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SOME SPECIAL CHARACTERSTICS OF LATTICE ORDERED COMMUTATIVE LOOPS

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ABSTRACT. In this paper, we have shown that a commutative l-group resembles like a commutative l-group, and most of the features of l-groups are retained by the l-loop. After developing some of the relevant properties of an l-loop, we have characterised its positive cone, obtained a necessary and sufficient condition for an l-loop to become an l-group and for it to be totally ordered.

1. INTRODUCTION

In 1967, G. Birkhoff in Lattice Theory, various properties of lattice ordered groups were established. In 1970, T. Evans described about lattice ordered loops and quasigroups. In 1990, Hala made a description on quasigroups and loops [1-3]. In view of this a lot of interest has been shown different authors develop these concepts in different algebraic systems [4-13]. In 2019, B.Sailaja, V.B.V.N. Prasad, developed exploring the axiom of excluded middle and axiom of contradiction in fuzzy sets. In 2020, R. Sunil Kumar and V.B.V.N. Prasad were gave some special characteristics of Atoms in Lattice ordered loops and in 2020, V.B.V.N.Prasad, T. Rama Rao and some authors were gave Some Basic Principles on Posets, Hasse diagrams and lattices [14-16].

In this manuscript, we have shown that a commutative l-group resembles like a commutative l-group, and most of the features of l-groups are retained by the

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l-loop. After developing some of the relevant properties of an l-loop, we have characterised its positive cone, obtained a necessary and sufficient condition for an l-loop to become an l-group and for it to be totally ordered.

2. Loops

Definition 2.1. *A commutative loop is a system A=(A, +, -) such that*

(i) *(A, +) is a commutative groupoid with identity element denoted by '0',*

(ii) *For each pair $a, b \in A$, there exists a unique $x \in A$ such that $x + b = a$ and we denote this x by $a - b$.*

Corollary 2.1. *In a commutative loop the cancellation laws hold.*

Corollary 2.2.

(i) $\forall a \in A$, $a - a = 0$, $a - 0 = a$ *and* $(a + a) + a = a + (a + a)$,

(ii) $\forall a, b \in A$, $(a + b) + a = a + (b + a)$,

(iii) $\forall a, b \in A$, $(a + b) - a = a + (b - a) = a - (a - b) = b$.

Lemma 2.1. *Any commutative loop A can be equationally defined as an algebra with binary operations + and -, satisfying condition (i) of definition 1 and condition (iii) of Corollary 2.2.*

Lemma 2.2. *A subset S of a commutative loop A is a sub-loop if and only if $\forall x, y \in S$, $x + y \in S$ and $x - y \in S$*

The proof of Lemma 2.1 is easy and a part of literature. The proof of Lemma 2.2 follows from Lemma 2.1. Throughout this paper a loop means a commutative loop, and very often we write $0 - a$ as $(-a)$.

3. Partially Ordered Loops

Definition 3.1. *A system $A = (A, +, -, \leq)$ is called a partially ordered loop or briefly a po-loop if*

(i) $(A, +, -)$ *is a loop,*

(ii) $(A, \leq)$ *is a poset in which every loop translation $x \to a + x$ is an order-automorphism.*

Lemma 3.1. *In any po-loop $A, \forall a, b, c \in A, a \leq b \Leftrightarrow a - c \leq b - c$ and also $c - b \leq c - a$ and hence $-b \leq -a, a - b \leq 0$.*

Proof. $a \leq b \Leftrightarrow c+(a-c) \leq c+(b-c) \Leftrightarrow a-c \leq b-c$ and $a \leq b \Leftrightarrow a+(c-a) \leq b+(c-a) \Leftrightarrow b+(c-b) \leq b+(c-a) \Leftrightarrow c-b \leq c-a$. Hence the proof. □

Lemma 3.2. *Except in the trivial case of $A = \{0\}$, a po-loop A cannot have universal bounds.*

Proof. Routine □

Remark 3.1. *Let A be a po-loop. The set $P = \{x \in A/x \geq 0\}$ is referred to as the positive cone of A. Also we write $-P = \{x \in A/0 - x \in P\}$.*

Lemma 3.3. *In a po-loop A with positive cone $P, \forall a, x \in A$,*
(i) $x \in P, x + P = P \Rightarrow x = 0$,
(ii) $a + (x + P) \subseteq (a + x) + P$,
(iii) $(x + P) - a \subseteq (x - a) + P$.

Proof.

(i) $x + P = P \Rightarrow -x \in P \Rightarrow x \in -P \Rightarrow x = 0$.
(ii) For $p \in P, a + (x + p) \geq a + x \Rightarrow [a + (x + p)] - (a + x) \in P$ and the proof follows.
(iii) For $p \in P, (x + p) - a \geq x - a \Rightarrow [(x + p) - a] - (x - a) \in P$ and the proof follows.

□

Theorem 3.1. *In po-loop A, the positive cone P has the following properties for all $a, x, b \in A$:*
(i) $P \cap -P = 0$,
(ii) $(a + P) + P = a + P$,
(iii) $a + (x + P) = (a + x) + P$,
(iv) $(x + P) - a = (x - a) + P$,
(v) $a \leq b \quad iff \quad b - a \in P$.
Conversely, if A is a loop and $P \subset A$ satisfying (i) to (iv), then by defining $a \geq b$ iff $(b - a) \in P$, A becomes po-loop in which P is precisely positive cone.

Proof.
(i) See remark to lemma 3.2.
(ii) Clearly $a+P \subseteq (a+P)+P$. For $p, q \in P, [(a+p)+q] \geq a \Rightarrow [(a+p)+q]-a \in P$

(iii) In view of Lemma 3.3(ii), it suffices to show $(a+x)+P \subseteq a+(x+P)$. Now $[(a+x)+P]-a \subseteq [(a+x)-a]+P = x+P$, by using Lemma 3.3(iii).

(iv) $[(x-a)+P]+a \subseteq [(x-a)+a]+P$ by Lemma (3.3) (ii) $= x+P$ and the proof follows by Lemma (3.3)(iii).

(v) Follows from Lemma (3.1). Conversely let P satisfies (i) to (iv).

Then clearly the condition (ii) leads to $a \leq b \Leftrightarrow b+P \subseteq a+P$, which together with Lemma (3.3) (i) show that '$\leq$' is a partial order relation. Clearly the map $x \to x+a$ is a bijection.

Now $x \leq y$ iff $y-x \in P$ iff $y \in x+P$ iff $y+a \in (x+P)+a = (x+a)+P$ iff $x+a \leq y+a$.

This completes the proof. □

Lemma 3.4. *In any po-loop A, the positive cone P is invariant under all inner automorphisms, namely,* $x \to (-a+x)+a$ *and* $x \to (-a)+(x+a)$.

Proof. follows easily from the definition of a po-loop. □

Lemma 3.5. *A po-loop A is directed if and only if every element is expressed as the difference of two positive elements in the sense that* $x \in A \Rightarrow x = a-b$*, where* a *and* b *are two positive elements in* A.

Proof. If A is directed, for $a \in A$, there exists $0 \leq c \in A$ such that $a \leq c$. Then $a = c-(c-a)$ gives the required representation. Conversely if $a, b \in A$, then $a = a' - a''$, $b = b' - b''$ where $0 \leq a'$, $a'', b', b'' \in A$ and $c = a' + b' \Rightarrow a, b, c \in A$. This completes the proof. □

Corollary 3.1. *Any directed loop is bidirected.*

Theorem 3.2. *In a commutative po-loop A, the mapping* $x \to a-x$ *is a dual order-automorphism for any* $a \in A$.

Proof. Since a-x exists $\forall a$, $x \in A$ and $x = y \Leftrightarrow a-x = a-y$, the mapping $x \to a-x$ is a bijection. Further $x \leq y \Leftrightarrow x+(a-x) \leq y+(a-x) \Leftrightarrow a \leq y+(a-x) \Leftrightarrow y+(a-y) \leq y+(a-x) \Leftrightarrow a-y \leq a-x$. Hence the proof. □

4. Lattice Ordered Loops

Definition 4.1. *In a po-loop* $A = (A, +, -, \leq)$*, if* $(A, \leq)$ *is a lattice, then we call the system* $A = (A, +, -, \leq)$ *a lattice ordered loop or briefly an l-loop.*

Corollary 4.1. *In any l-loop* $A, \forall a, b, c \in A$,

(i) $a + (b \vee c) = (a + b) \vee (a + c)$
$a + (b \wedge c) = (a + b) \wedge (a + c)$
(ii) $a - (b \vee c) = (a - b) \wedge (a - c)$
$a - (b \wedge c) = (a - b) \vee (a - c)$
(iii) $(a \vee b) - c = (a - c) \vee (b - c)$
$(a \wedge b) - c = (a - c) \wedge (b - c)$

Proof.
(i) is a part of literature.
(ii) Follows from theorem 3.2.
(iii) $c + [(a \vee b) - c] = a \vee b = [c + (a - c)] \vee [c + (b - c)] = c + [(a - c) \vee (b - c)] \Rightarrow (a \vee b) - c = (a - c) \vee (b - c)$. Similarly $(a \wedge b) - c = (a - c) \wedge (b - c)$. □

Theorem 4.1. *A po-loop A is an l-loop if and only if* $a \vee 0$ *exists* $\forall a \in A$.

Proof. Necessity is trivial. To prove the converse it is enough to observe that $(a - b) \vee 0 + b$ acts as $a \vee b$. The existence of $a \wedge b$ now follows from Corollary 4.1 (ii).

It is well known that in an l-loop the lattice becomes automatically distributive. We mention below some more results that hold in l-loops. □

Lemma 4.1. *In any l-loop* A, *for all* $a, b, c \in A$,

(i) $(a - b) \vee 0 + a \wedge b = a$ *and as such* $a \vee 0 + a \wedge 0 = a$.
(ii) $(a - b) \vee (b - a) + a \wedge b = a \vee b$.
(iii) $(a - b) \wedge (b - a) = a \wedge b - a \vee b$.
(iv) $a \wedge b + a \vee b = a + b$.

Proof.
(i) $(a - b) \vee 0 + a \wedge b = [a - a \wedge b] + a \wedge b = a$ (by Corollary 4.1(i)).

(ii) $(a \vee b) - (a \wedge b) = [(a \vee b) - a] \vee [(a \wedge b) - b] = (a - b) \vee (b - a) \vee 0$ and $a \wedge b + [(a - b) \vee (b - a)] = [a + \{(a - b) \vee (b - a)\}] \wedge [b + (a - b) \vee (b - a)] \geq [a + (b - a)] \wedge [b + (a - b)] = a \wedge b \Rightarrow (a - b) \vee (b - a) \geq 0$. This completes the proof of (ii).

(iii) Similar to(ii).

(iv) Found in the literature. □

Lemma 4.2. *In an l-loop A, if* $b \wedge c = 0$, *then* $(b - c) \vee 0 = b$ *and* $(b - c) \wedge 0 = -c$

Proof. Follows from Corollary 4.1(ii). □

Corollary 4.2. *In an l-loop A, for* $x, y \in A$*,* $x \wedge y = 0 \Rightarrow x - y = x + (0 - y)$.

Proof. $x - y = (x - y) \vee 0 + (x - y) \wedge 0 = x + (0 - y)$. □

We write $|x| = x \vee (-x)$. Then it is easy to prove the following:

Lemma 4.3. *In any l-loop* A*, for all* $a, x \in A, -a \leq x \leq a \Leftrightarrow |x| \leq a$.

Lemma 4.4. *In any l-loop* A*, for all* $a, x, y \geq 0$*,*
(i) $a \wedge (x + y) \leq a \wedge x + a \wedge y$*;*
(ii) $a \vee (x + y) \leq a \vee x + a \vee y$.

Proof.
(i) $\{a \wedge (x + y)\} - (a \wedge x) = [\{a \wedge (x + y)\} - a] \vee [\{a \wedge (x + y)\} - x]$
$= [\{0 \wedge (x + y)\} - a] \vee [(a - x) \wedge \{(x + y) - x\}]$
$\leq 0 \vee [(a - x) \wedge y] \leq 0 \vee (a \wedge y) = (0 \vee a) \wedge (0 \vee y) = a \wedge y$
$\Rightarrow a \wedge (x + y) \leq a \wedge x + a \wedge y$.
(ii) Follows from the definition of an l-loop. □

Theorem 4.2. *An l-loop A is an l-group if and only if* $a - c \leq (a - b) + (b - c)$ *for all* $a, b, c \in A$.

Proof. Necessity is trivial. Conversely by hypothesis we have:

$$[a + (b + c)] - c \leq [\{a + (b + c)\} - (b + c)] + [(b + c) - c] = a + b.$$

So, $a+(b+c) \leq (a+b)+c$, and $\{(a+b)+c\}-a \leq [\{(a+b)+c\}-(a+b)]+[(a+b)-a] = b + c$.

In a similar way $(a + b) + c \leq a + (b + c)$.

This completing the proof. □

Theorem 4.3. *Let A be an l-loop. A is totally ordered if and only if* $a, b > 0, a \wedge b \neq 0$.

Proof. If A is totally ordered and $a, b > 0$, then clearly $a \wedge b \neq 0$. Conversely let $\forall a, b > 0, a \wedge b \neq 0$.

If A is not totally ordered,$\exists$ atleast one pair $x, y \in A$,such that $x \| y$,that is $(x - y) \vee 0 \neq 0$ and $(y - x) \vee 0 \neq 0$.

Taking $a = (x - y) \vee 0 > 0$ and $b = (y - x) \vee 0 > 0$, We get $a \wedge b = [(x-y) \vee 0] \wedge [(y-x) \vee 0] = [(x-y) \wedge (y-x)] \vee 0 = 0$ since $(x-y) \wedge (y-x) \leq 0 \; \forall x, y \in A$ by Lemma 4.1(iii).

But this contradicts the hypothesis. Hence A is totally ordered. □

Now we state the following results without proof:

Lemma 4.5. *If the Algebra $(A, +, -, \vee)$ is a loop with + and – and a join semi lattice with $\vee$, and if $a + (b \vee c) = (a + b) \vee (a + c)$, then $(A, +, -, \vee)$ is an l-loop.*

Lemma 4.6. *Every l-loop is isomorphic with an l-sub loop of an unrestricted direct product of sub directly irreducible l-loops.*

In a commutative loop we can write $na = a + a + a....... + a$, n times, where n is a positive integer and we define $(-n)a = n(-a)$.

Lemma 4.7. *Every l-loop is semi closed in the sense that $n \in Z^+, na \geq 0 \Leftrightarrow a \geq 0$.*

Corollary 4.3. $na \leq 0 \Rightarrow a \leq 0$.

Corollary 4.4. $na = 0 \Leftrightarrow a = 0$.

Corollary 4.5. *Any l-loop is torsion free in the sense that every element except zero has infinite order.*

Definition 4.2. *A po-loop is called integrally closed if and only if $na \leq b \; \forall n \in N \Rightarrow a \leq 0$. Then we have,*

Lemma 4.8. *A sub loop of a (conditionally) complete l-loop is integrally closed.*

5. Conclusion

In this manuscript, we have shown that a commutative l-group resembles like a commutative l-group, and most of the features of l-groups are retained by the l-loop. After developing some of the relevant properties of an l-loop, we have characterised its positive cone, obtained a necessary and sufficient condition for an l-loop to become an l-group and for it to be totally ordered and these concepts may be applied for different algebraic structures in future.

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