfrac{kp}{\Delta_a(G)+k} \leq \gamma_k^a(G)$.*

Proof. Let G be any semigraph with no middle end vertices.
Case (i): Let $D = V_e$. Then D is a minimum adjacent $2-$ dominating set of G. The maximum adjacent degree of each vertex in D is at most Δ_a. Also since each vertex in $V - D$ is adjacent to at least two vertices in D:

$$2\left(p - \gamma_2^a\right) \leq \Delta_a \gamma_2^a,$$

i.e.,

$$2p \leq \left(\Delta_a + 2\right)\gamma_2^a.$$

Hence,

$$\frac{2p}{\Delta_a(G)+2} \leq \gamma_2^a(G).$$

Case (ii): Let $D = V_m$. Then D is an adjacent $k-$ dominating set of G. Then D contains a minimum adjacent $k-$ dominating set S. The maximum adjacent degree of each vertex in S is at most Δ_a. Also since each vertex in $V - S$ is adjacent to at least k vertices in S. $k\left(p - \gamma_k^a\right) \leq \Delta_a \gamma_k^a$.
i.e $kp \leq \left(\Delta_a + k\right)\gamma_k^a$. Hence $\dfrac{kp}{\Delta_a(G)+k} \leq \gamma_k^a(G)$. $\square$

The following result is immediate.

Corollary 3.3. *If G is a complete semigraph, then every set of k vertices is an adjacent $k-$dominating set.*

Theorem 3.7. *If G is a connected $k-$uniform semigraph with q edges and no middle end vertices, then $\gamma_{k-2}^a(G) = q(k-2)$, where $k \neq 4$.*

Proof. Since each edge has exactly $(k-2)$ middle vertices. Therefore every vertex not in V_m is adjacent to $(k-2)$ middle vertices from V_m. i.e. V_m is a minimal adjacent $(k-2)$ dominating set. Hence $\gamma_{k-2}^a(G) = q(k-2)$. The equality in Theorem 3.7 is not satisfied for $4-$uniform semigraphs with more than one edge. $\square$

Example 4. *Let G be a $4-$uniform semigraph with $q > 1$, then $\gamma_2^a(G) < q(k-2) = 2q$.*

4. Conclusion

The notion of adjacency $k-$domination and end vertex adjacency $k-$domination in semigraph have been introduced and inequalities involving three domination parameters such as adjacency $k-$domination, end vertex adjacency $k-$domination and adjacency domination have been obtained. The necessary and sufficient condition for minimal adjacent $k-$dominating set is also discussed and adjacency $k-$domination number is determined for various semigraphs.

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