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AN INVESTIGATION OF COMPOSITION OPERATORS BETWEEN EXPONENTIAL WEIGHTED SPACES

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ABSTRACT. Exponential weight plays the main role in the study of boundedness and compactness of variations held by composition operators. In this paper Hardy's and Bergman spaces' exponential weight were introduced, and applied an exponential weight function to find the difference of composition operators between weighted Bergman spaces and weighted Hardy spaces of analytic functions that were defined on unit circle D. In particular, as an application, the condition for boundedness and compactness of the difference operated between weighted Hardy spaces and weighted Bergman spaces has been discussed.

1. INTRODUCTION

Suppose that Φ and Ψ are identity functions of the unit circle D on a z-plane. The related functions are generated by linear composition operators $C_\Phi(f) = f\ o\ \Phi$, $C_\Psi(f) = f\ o\ \Psi$ suitable. $C_\Psi(f) = f\ o\ \Psi$ can be expressed in the space $H(D)$ related to analytic functions on D. Furthermore, suppose α and β are severely constructive bounded continuous functions (expressed as weights) on D, the difference between the composition operators $C_\Phi - C_\Psi$ can be expressed as a weighted Bergman space given by,

$$B^p_\alpha = \left\{ f \in H(D); \|f\|^p_\alpha := \left(\int_D |f(z)|^p \alpha(z) dA(z) \right)^{\frac{1}{P}} < \infty \right\}, \quad P \in [1, \infty),$$

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where $dA(z)$ is the area measure so that the area of D is 1 and the weighted Hardy space of analytic functions is,

$$H^p_\beta := \left\{ f \in H(D); \|f\|_\beta := sup_{z\in D}\beta(z)|(z)| < \infty \right\}.$$

Studies have been conducted on composition operators as well as weighted composition operators on different spaces of analytic functions, as shown in [2–4, 7, 21]. Additional details of composition operators can be found in the studies by [8, 17, 19]. The boundedness and compactness of differences in composition operators on different spaces of analytic functions has been a subject of study by a number of authors [5, 12, 15, 16, 18, 23]. This paper identifies the differences of boundedness and compactness that are experimented within composition operators that can be interpreted across exponential weights.

2. Preliminaries

We start off by obtaining several geometric pieces of information about the unit circle. The value $a \in D$ is set, and then later is used to examine the automorphism $\varphi_a(z) := \frac{a-z}{1-\bar{a}z}, z \in D$ which exchanges 0 and a. The pseudo-hyperbolic metric is then determined by

$$d(z,a) := |\varphi_a(z)| = \left|\frac{a-z}{1-\bar{a}z}\right|, \quad \forall z, a \in D.$$

Furthermore, verification is done to ensure that

$$-\varphi'_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}, \quad z \in D.$$

Afterwards, further details on weights and weighted spaces are obtained. D^p_β is used to denote the closed unit disc of H^p_β. The radial weights β are used to justify that $\beta(z) = \beta(|z|)\forall z \in D$. The results that relate to weighted spaces of analytic functions can be secured by using connected weights. For a weight β, the connected weight $\tilde{\beta}$ is described by the following equation:

$$\tilde{\beta}(z) := \frac{1}{sup\{|f(z)|; f \in D^p_\beta\}}, z \in D.$$

The related weights are continuous in nature, such that $\tilde{\beta} \geq \beta > 0$ [1]. Moreover, for any $z \in D$ whereas $f_z \in H^p_\beta, \|f_z\|_\beta \leq 1$, so that $|f_z(z)| = \frac{1}{\tilde{\beta}(z)}$. A weight α is described as being fundamental if there is possibility of a constant

$C > 0$ and $z \in D$ whereby $\alpha(z) \leq \tilde{\alpha}(z) \leq C\alpha(z)$. The studies [1–3] deal with the instances where weights are fundamental. In this paper, the following circumstance which is due to Lusky [14] serves a significant purpose. We define, a satellite weight α which convinces (L1) if $\underset{b}{inf} \frac{\alpha(1-2^{-b-1})}{\alpha(1-2^{-b})} > 0$. This position is similar to the circumstance in [9]. $0 < r < 1$ and $1 < C < \infty$ with $\frac{\alpha(z)}{\alpha(u)} \leq C$ for every $u, z \in D$ with $d(z, u) \leq r$.

The proof of this comparison is provided in [9, 17]. According to [1], stellate weights convince (L1) are considered to be fundamental.

Weights can be considered in the following equations. Suppose that α become an analytic map on D, non-disappearance purely positive on $[0, 1]$ and satisfying $\lim_{r \to 1} \alpha(r) = 0$. Then we describe the weight α as follows: $\alpha(z) := m\left(e^{-\frac{1}{\left(1-z\right)^{a}}}\right), a \geq 1$ and all $z \in D$. Furthermore, we give an explanation about the instances of weights of this type, otherwise referred to as the exponential weight which is defined as: $\alpha(z) := e^{-\frac{1}{\left(1-z\right)^{a}}}, a \geq 1$. Then we obtained the weight $\alpha(z) := e^{-\frac{1}{\left(1-|z|^{2}\right)^{a}}}$.

The above example also convince the circumstance (L1) [14]. Therefore, the kind of weights we prepare here include the classical examples, which have been investigated by previous researchers. The differences between composition operators on weighted Bergman spaces were examined by Boo Rim Choe, Hyungwoon Koo and Wayne Smith over the upper half-plane [6]. Wayne Smith, Hyungwoon Koo and Boo Rim Choe likewise examined the contrasts between boundedness and compactness of the differences. These phenomena were of two generalized weighted composition operators acting from the Bloch space to Bers-type spaces [13]. The compact distinctions of weighted composition operators on standard weighted Bergman spaces were described by Maocai Wang, Xingxing Yao, and Fen Chen. They portrayed the necessary conditions for the differences of weighted composition operators to be compact [22]. Yecheng Shi, Songxiao Li show the boundedness, compactness and specified standards of differences of composition operators on Bloch type spaces using the Bloch type norm to characterize these [20].

Whereas there have been extensive studies regarding the composition operators in various spaces, there have not been many results pertaining to exponential weight spaces. In this paper we use information from related literature and formulate the methods of E. Wolf [23] by using exponential weight, weighted Hardy and weighted Bergman spaces. For all $u, z \in D$ we need to create a map $\alpha_u(z) := \alpha(\bar{u}z)$. Given that ν is analytic on D, the map α_u is assumed to be analytic on D.

3. Some Lemmas

The following lemma is common knowledge when using standard weights [10, 11, 23], but has not been applied in the cases of exponential weights. The following consequences are valid for all $P \in \big[1, \infty\big)$.

Lemma 3.1. *Suppose that* α *is a stellate weight defined as* $\alpha(z) := m\Big(e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}\Big)$, $\forall z \in D$ *so that* $sup_{u\in D} sup_{z\in D} \frac{\alpha(z)\big|\alpha_u(\varphi_u(z))\big|}{\alpha\big(\varphi_u(z)\big)} \leq C < \infty$. *Further*

$$|f(z)| \leq \frac{C^{\frac{1}{P}}}{\alpha(0)^{\frac{1}{P}} e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}} \alpha(z)^{\frac{1}{p}}} \|f\|^p_\alpha, \forall z \in D, \forall f \in B^p_\alpha.$$

Proof. Suppose that $a \in D$ and consider

$$T_a : B^p_\alpha \to B^p_\alpha, T_a(f(z)) = f(\varphi_a(z))\varphi'_a(z)^{\frac{2}{P}} \alpha_a(\varphi_a(z))^{\frac{1}{P}}.$$

By changing of variables we get

$$\begin{aligned}
\|T_a f\|^p_{\alpha,p} &= \int_D \alpha(z)|f(\varphi_a(z))|^p|\varphi'_a(z)|^2|\alpha_a(\varphi_a(z))|dA(z),\\
&= \int_D \frac{\alpha(z)|(\alpha_a(\varphi_a(z)))|}{\alpha(\varphi_a(z))}|f(\varphi_a(z))|^p|\varphi'_a(z)|^2\alpha(\varphi_a(z))dA(z)\\
&\leq sup_{z\in D}\frac{\alpha(z)\alpha_a(\varphi_a(z))}{\alpha(\varphi_a(z))}\int_D |f(\varphi_a(z))|^p|\varphi'_a(z)|^2\alpha(\varphi_a(z))dA(z)\\
&\leq C\int_D \alpha(t)|f(t)|^p dA(t) = C\|f\|^p_{\alpha,p}.
\end{aligned}$$

By using mean value theorem and substituting $g(z) := T_a(f(z))$ we receive

$$\alpha(0)|g(0)|^p \leq \int_D \alpha(z)|g(z)|^p dA(z) = \|g\|^p_{\alpha,p} \leq C\|f\|^p_{\alpha,p}.$$

Therefore,

$$\alpha(0)|g(0)|^p = \alpha(0)|f(a)|^p e^{-\frac{1}{\left(1-|a|^2\right)^{\frac{2}{P}}}}\alpha(a) \leq C\|f\|^p_{\alpha,p}.$$

Hence,

$$\big|f(a)\big| \leq C^{\frac{1}{p}} \frac{\big\|f\big\|^p_{\alpha}}{\alpha(0)^{\frac{1}{P}} e^{-\frac{1}{\left(1-|a|^2\right)^{\frac{2}{P}}}} \alpha(a)^{\frac{1}{p}}}.$$

Because a was arbitrary, the statement follows. □

Lemma 3.2. *Suppose that* α *is a stellate weight defined as* $\alpha(z) := m\Bigg(e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}\Bigg)$, $\forall z \in D$ *so that* $sup_{u\in D} sup_{z\in D} \frac{\alpha(z)\left|\alpha_u\left(\varphi_u(z)\right)\right|}{\alpha\left(\varphi_u(z)\right)} \leq C < \infty$. *Let* α *convince circumstance (L1). Further there exist* $r \in \big(0,1\big)$ *and* $N > 0$ *so that* $f \in B^p_\alpha$ *implies,*

$$|f(z) - f(u)| \leq \frac{4MC^{\frac{1}{p}}}{\alpha(0)^{\frac{1}{p}}} \frac{\|f\|^p_\alpha}{re^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}} \alpha(z)^{\frac{1}{p}}} d(z,u), \forall z, u \in D \text{ and } (z,u) \leq \frac{r}{2}.$$

Proof. Because α has circumstance (L1) and the weight $\mu(z) = e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}$ are convinces circumstance(L1), $r \in (0,1), N_1 < \infty$ and $N_2 < \infty$ can be found so that

$$\frac{\alpha(z)}{\alpha(u)} \leq N_1 \quad \text{and} \quad \frac{e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}}{e^{-\frac{1}{\left(1-|u|^2\right)^{\frac{2}{P}}}}} \leq M_2, \qquad \forall z, u \in D \text{ and } d(z,u) \leq r.$$

Suppose that $u \in D$ be fixed. Since $\varphi_u\big(\varphi_u(z)\big) = z$ and $\varphi_u(0) = u$, therefore

$$|f(z) - f(u)| = |f(\varphi_u(\varphi_u(z))) - f(\varphi_u(\varphi_u(u)))|.$$

By using Lemma 3.1 and $|z| = d(\varphi_u(z), u) \le r$ we get

$$\begin{aligned}
|f(\varphi_u(z))| &\le \frac{C^{\frac{1}{P}}\|f\|_\alpha^p}{\alpha(0)^{\frac{1}{p}}\alpha(\varphi_u(z))^{\frac{1}{p}} e^{-\frac{1}{\left(1-|\varphi_u(z)|^2\right)^{\frac{2}{P}}}}} \\
&= \frac{C^{\frac{1}{p}}}{\alpha(0)^{\frac{1}{p}}} \frac{\|f\|_\alpha^p}{\alpha(u)^{\frac{1}{p}} e^{-\frac{1}{\left(1-|u|^2\right)^{\frac{2}{P}}}}} \frac{\alpha(u)^{\frac{1}{p}}(1-|u|^2)^{\frac{2}{p}}}{\alpha(\varphi_u(z))^{\frac{1}{p}} e^{-\frac{1}{\left(1-|\varphi_u(z)|^2\right)^{\frac{2}{P}}}}} \\
&\le \frac{C^{\frac{1}{p}} N_1^{\frac{1}{p}} N_2^{\frac{1}{p}}}{\alpha(0)^{\frac{1}{p}}} \frac{\|f\|_\alpha^p}{\alpha(u)^{\frac{1}{p}} e^{-\frac{1}{\left(1-|u|^2\right)^{\frac{2}{P}}}}}.
\end{aligned}$$

Suppose that $g_u(z) := f(\varphi_u(z)$. Consequently, for

$$d(z,u) = |\varphi_u(z)| \le \frac{r}{2}, \theta \in D \text{ and } |\theta| \le |\varphi_u(z)| \le \frac{r}{2}$$

so that

$$\begin{aligned}
|f(z) - f(u)| &= \left|g_u\big(\varphi_u(z)\big) - g_u(0)\right| \\
&\le |\varphi_u(z)| \left| \int_0^1 \left[\frac{\delta}{\delta t} g_u\right]\big(t\varphi_u(z)\big) dt \right| \\
&\le \left[\frac{\delta}{\delta t} g_u(\theta)\right] |\varphi_u(z)| = |\varphi_u(z)| \frac{1}{2\pi} \left| \int_{|\xi|=r} \frac{g_u(\xi)}{(\xi-\theta)^2} d\xi \right|.
\end{aligned}$$

Lastly,

$$\begin{aligned}
|f(z) - f(u)| &\le \frac{C^{\frac{1}{p}} N_1^{\frac{1}{p}} N_2^{\frac{2}{p}}}{\alpha(0)^{\frac{1}{p}}} |\varphi_u(z)| r \frac{\|f\|_\alpha^p}{(r - |\varphi_u(z)|^2)} \frac{1}{\alpha(u)^{\frac{1}{p}} e^{-\frac{1}{\left(1-|u|^2\right)^{\frac{2}{P}}}}} \\
&\le \frac{4C^{\frac{1}{p}} N_1^{\frac{1}{p}} N_2^{\frac{2}{p}} \|f\|_\alpha^p}{r\alpha(0)^{\frac{1}{p}} \alpha(u)^{\frac{1}{p}} e^{-\frac{1}{\left(1-|u|^2\right)^{\frac{2}{P}}}}} d(z,u).
\end{aligned}$$

By choosing $N := N_1^{\frac{1}{p}} N_2^{\frac{2}{p}}$ get the required. □

Lemma 3.3. *Suppose that α is exponential weight defined as*

$$\alpha(z) := m\Big(e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}\Big), \quad \forall z \in D$$

so that $sup_{u\in D}sup_{z\in D}\frac{\alpha(z)\left|\alpha_u(\varphi_u(z))\right|}{\alpha\left(\varphi_u(z)\right)} \leq C < \infty$ *and* α *are convinces circumstance (L1). Further* $\forall f \in B^p_\alpha$ *there exist* $C_\alpha > 0$ *so that*

$$|f(z)-f(u)| \leq C_\alpha \|f\|^p_\alpha max\left\{\frac{1}{\alpha(z)^{\frac{1}{p}}e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}}, \frac{1}{\alpha(u)^{\frac{1}{p}}e^{-\frac{1}{\left(1-|u|^2\right)^{\frac{2}{P}}}}}\right\}d(z,u)$$

for all $z,u\in D$.

Proof. From Lemma 3.2 there is a constant $s\in\left(0,1\right)$ and $N<\infty$ so that

$$|f(z)-f(u)| \leq \frac{4NC^{\frac{1}{p}}}{\alpha(0)^{\frac{1}{p}}}\frac{\|f\|^p_\alpha}{\alpha(z)^{\frac{1}{p}}e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}}d(z,u), \forall z,u\in D \text{ and } d(z,u)\leq \frac{s}{2}.$$

Following, if $d(z,u)>\frac{s}{2}$, so

$$\begin{aligned}|f(z)-f(u)| &\leq \frac{2C^{\frac{1}{p}}}{\alpha(0)^{\frac{1}{p}}}\|f\|^p_\alpha max\left\{\frac{1}{\alpha(z)^{\frac{1}{p}}e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}}, \frac{1}{\alpha(u)^{\frac{1}{p}}e^{-\frac{1}{\left(1-|u|^2\right)^{\frac{2}{P}}}}}\right\}\\ &\leq \frac{4}{s}\frac{C^{\frac{1}{p}}}{\alpha(0)^{\frac{1}{p}}}\|f\|^p_\alpha max\left\{\frac{1}{\alpha(z)^{\frac{1}{p}}e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}}, \frac{1}{\alpha(u)^{\frac{1}{p}}e^{-\frac{1}{\left(1-|u|^2\right)^{\frac{2}{P}}}}}\right\}d(z,u).\end{aligned}$$

Thus, $C_\alpha := \|f\|^p_\alpha max\left\{\frac{4NC^{\frac{1}{p}}}{s\alpha(0)^{\frac{1}{p}}}, \frac{4C^{\frac{1}{p}}}{s\alpha(0)^{\frac{1}{p}}}\right\}$. Therefore,

$$|f(z)-f(\alpha)| \leq C_\alpha max\left\{\frac{1}{\alpha(z)^{\frac{1}{p}}e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}}, \frac{1}{\alpha(u)^{\frac{1}{p}}e^{-\frac{1}{\left(1-|u|^2\right)^{\frac{2}{P}}}}}\right\}d(z,u), \forall z,u\in D$$

and get the required. □

4. Main Results

Theorem 4.1. *Let* β *be an arbitrary exponential weight and* α *be exponential weight (i.e.* $\alpha(z) := m\left(e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{P}}}}\right)$ *for each* $z\in D$*) so that*

$$sup_{u\in D}sup_{z\in D}\frac{\alpha(z)\left|\alpha_u\left(\varphi_u(z)\right)\right|}{\alpha\left(\varphi_u(z)\right)} \leq C < \infty$$

and so that α has circumstance (L1). Furthermore, let $\emptyset$ and Ψ be analytic transformations of D. Then the difference $C_\emptyset - C_\Psi : B^p_\alpha \to H^p_\beta$ is bounded if and only if

$$\left\{\frac{1}{\alpha(\emptyset(z))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\emptyset(z)|^2\right)^{\frac{2}{P}}}}}, \frac{1}{\alpha(\Psi(z))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\Psi(z)|^2\right)^{\frac{2}{P}}}}}\right\} d(\emptyset(z), \Psi(z)) < \infty.$$

Proof. From [1] by making out that under the given hypothesis α and $\tilde{\alpha}$ are equivalent, i.e. A constant $b > 0$ can be obtained so that $\alpha(z) \le \tilde{\alpha}(z) \le b\alpha(z)$ for any $z \in D$. Leading, assume that the difference is bounded and wish to indicate that

$$sup_{z\in D}\frac{\beta(z)}{\alpha(\emptyset(z))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\emptyset(z)|^2\right)^{\frac{2}{P}}}}} < \infty \text{ indirectly.}$$

Suppose that there is a sequence $(z_n)_n \subset D$ so that $|\emptyset(z_n)| \to 1$ and

$$\frac{\beta(z_n)}{\alpha(\emptyset(z_n))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\emptyset(z_n)|^2\right)^{\frac{2}{P}}}}} \ge n$$

for any $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and select $f^p_n \in D^p_\beta$ so that $|f_n(\emptyset(z_n))|^p = \frac{1}{\tilde{\alpha}\left(\emptyset(z_n)\right)}$. Directly set $g_n(z) := f_n(z)\varphi'_{\emptyset(z_n)}(z)^{\frac{2}{p}}\varphi_{\Psi(z_n)}(z), \forall z \in D$.

Further interchange of variables yields

$$\begin{aligned}
\|g\|^p_{\alpha,p} &= \int_D |g_n(z)|^p\alpha(z)dA(z)\\
&= \int_D |f_n(z)|^p|\varphi'_{\emptyset(z_n)}(z)|^2|\varphi_{\Psi(z_n)}(z)|^p\alpha(z)dA(z)\\
&\le sup_{z\in D}\alpha(z)|f_n(z)|^p sup_{z\in D}|\varphi_{\Psi(z_n)}(z)|^p\int_D |\varphi'_{\emptyset(z_n)}(z)|^2 dA(z)\\
&= \int_D dA(t) = 1.
\end{aligned}$$

Naturally, $(g_n)_n$ belongs to the closed unit ball of B^p_α and by boundedness of the difference a constant $c > 0$ has been detected, so that

$$C \ge \beta(z_n)|g_n(\emptyset(z_n)) - g_n(\Psi(z_n))| = \beta(z_n)\frac{d(\emptyset(z_n), \Psi(z_n))}{\tilde{\alpha}(\emptyset(z_n))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\emptyset(z_n)|^2\right)^{\frac{2}{P}}}}} \ge n,$$

$\forall n \in \mathbb{N}$, which is objection. Obtaining of $sup_{z\in D}\frac{\beta(z)}{\alpha(\Psi(z))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\Psi(z)|^2\right)^{\frac{2}{P}}}}} < \infty$ analogously.

For the converse, by using Lemma 3. 3 and be able to conclude that there exists a constant $C_\alpha > 0$ so that

$$\|c_\emptyset - c_\Psi\| = sup_{z\in D}\beta(z)sup\big\{|f(\emptyset(z)) - f(\Psi(z))|; f \in B^p_\alpha, \|f\|^p_\alpha \le 1\big\}$$
$$\le sup_{z\in D}C_p max\left\{\frac{\beta(z)}{\alpha(\emptyset(z))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\emptyset(z)|^2\right)^{\frac{2}{P}}}}}, \frac{\beta(z)}{\alpha(\Psi(z))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\Psi(z)|^2\right)^{\frac{2}{P}}}}}\right\}$$
$$d(\emptyset(z), \Psi(z)) < \infty.$$

Therefore the difference is bounded. □

Theorem 4.2. *Let β be an arbitrary weight and α be a weight as describe in the previous section (i.e. $\alpha(z) := m\big(e^{-\frac{1}{\left(1-|z|^2\right)^{\frac{2}{p}}}}\big)$ for any $z \in D$) so that*

$$sup_{u\in D}sup_{z\in D}\frac{\alpha(z)\big|\alpha_u\big(\varphi_u(z)\big)\big|}{\alpha\big(\varphi_u(z)\big)} \le C < \infty$$

and α convince circumstance (L1). Furthermore, let $\emptyset$ and Ψ be analytic transformations of D with

$$max\big\{\|\emptyset\|_p, \|\Psi\|_p\big\} = 1.$$

Then the difference $C_\emptyset - C_\Psi : B^p_\alpha \to H^p_\beta$ is compact if and only if the following circumstances are convinced:

$$\lim sup_{|\emptyset(z)|\to 1}\frac{\beta(z)}{\alpha(\emptyset(z))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\emptyset(z)|^2\right)^{\frac{2}{P}}}}}d(\emptyset(z), \Psi(z)) = 0 \tag{4.1}$$

and

$$\lim sup_{|\Psi(z)|\to 1}\frac{\beta(z)}{\alpha(\Psi(z))^{\frac{1}{p}}e^{-\frac{1}{\left(1-|\Psi(z)|^2\right)^{\frac{2}{P}}}}}d(\emptyset(z), \Psi(z)) = 0 \tag{4.2}$$

Proof. Observe that under the given hypothesis α and $\tilde{\alpha}$ are equivalent. Initially, suppose that equations 4.1 and 4.2 hold. Let $(f_n)_n$ be a bounded sequence in B^p_α that converges to zero uniformly on compact subsets of D.

Let $N = sup_n \|f_n\|_\alpha^p < \infty$. Given $\varepsilon > 0$, there is $r > 0$ so that if $|\varphi(z)| \geq r$, so

$$\frac{\beta(z)}{\alpha(\emptyset(z))^{\frac{1}{p}} e^{-\frac{1}{\left(1-|\emptyset(z)|^2\right)^{\frac{2}{P}}}}} d(\emptyset(z), \Psi(z)) < \frac{\varepsilon}{3NC_\alpha}$$

and if $|\Psi(z)| \geq r$, further

$$\frac{\beta(z)}{\alpha(\Psi(z))^{\frac{1}{p}} e^{-\frac{1}{\left(1-|\Psi(z)|^2\right)^{\frac{2}{P}}}}} d(\emptyset(z), \Psi(z)) < \frac{\varepsilon}{3NC_\alpha}.$$

However, formerly $f_n \to 0$ uniformly on $\{\mu; |\mu| \leq r\}$ there is an $n_0 \in \mathbb{N}$ so that, if $|\emptyset(z)| \leq r$ and $n \geq n_0$, in that case $\left|f_n\big(\emptyset(z)\big)\right| < \frac{\varepsilon}{3M}$ and if $|\Psi(z)| \leq r$ and $n \geq n_0$, in that case $\left|f_n\big(\Psi(z)\big)\right| < \frac{\varepsilon}{3M}$, where $M = sup_{z\in D}\alpha(z)$. So, by applying Lemma 3. 3, setting $X := \{z \in D; |\emptyset(z)| \leq r\}$ and $Y := \{z \in D; |\Psi(z)| \leq r\}$. Therefore

$$\begin{aligned}
&sup_{z\in D}\beta(z)|C_\emptyset f_n(z) - C_\Psi f_n(z)| = sup_{z\in D}\beta(z)|f_n(\emptyset(z)) - f_n(\Psi(z))| \\
\leq& sup_{z\in X\cap Y}\beta(z)|f_n(\emptyset(z)) - f_n(\Psi(z))| \\
&+ sup_{z\in D\setminus(X\cap Y)}\beta(z)|f_n(\emptyset(z)) - f_n(\Psi(z))| \\
\leq& sup_{z\in D\setminus(X\cap Y)}\beta(z)|f_n(\emptyset(z)) - f_n(\Psi(z))| + sup_{z\in X}\beta(z)|f_n(\emptyset(z))| \\
&+ sup_{z\in Y}\beta(z)|f_n(\Psi(z))| \\
\leq& sup_{z\in D\setminus(X\cap Y)} max\left\{\frac{C_n\|f_n\|_{v,p}\beta(z)}{\alpha(\emptyset(z))^{\frac{1}{p}} e^{-\frac{1}{\left(1-|\emptyset(z)|^2\right)^{\frac{2}{P}}}}}, \frac{C_n\|f_n\|_{v,p}\beta(z)}{\alpha(\Psi(z))^{\frac{1}{p}} e^{-\frac{1}{\left(1-|\Psi(z)|^2\right)^{\frac{2}{P}}}}}\right\} \\
&d(\emptyset(z), \Psi(z)) + \frac{2\varepsilon}{3} \leq \varepsilon
\end{aligned}$$

for all $n \geq n_0$. Conversely, let $C_\varphi - C_\Psi : B_\alpha^p \to B_\beta^p$ is compact and equation 4.1 does not hold, so there are $\delta > 0$ and $(z_n)_n \subset D$ with $|\emptyset(z_n)| \to 1$ so that

$$\frac{\beta(z_n)}{\tilde{\alpha}(\emptyset(z_n))^{\frac{1}{p}} e^{-\frac{1}{\left(1-|\emptyset(z_n)|^2\right)^{\frac{2}{P}}}}} d(\emptyset(z_n), \Psi(z_n)) \geq \delta$$

for every n. Since $|\emptyset(z_n)| \to 1$, there exist natural numbers $a(n)$ and also $\lim_{n\to\infty} a(n) = \infty$ and so that $|\emptyset(z_n)|^{a(n)} \geq \frac{1}{2}$ for every n. So, for all $n \in \mathbb{N}$ by thinking out the function g_n

$$g_n(z) := f_n(z)\varphi'_{\emptyset(z_n)}(z)^{\frac{2}{p}} z^{a(n)},$$

where f_n is chosen like in the proof of Theorem 4.1 i.e. by choosing $f_n^p \in D_\alpha^p$ so that $|f_n(\emptyset(z_n))|^p = \frac{1}{\tilde{\alpha}\left(\emptyset(z_n)\right)}$. Further $(g_n)_n$ is norm bounded and $g_n \to 0$ pointwise as a result of the factor $z^{a(n)}$. Therefore, it follows that a subsequence of $((C_\emptyset - C_\Psi)g_n)_n$ goes to 0 in B_β^p. Nevertheless

$$\begin{aligned}\|(C_\emptyset - C_\Psi)g_n\|_\beta &\geq \beta(z_n)|(C_\emptyset - C_\Psi)g_n(z_n)| \\ &= \beta(z_n)|g_n(\emptyset(z_n)) - g_n(\Psi(z_n))| \\ &= \frac{\beta(z_n)|\emptyset(z_n)|^{a(n)}}{\left(1-|\emptyset(z_n)|^2\right)^{\frac{2}{p}}\alpha\left(\emptyset(z_n)\right)^{\frac{1}{p}}} d(\emptyset(z), \Psi(z)) \geq \frac{1}{2}\delta\end{aligned}$$

which is a contradiction.

Prove of circumstance equation 4.2 similarly. □

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