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DEGREE EXPONENT SUBTRACTION ENERGY

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ABSTRACT. The ordinary energy of a graph G is defined as the sum of the absolute eigenvalues of its adjacency matrix. In this paper we introduced a degree exponent subtraction matrix and we investigate its bounds for spectral radius of spectrum, partial sum of absolute eigenvalues and energy.

1. INTRODUCTION

The energy of a graph G is closely related with the total π -electron energy of molecules [2, 4]. This has motivated researchers to introduce different matrices associated with the graph and study their energies such as, Laplacian energy [1, 5], distance energy [7], degree sum energy [11], degree subtraction energy [10], degree exponent energy [9] etc. In this paper, we introduce degree exponent subtraction matrix and obtain its bounds for spectral radius, partial sum of absolute eigenvalues and energy.

Let G be a simple, finite, undirected, nontrivial graph of order n and size m. Let $V(G) = \{v_1, v_2, \ldots, v_j, \ldots, v_n\}$ be a vertex set. Let $d_j = deg_G(v_j)$ be the degree of a vertex v_j in G. Let eigenvalues of adjacency matrix [6] be $\lambda_1, \lambda_2, \cdots, \lambda_j, \cdots, \lambda_n$. Then the energy $\epsilon(G)$ of G is defined as $\epsilon(G) = \sum_{j=1}^{n} |\lambda_j|$ [3]. The adjacency eigenvalues of a complete graph K_n are $n-1$

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and -1 $(n-1)$times. The adjacency eigenvalues of a complete bipartite graph $K_{p,q}$ are $\sqrt{pq}$, 0 $(p+q-2)$times and $-\sqrt{pq}$.

The degree exponent subtraction matrix (DES) of a graph G is $n \times n$ matrix, defined as $DES(G) = [des_{jk}]$ where

$$des_{jk} = \begin{cases} d_j^{d_k} - d_k^{d_j} & j \neq k \\ 0 & \text{otherwise} \end{cases}.$$

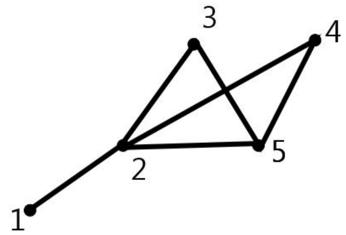
Characteristic polynomial of $DES(G)$ is defined as

$$\phi(G, y) = det\left(yI_n - DES(G)\right),$$

where I_n is unit matrix of order n. The roots of $\phi(G : y) = 0$ are called DES-eigenvalues which are labeled as $y_1, y_2, \ldots, y_j, \ldots, y_n$. The energy of degree exponent subtraction matrix of G is defined as

$$DESE(G) = \sum_{j=1}^{n} |y_j|.$$

Example 1.



$$DES(G) = \begin{bmatrix} 0 & -3 & -1 & -1 & -2 \\ 3 & 0 & 0 & 0 & -17 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 2 & 17 & 1 & 1 & 0 \end{bmatrix}$$

Graph and its DES matrix

Characteristic polynomial of above matrix is

$$\phi(G, y) = y^5 + 306y^3 + 392y$$

$$spec(DES(G)) = \begin{pmatrix} 0 & 17.4560i & -17.4560i & 1.1342i & -1.1342i \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

where $i = \sqrt{-1}$,

$$DESE(G) \approx 37.1804.$$

2. Preliminaries

Theorem 2.1. *Cauchy-Schwarz inequality [12] states that if* $(a_1, a_2, \dots, a_n)$ *and* $(b_1, b_2, \dots, b_n)$ *are n real vector, then*

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right). \tag{2.1}$$

Theorem 2.2. *Ozeki's inequality [8], if* a_j *and* b_j *are nonnegative real numbers, then*

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \frac{n^2}{4}\left(M_1 M_2 - m_1 m_2\right)^2 \tag{2.2}$$

where $M_1 = max_{1\leq j\leq n}(a_j)$, $M_2 = max_{1\leq j\leq n}(b_j)$, $m_1 = min_{1\leq j\leq n}(a_j)$, $m_2 = min_{1\leq j\leq n}(b_j)$.

Theorem 2.3. *Polya-Szego inequality [8], if* a_j *and* b_j *are non-negative real numbers, then*

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 \left(\sum_{j=1}^{n} a_j b_j\right)^2, \tag{2.3}$$

where M_1, M_2 *and* m_1, m_2 *are defined similarly to theorem (2.2).*

Theorem 2.4. *[8] If* a_j *and* b_j *are non-negative real numbers, then*

$$\left|n\sum_{j=1}^{n} a_j b_j - \sum_{j=1}^{n} a_j \sum_{j=1}^{n} b_j\right| \leq \alpha(n)(S-s)(T-t), \tag{2.4}$$

where s, t, S *and* T *are real constants such that* $s \leq a_j \leq S$ *and* $t \leq b_j \leq T$ *for each* $j, 1 \leq j \leq n$. $\alpha(n) = n\lfloor \frac{n}{2} \rfloor \left(1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor\right)$.

Theorem 2.5. *[8] If* a_j *and* b_j *are non-negative real numbers, then*

$$\sum_{j=1}^{n} b_j^2 + CD\sum_{j=1}^{n} a_j^2 \leq (C+D)\sum_{j=1}^{n} a_j b_j, \tag{2.5}$$

where C *and* D *are real constants such that* $Ca_j \leq b_j \leq Da_j$ *for each* $j = 1, 2, \dots, n$.

3. Bounds for spectral radius of DES(G)

Lemma 3.1. *If G is a graph with n vertices and m edges then the eigenvalues $y_j, 1 \leq j \leq n$, of the $DES(G)$ satisfy the following relations*

$$(i)\ \sum_{j=1}^{n} y_j = 0; \qquad (ii)\ \sum_{j=1}^{n} y_j^2 = -2P; \qquad (iii)\ \sum_{j=1}^{n} |y_j|^2 = 2P;$$

where $P = \sum_{1\leq j<k\leq n} \left({d_j}^{d_k} - {d_k}^{d_j}\right)^2$.

Proof. Since $\sum_{j=1}^{n} y_j = trace(DES(G)) = 0$,

$$\sum_{j=1}^{n} y_j^2 = trace(DES(G)^2) = -2 \sum_{1\leq j<k\leq n} \left({d_j}^{d_k} - {d_k}^{d_j}\right)^2 = -2P.$$

Next, having in mind that the eigenvalues y_j are purely imaginary or zeros, it follows that

$$\sum_{j=1}^{n} |y_j|^2 = 2 \sum_{1\leq j<k\leq n} \left({d_j}^{d_k} - {d_k}^{d_j}\right)^2 = 2P.$$

□

Lemma 3.2. *Let G be a graph with n vertices, m_1 edges. Let λ_j be adjacency eigenvalues of G such as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let H be another graph with n vertices, m_2 edges with vertex degree $d_j, j = 1, 2, \cdots, n$. Let y_j be $DES(H)$-eigenvalues of H which are given as $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$. Then*

$$\sum_{j=1}^{n} (\lambda_j |y_j|) \leq \sqrt{4m_1 P}.$$

Proof. Substitute $a_j = \lambda_j$ and $b_j = |y_j|$ in Theorem 2.1, we get

$$\left(\sum_{j=1}^{n} (\lambda_j |y_j|)\right)^2 \leq \left(\sum_{j=1}^{n} {\lambda_j}^2\right)\left(\sum_{j=1}^{n} |y_j|^2\right). \tag{3.1}$$

Since $\sum_{j=1}^{n} |y_j|^2 = 2P$ and $\sum_{j=1}^{n} {\lambda_j}^2 = 2m_1$. On substituting and simplifying equation (3.1) we get required result. □

Theorem 3.1. *Let G be a graph with n vertices and m edges with vertex degree $d_j, j = 1, 2, \cdots, n$. Let absolute DES-eigenvalues be $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$. Then*

$$|y_1| \leq \sqrt{\frac{2a}{a-1} P} + \frac{1}{a-1} \sum_{j=2}^{a} |y_{n-a+j}| \qquad (2 \leq a \leq n). \tag{3.2}$$

Proof. Let $|y_1|, |y_2|, \cdots, |y_{n-a+1}|, |y_{n-a+2}|, \cdots, |y_n|$ be the absolute DES-eigenvalues of G. Let $H = K_a \cup \overline{K_{n-a}}$. Then adjacency eigenvalues of H are $a-1$, $0\ (n-a\ times)$ and $-1\ (a-1\ times)$. The number of edges of H, $m_1 = \frac{a(a-1)}{2}$. By using Lemma 3.2, we get

$$\sum_{j=1}^{n}(\lambda_j|y_j|) \leq \sqrt{4m_1P}$$
$$(a-1)|y_1| - \sum_{j=n-a+2}^{n}(|y_j|) \leq \sqrt{2Pa(a-1)}$$

$$|y_1| \leq \sqrt{\frac{2Pa}{(a-1)}} + \frac{1}{a-1}\sum_{j=n-a+2}^{n}(|y_j|) \tag{3.3}$$

from equation (3.3) we get our required result (3.2). □

Remark 3.1. *Since $DES(G)$ is skew symmetric matrix therefore, $|y_1| = |y_2|$. Hence $|y_1|$,$|y_2|$ have the same upper bound.*

Corollary 3.1. *If G be a graph with n vertices, m edges having vertex degrees $d_1, d_2, \cdots, d_n$, then*

$$|y_1| \leq \sqrt{\frac{2P(n-1)}{n}} + \frac{DESE(G)}{n}. \tag{3.4}$$

Proof. Putting $a = n$ in above equation (3.2), we get

$$|y_1| \leq \sqrt{\frac{2nP}{n-1}} + \frac{1}{n-1}\sum_{j=2}^{n}|y_j|$$
$$\leq \sqrt{\frac{2nP}{n-1}} + \frac{1}{n-1}(DESE(G) - |y_1|).$$

On simplifying we get result (3.4). □

Remark 3.2. *The equality of (3.4) holds for regular graphs. As $P = 0$ So $|y_1| = 0$.*

4. Bounds for partial sum of absolute eigenvalue of $DES(G)$

Theorem 4.1. *If G is a graph with n vertices and m edges, with vertex degrees $d_1, d_2, \cdots, d_n$ and DES-eigenvalues $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$, then*

$$\sum_{j=1}^{k} |y_j| \leq \sqrt{\frac{2k(a-1)P}{a}} + \frac{DESE(G)}{a} \qquad 1 \leq k \leq n. \tag{4.1}$$

Proof. Let $|y_1|, |y_2|, \cdots, |y_k|, |y_{k+1}|, \cdots, |y_n|$ be the absolute DES-eigenvalues of G. Let H be the union of k copies of complete graph K_a, that is $H = \cup_k K_a$ where $ka = n$. The adjacency eigenvalues of H are $a-1$ $(k \ \ times)$, -1 $(n-k \ \ times)$. The number of vertices and edges of H are $n = ak$ and $\frac{ka(a-1)}{2}$ respectively. Using Lemma 3.2, we get

$$\begin{aligned}
(a-1)\sum_{j=1}^{k} |y_j| - \sum_{j=k+1}^{n} |y_j| &\leq \sqrt{\frac{4ka(a-1)P}{2}} \\
a\sum_{j=1}^{k} |y_j| - \sum_{j=1}^{n} |y_j| &\leq \sqrt{2ka(a-1)P} \\
a\sum_{j=1}^{k} |y_j| &\leq \sqrt{2ka(a-1)P} + DESE(G) \\
\sum_{j=1}^{k} |y_j| &\leq \sqrt{\frac{2k(a-1)P}{a}} + \frac{DESE(G)}{a}.
\end{aligned}$$

Thus, we obtain the bound for the sum of k absolute DES-eigenvalues of a G. If $k = 1$ we observe that the equation (4.1) gets reduced to equation (3.4). □

Theorem 4.2. *Let G be a graph with n vertices and m edges with vertex degree $d_j, j = 1, 2, \cdots, n$ and absolute DES-eigenvalues be $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$. Then*

$$\sum_{j=1}^{k} (|y_j| - |y_{n-k+j}|) \leq \sqrt{4kP}.$$

Proof. Let $|y_1|, |y_2|, \cdots, |y_k|, |y_{k+1}|, \cdots, |y_n|$ be the absolute DES-eigenvalues of G. Let H be the union of k copies of complete bipartite graph $K_{a,b}$, $H = \cup_k K_{a,b}$ where $n = ka$. Then adjacency eigenvalues of H are $\sqrt{ab}$ of multiplicity k, $zero$ of multiplicity $n-2k$ and $-\sqrt{ab}$ of multiplicity k. The number of edges of H is kab. By using Lemma 3.2 we get

$$\sqrt{ab}\sum_{j=1}^{k}|y_j| - \sqrt{ab}\sum_{j=n-k+1}^{n}|y_j| \leq \sqrt{4kabP}$$

$$\sqrt{ab}\sum_{j=1}^{k}|y_j| - \sqrt{ab}\sum_{j=1}^{k}|y_n - k + j| \leq \sqrt{4kabP}$$

$$\sum_{j=1}^{k}(|y_j| - |y_{n-k+j}|) \leq \sqrt{4kP}.$$

□

5. Bounds for Degree exponent subtraction energy

Theorem 5.1. *If G is a graph with n vertices, then*

$$\sqrt{2P} \leq DESE(G) \leq \sqrt{2nP}.$$

Proof. Putting $a_j = 1$ and $b_j = |y_j|$ in Theorem 2.1 we get

$$\left(\sum_{j=1}^{n}|y_j|\right)^2 \leq n\sum_{j=1}^{n}|y_j|^2$$

$$DESE(G)^2 \leq 2nP$$

$$DESE(G) \leq \sqrt{2nP},$$

which is the upper bound for $DES(G)$ and we know that

$$\left(\sum_{j=1}^{n}|y_j|\right)^2 \geq \sum_{j=1}^{n}|y_j|^2$$

$$DESE(G)^2 \geq 2P$$

$$DESE(G) \geq \sqrt{2P}.$$

Therefore, we get

$$\sqrt{2nP} \leq DESE(G) \geq \sqrt{2P}.$$

□

Theorem 5.2. *Let G be a simple connected graph and let $det(DES(G))$ be the absolute value of the determinant of degree exponent subtraction matrix. Then*

$$\sqrt{2P + n(n-1)(det(DES(G)))^{\frac{2}{n}}} \quad \leq \quad DESE(G) \quad \leq \quad \sqrt{2nP}.$$

Proof. We know that

$$\begin{aligned} \left(\sum_{j=1}^{n} |y_j|\right)^2 &= \sum_{j=1}^{n} |y_j|^2 + 2\sum_{j<k} |y_j||y_k| \\ DESE(G)^2 &= 2P + 2\sum_{j<k} |y_j||y_k| \end{aligned}$$

$$= 2P + \sum_{j \neq k} |y_j||y_k|. \tag{5.1}$$

We know that the arithmetic mean is always greater than or equal to the geometric mean.

$$\begin{aligned} \frac{1}{n(n-1)} \sum_{j \neq k} |y_j||y_k| &\geq \left(\prod_{j \neq k} |y_j||y_k|\right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{j \neq k} |y_j|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\ &= \left(\prod_{j \neq k} |y_j|^{\frac{2}{n}}\right) \\ &= (DESE(G))^{\frac{2}{n}} \end{aligned}$$

$$\sum_{j \neq k} |y_j||y_k| \quad \leq \quad n(n-1)(DESE(G))^{\frac{2}{n}} \tag{5.2}$$

from equation (5.1) and equation (5.2), we get

$$DESE(G) \quad \geq \quad \sqrt{2P + n(n-1)(DESE(G))^{\frac{2}{n}}}. \tag{5.3}$$

Let a non-negative quantity Z. Which is given as

$$\begin{aligned} Z &= \sum_{j=1}^{n}\sum_{j=1}^{n}(|y_j|-|y_k|)^2 \\ Z &= n\sum_{j=1}^{n}|y_j|^2+n\sum_{k=1}^{n}|y_k|^2-2\left(\sum_{j=1}^{n}|y_j|\right)\left(\sum_{k=1}^{n}|y_k|\right) \\ Z &= 4nP-2(DESE(G))^2. \end{aligned}$$

Since $Z \geq 0$,

$$DESE(G) \leq \sqrt{2nP}, \tag{5.4}$$

from equation (5.3) and equation (5.4) we get required result. □

Theorem 5.3. *If G is a graph of order n, Let $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$ are the absolute eigenvalues of $DES(G)$. Then,*

$$DESE(G) \geq \sqrt{2nP-\frac{n^2}{4}(|y_1|-|y_n|)^2},$$

where $|y_1|$ and $|y_n|$ are maximum and minimum of the absolute value of $y_j's$.

Proof. In inequality (2.2) let us take $a_j = 1$ and $b_j = |y_j|$, then we get

$$\begin{aligned} n\sum_{j=1}^{n}|y_j|^2-\left(\sum_{j=1}^{n}|y_j|\right)^2 &\leq \frac{n^2}{4}(|y_1|-|y_n|)^2 \\ 2nP-(DESE(G))^2 &\leq \frac{n^2}{4}(|y_1|-|y_n|)^2 \\ DESE(G) &\geq \sqrt{2nP-\frac{n^2}{4}(|y_1|-|y_n|)^2}. \end{aligned}$$

□

Corollary 5.1. *If G is graph with odd order, then*

$$DESE(G) \geq \sqrt{2nP-\frac{n^2}{4}(|y_1|)^2}.$$

Theorem 5.4. *If G be a graph of order n and $|y_1| \geq |y_2| \geq \cdots \geq |y_n| > 0$ are the absolute eigenvalues of $DES(G)$, then*

$$DESE(G) \geq \frac{2\sqrt{2nP|y_1||y_n|}}{|y_1|+|y_n|}.$$

Proof. Suppose $a_j = 1$ and $b_j = |y_j|$ then from inequality Theorem 2.3, we get

$$\sum_{j=1}^{n} 1^2 \sum_{j=1}^{n} |y_j|^2 \quad \leq \quad \frac{1}{4}\left(\sqrt{\frac{|y_1|}{|y_n|}}+\sqrt{\frac{|y_n|}{|y_1|}}\right)^2 \left(\sum_{j=1}^{n} |y_j|\right)^2. \tag{5.5}$$

Since $\sum_{j=1}^{n} 1^2 = n, \sum_{j=1}^{n} |y_j|^2 = 2P$ and $\sum_{j=1}^{n} |y_j| = DESE(G)$. On simplifying equation (5.5) we get required result. □

Theorem 5.5. *Let G be a graph of order n. Let $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$ are the absolute eigenvalues of $DES(G)$. Then*

$$DESE(G) \geq \sqrt{2nP - \alpha(n)\left(|y_1| - |y_n|\right)^2},$$

where $\alpha(n) = n\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right)$.

Proof. Suppose $a_j = b_j = |y_j|, s = t = |y_n|, S = T = |y_1|$ then from inequality Theorem 2.4, we get

$$\left| n\sum_{j=1}^{n} |y_j|^2 - \left(\sum_{j=1}^{n} |y_j| \sum_{j=1}^{n} |y_j|\right)\right| \quad \leq \quad \alpha(n)\left(|y_1| - |y_n|\right)^2. \tag{5.6}$$

Since $\sum_{j=1}^{n} |y_j|^2 = 2P$ and $\sum_{j=1}^{n} |y_j| = DESE(G)$. On simplifying equation (5.6) we get required result. □

Corollary 5.2. *If G is graph with odd order, then*

$$DESE(G) \geq \sqrt{2nP - \alpha(n)\left(|y_1|\right)^2}.$$

Remark 5.1. *Since $\alpha(n) \leq \frac{n^2}{4}$, the lower bound Theorem 2.2 is more sharper than the lower bound Theorem 2.4.*

Theorem 5.6. *Let G be a graph of order n. Let $|y_1| \geq |y_2| \geq \cdots \geq |y_n|$ are the absolute eigenvalues of $DES(G)$.Then*

$$DESE(G) \geq \frac{|y_1||y_n|n + 2P}{|y_1| + |y_n|}.$$

Proof. Substituting $a_j = 1, b_j = |y_j|$, $C = |y_n|, D = |y_1|$ then from inequality (2.5), we get

$$\sum_{j=1}^{n} |y_j|^2 + |y_n||y_1| \sum_{j=1}^{n} 1^2 \quad \leq \quad \left(|y_n| + |y_1|\right)\left(\sum_{j=1}^{n} |y_j|\right). \tag{5.7}$$

Since $\sum_{j=1}^{n} |y_j|^2 = 2P$ and $\sum_{j=1}^{n} |y_j| = DESE(G)$. On simplifying equation (5.7) we get required result. □

6. CONCLUSION

In this paper we have introduced a new matrix called degree exponent subtraction in which we found bounds for spectral radius of spectrum and partial sum of absolute eigenvalues of $DES(G)$, Here we got sharper lower bound of $DESE(G)$ even though we have skew-symmetric matrix.

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