Advances in Mathematics: Scientific Journal **9** (2020), no.11, 9179–9186
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.11.23 Spec. Iss. on ICMMSIS-2020

STABILITY ANALYSIS IN BANACH SPACES FOR A GENERAL ADDITIVE FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we present the Hyers-Ulam stability of generalized additive functional equation in Banach spaces and stability results have been obtained by a classical direct method by various general control functions.

1. INTRODUCTION

The origin of stability for functional equation is related with a question of Ulam in 1940 [12] concerning the stablity of establishment homomorphisms and certifiably answered for Banach spaces through Hyers [5]. In this way, the aftereffect of Hyers got generalized by Aoki [1] for additive mappings. The paper of T. Aoki has given a ton of impact inside the advancement of generalized Hyers-Ulam stability of functional equations. During the past eight decades, the annoyances inconveniences of a few utilitarian conditions had been broadly explored by various creators [2–4, 9, 11]. The wording generalized Ulam - Hyers stability starts from these authentic foundations.

These wordings are likewise actualized to the envelope of other functional equations and it has been broadly researched by some of creators and there are innumerable colossal results bearing on this difficulty by and large with different additive functional equations (see [6, 7]) and references cited there in and also

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2020 *Mathematics Subject Classification.* 39B52, 39B82.

Key words and phrases. Additive functional equation, generalized Ulam - Hyers stability, Banach spaces.

Second auther [8, 10] built up the Hyers-Ulam stability of linear and nonlinear mappings.

In this paper, we introduce generalized additive functional equation and estabilish the solution, generalized Ulam - Hyers stability of various general control functions of this equation

$$\begin{aligned} & h\left(K^{\mathcal{M}+\mathcal{N}}x+K^{\mathcal{L}}y\right)+h\left(K^{\mathcal{M}+\mathcal{N}}y+K^{\mathcal{L}}z\right)+K^{\mathcal{M}+\mathcal{N}}h(x-y)+K^{\mathcal{L}}h(y-z) \\ & \quad =2\left(K^{\mathcal{M}+\mathcal{N}}h(x)+K^{\mathcal{L}}h(y)\right) \end{aligned} \tag{1.1}$$

with $K^{\mathcal{M}+\mathcal{N}}, K^{\mathcal{L}} \neq 0$ in Banach spaces using direct method.

2. General Solution

All through this segment let us take (X, Y) be real vector spaces. The confirmation of the accompanying lemma are inconsequential, consequently the subtleties of the verification is precluded.

Lemma 2.1. *Let $h : X \to Y$ be an odd mapping satisfies the FE*

$$h(x+y)=h(x)+h(y), \tag{2.1}$$

if $h : X \to Y$ satisfies the FE (1.1) $\forall x, y, z \in X$.

Proof. An odd function $h : X \to Y$ satisfy the FE (2.1). Let $x = y = 0$ in (2.1), we obtain $f(0) = 0$. Interchanging x by $-y$ in (2.1), we reach $h(-y) = -h(y)$ $\forall y \in X$. By Replacing y by x in (2.1), we arrive

$$h(2x)=2h(x),$$

for all $x \in X$. By induction of n, we arrive

$$h(nx)=n\,h(x). \tag{2.2}$$

Taking $x = K^{\mathcal{M}+\mathcal{N}}x, y = K^{\mathcal{L}}y$ in (2.1) and using (2.2), we get

$$h\left(K^{\mathcal{M}+\mathcal{N}}x+K^{\mathcal{L}}y\right)=K^{\mathcal{M}+\mathcal{N}}h(x)+K^{\mathcal{L}}h(y). \tag{2.3}$$

Taking $y = K^{\mathcal{M}+\mathcal{N}}y, z = K^{\mathcal{L}}z$ in (2.1) and using (2.2), we arrive

$$h\left(K^{\mathcal{M}+\mathcal{N}}y+K^{\mathcal{L}}z\right)=K^{\mathcal{M}+\mathcal{N}}h(y)+K^{\mathcal{L}}h(z), \tag{2.4}$$

$\forall x, z \in X$. Consider y by $-y$ in (2.1), and using $h(-y) = -h(y)$, we have

$$h(x - y) = h(x) - h(y), \tag{2.5}$$

$\forall x, y \in X$. Multiply both side by $K^{\mathcal{M}+\mathcal{N}}$ in (2.5), we arrive

$$K^{\mathcal{M}+\mathcal{N}}h(x - y) = K^{\mathcal{M}+\mathcal{N}}h(x) - K^{\mathcal{M}+\mathcal{N}}h(y), \tag{2.6}$$

$\forall x, y \in X$. let $x = y, y = -z$ in (2.1), and using $h(-y) = -h(y)$, we arrive

$$h(y - z) = h(y) - h(z), \tag{2.7}$$

$\forall y, z \in X$. Both side multiply by $K^{\mathcal{L}}$ in (2.7), we get

$$K^{\mathcal{L}}h(y - z) = K^{\mathcal{L}}h(y) - K^{\mathcal{L}}h(z), \tag{2.8}$$

$\forall y, z \in X$. Adding (2.3), (2.4), (2.6) and (2.8), we reach (1.1). □

3. Stability Results

Let us take $\mathbb{H}^*$ normed linear space and $\mathbb{I}^*$ Banach space. Define

$$\begin{aligned} D_{\mathcal{L}}^{\mathcal{M}+\mathcal{N}}h(x, y, z) &= h\left(K^{\mathcal{M}+\mathcal{N}}x + K^{\mathcal{L}}y\right) + h\left(K^{\mathcal{M}+\mathcal{N}}y + K^{\mathcal{L}}z\right) \\ &\quad + K^{\mathcal{M}+\mathcal{N}}h(x - y) + K^{\mathcal{L}}h(y - z) - 2\left(K^{\mathcal{M}+\mathcal{N}}h(x) + K^{\mathcal{L}}h(y)\right), \end{aligned}$$

where $K^{\mathcal{M}+\mathcal{N}}, K^{\mathcal{L}} \neq 0 \ \forall x, y, z \in X$.

Theorem 3.1. *Suppose that $h : \mathbb{H}^* \to \mathbb{I}^*$ be a function satisfying the inequality*

$$\left\| D_{\mathcal{L}}^{\mathcal{M}+\mathcal{N}}h\left(x, y, z\right) \right\| \leq \mathfrak{H}\left(x, y, z\right), \tag{3.1}$$

$\forall x, y, z \in \mathbb{H}^*$ *and Let $\mathfrak{H} : \mathbb{H}^* \times \mathbb{H}^* \times \mathbb{H}^* \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\mathfrak{H}\left(\chi^n x, \chi^n y, \chi^n z\right)}{\chi^n} = 0,$$

$\forall x, y, z \in \mathbb{H}^*$. *Then, there occurs a distinct additive function $A : \mathbb{H}^* \to \mathbb{I}^*$ and satisfying the FE (1.1) such that*

$$\|A_t(x) - h(x)\| \leq \frac{1}{2} \sum_{s=1}^{\infty} \frac{\mathfrak{H}(\chi^{(s-1)}x, \chi^{(s-1)}x, \chi^{(s-1)}x)}{\chi^s},$$

$\forall x \in \mathbb{H}^*$. *If $A_t(x)$ is the additive mapping defined as*

$$A_t(x) = \lim_{n\to\infty} \frac{h(\chi^n x)}{\chi^n},$$

$\forall x \in \mathbb{H}^*$ *where $\chi = \left(K^{\mathcal{M}+\mathcal{N}} + K^{\mathcal{L}}\right)$.*

Proof. Considering (x, y, z) by $(0, 0, 0)$ in (3.1), then we have

$$\left\| \left(2 - K^{\mathcal{M}+\mathcal{N}} - K^{\mathcal{L}}\right) h(x) \right\| = 0$$

or $h(x) = 0$. Switching $x = y = z = x$ in (3.1), we get

$$\|h(\chi x) - \chi h(x)\| \leq \frac{1}{2} \mathfrak{H}(x, x, x) \tag{3.2}$$

$\forall x \in \mathbb{H}^*$, where $\chi = \left(K^{\mathcal{M}+\mathcal{N}} + K^{\mathcal{L}}\right)$. We replace x by $\chi^{s-1}x$ (for $s \in \mathbb{N}$ and $s \geq 1$) in (3.2) , and we obtain

$$\left\| h(\chi^s x) - \chi h(\chi^{s-1}x) \right\| \leq \frac{1}{2} \mathfrak{H}\left(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x\right)$$

$\forall x \in \mathbb{H}^*$. Multiplying both sides of the above inequality by $\dfrac{1}{\chi^s}$ and we have the result by adding n inequalities then

$$\sum_{s=1}^{n} \frac{1}{\chi^s} \left\| h(\chi^s x) - \chi h(\chi^{s-1}x) \right\| \leq \frac{1}{2} \sum_{s=1}^{n} \frac{\mathfrak{H}(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x)}{\chi^s}.$$

Utilizing the triangle inequality

$$|a + b| \leq |a| + |b|,$$

and we reach left side of the inequality after simpliying

$$\left\| \frac{1}{\chi^n} h(\chi^n x) - h(x) \right\| \leq \frac{1}{2} \sum_{s=1}^{n} \frac{\mathfrak{H}(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x)}{\chi^s}. \tag{3.3}$$

Since

$$\sum_{s=1}^{n} \frac{\mathfrak{H}(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x)}{\chi^s} \leq \sum_{s=1}^{\infty} \frac{\mathfrak{H}(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x)}{\chi^s},$$

the inequality (3.3) yields

$$\left\| \frac{1}{\chi^n} h(\chi^n x) - h(x) \right\| \leq \frac{1}{2} \sum_{s=1}^{\infty} \frac{\mathfrak{H}(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x)}{\chi^s}$$

$\forall x \in \mathbb{H}^*$. By induction it will be demonstrated that (3.3) is exist $\forall$ $\mathbb{N}$. If $m > n > 0$, then $m - n \in \mathbb{N}$ and supplanting n by $m - n$ in (3.3), we have

$$\left\| \frac{1}{\chi^{m-n}} h\left(\chi^{m-n}x\right) - h(x) \right\| \leq \frac{1}{2} \sum_{s=1}^{\infty} \frac{\mathfrak{H}(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x)}{\chi^s},$$

which is

$$\left\|\frac{1}{\chi^m}h\left(\chi^{m-n}x\right)-\frac{1}{\chi^n}h(x)\right\|\leq\frac{1}{2\chi^n}\sum_{s=1}^{\infty}\frac{\mathfrak{H}\left(\chi^{s-1}x,\chi^{s-1}x,\chi^{s-1}x\right)}{\chi^s}\tag{3.4}$$

$\forall x\in\mathbb{H}^*$. Interchanging x by $\chi^n x$ in (3.4), we obtain

$$\left\|\frac{1}{\chi^m}h\left(\chi^{m}x\right)-\frac{1}{\chi^n}h(\chi^n x)\right\|\leq\frac{1}{2\chi^n}\sum_{s=1}^{\infty}\frac{\mathfrak{H}\left(\chi^{s+n-1}x,\chi^{s+n-1}x,\chi^{s+n-1}x\right)}{\chi^s}.\tag{3.5}$$

Since

$$\lim_{n\to\infty}\frac{1}{2\chi^n}=0,$$

and hence from (3.5), we obtain

$$\lim_{n\to\infty}\left\|\frac{1}{\chi^m}h\left(\chi^{m}x\right)-\frac{1}{\chi^n}h(\chi^n x)\right\|=0.$$

Therefore

$$\left\{\frac{h(\chi^n x)}{\chi^n}\right\}_{n=1}^{\infty}$$

is Cauchy sequence. Then the sequence has a limit in $\mathbb{H}^*$. We Define

$$A_t(x)=\lim_{n\to\infty}\frac{h(\chi^n x)}{\chi^n}$$

$\forall x\in\mathbb{H}^*$. First we showthat $A:\mathbb{H}^*\to\mathbb{H}^*$ is additive. Consider

$$\begin{aligned}
&\Big\|A_t\left(K^{\mathcal{M}+\mathcal{N}}x+K^{\mathcal{L}}y\right)+A_t\left(K^{\mathcal{M}+\mathcal{N}}y+K^{\mathcal{L}}z\right)+K^{\mathcal{M}+\mathcal{N}}A_t(x-y)\\
&\qquad\qquad+K^{\mathcal{L}}A_t(y-z)-2\left(K^{\mathcal{M}+\mathcal{N}}A_t(x)+K^{\mathcal{L}}A_t(y)\right)\Big\|\\
&=\frac{1}{\chi^n}\Big\|h\left(\chi^nK^{\mathcal{M}+\mathcal{N}}x+\chi^nK^{\mathcal{L}}y\right)+h\left(\chi^nK^{\mathcal{M}+\mathcal{N}}y+\chi^nK^{\mathcal{L}}z\right)\\
&\qquad+K^{\mathcal{M}+\mathcal{N}}h(\chi^nx-\chi^ny)+K^{\mathcal{L}}h(\chi^ny-\chi^nz)\\
&\qquad\qquad-2\left(K^{\mathcal{M}+\mathcal{N}}h(\chi^nx)+K^{\mathcal{L}}h(\chi^ny)\right)\Big\|\\
&\leq\lim_{n\to\infty}\frac{1}{\chi^n}\mathfrak{H}\left(\chi^nx,\chi^ny,\chi^nz\right)=0.
\end{aligned}$$

Hence,

$$\begin{aligned}
A\left(K^{\mathcal{M}+\mathcal{N}}x+K^{\mathcal{L}}y\right)+A\left(K^{\mathcal{M}+\mathcal{N}}y+K^{\mathcal{L}}z\right)+K^{\mathcal{M}+\mathcal{N}}A(x-y)+K^{\mathcal{L}}A(y-z)\\
=2\left(K^{\mathcal{M}+\mathcal{N}}A_t(x)+K^{\mathcal{L}}A(y)\right)
\end{aligned}$$

$\forall x \in \mathbb{H}^*$. Next, we consider

$$\begin{aligned}||A_t(x) - h(x)|| &= ||\lim_{n\to\infty} \frac{h(\chi^n x)}{\chi^n} - h(x)|| \\ &= \lim_{n\to\infty} ||\frac{h(\chi^n x)}{\chi^n} - h(x)|| \\ &\leq \lim_{n\to\infty} \frac{1}{2} \sum_{s=1}^{\infty} \frac{\mathfrak{H}\left(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x\right)}{\chi^s}.\end{aligned}$$

Hence, we get

$$||A_t(x) - h(x)|| \leq \frac{1}{2} \sum_{s=1}^{\infty} \frac{\mathfrak{H}\left(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x\right)}{\chi^s},$$

for all $x \in \mathbb{H}^*$.

Next, we show A_t is unique. Then there occur another mapping $B_t : \mathbb{H}^* \to \mathbb{H}^*$ we have

$$||B_t(x) - h(x)|| \leq \frac{1}{2} \sum_{s=1}^{\infty} \frac{\mathfrak{H}\left(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x\right)}{\chi^s}.$$

Hence

$$\begin{aligned}||B_t(x) - A_t(x)|| &\leq ||B_t(x) - h(x)|| + ||A_t(x) - h(x)|| \\ &\leq \frac{1}{2} \sum_{s=1}^{n} \frac{\mathfrak{H}\left(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x\right)}{\chi^s} \\ &\quad + \frac{1}{2} \sum_{s=1}^{\infty} \frac{\mathfrak{H}\left(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x\right)}{\chi^s} \\ &= \sum_{s=1}^{\infty} \frac{\mathfrak{H}\left(\chi^{s-1}x, \chi^{s-1}x, \chi^{s-1}x\right)}{\chi^s}.\end{aligned}$$

Since the additive mappings are A_t and B_t then we see

$$\begin{aligned}||A_t(x) - B_t(x)|| &= \frac{1}{\chi^n}||A_t(\chi^n x) - B_t(\chi^n x)|| \\ &\leq \frac{1}{\chi^n} \sum_{s=1}^{\infty} \frac{\mathfrak{H}\left(\chi^{s+n-1}x, \chi^{s+n-1}x, \chi^{s+n-1}x\right)}{\chi^s}. \qquad (3.6)\end{aligned}$$

Hence taking the limit as $n \to \infty$ we get from (3.6),

$$\lim_{n\to\infty} ||A_t(x) - B_t(x)|| \leq \lim_{n\to\infty} \frac{1}{\chi^n} \sum_{s=1}^{\infty} \frac{\mathfrak{H}\left(\chi^{s+n-1}x, \chi^{s+n-1}x, \chi^{s+n-1}x\right)}{\chi^s}.$$

Hence

$$||A_t(x) - B_t(x)|| \leq 0.$$

Therefore $A_t(x) = B_t(x)\ \forall\ x \in \mathbb{H}^*$. Hence A_t is unique. □

Corollary 3.1. *Let* $\mathfrak{U}$ *and* p *be nonnegative real numbers. Let a function* $h : \mathbb{H}^* \to \mathbb{H}^*$ *fulfills the inequality*

$$(3.7)\qquad \left\|D_{\mathcal{L}}^{\mathcal{M}+\mathcal{N}} h(x,y,z)\right\| \leq \begin{cases} \mathfrak{U}, & \\ \mathfrak{U}\left\{||x||^p + ||y||^p + ||z||^p\right\}, & p \neq 1; \\ \mathfrak{U}||x||^p||y||^p||z||^p, & 3p \neq 1; \\ \mathfrak{U}\left\{||x||^p||y||^p||z||^p + \left\{||x||^{3p} + ||y||^{3p} + ||z||^{3p}\right\}\right\}, & 3p \neq 1; \end{cases}$$

$\forall\ x, y, z \in \mathbb{H}^*$. *Then the mapping* $A_t : \mathbb{H}^* \to \mathbb{H}^*$ *and we see*

$$\|h(x) - A_t(x)\| \leq \begin{cases} \dfrac{\mathfrak{U}}{2|\chi - 1|}, \\ \dfrac{3\ \mathfrak{U}||x||^p}{2|\chi - \chi^p|}, \\ \dfrac{\mathfrak{U}||x||^{3p}}{2|\chi - \chi^{3p}|}, \\ \dfrac{2\ \mathfrak{U}||x||^{3p}}{|\chi - \chi^{3p}|} \end{cases}$$

$\forall\ x\ \in\ \mathbb{H}^*$.

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