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FUZZY EQUITABLE EDGE COLORING OF SOME SIMPLE GRAPHS

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ABSTRACT. In our earlier works, we have discussed about the equitable edge coloring of various classes of some simple graphs (or crisp graphs). In this Paper we are going to state and discuss the Fuzzy equitable edge coloring of some classes of simple graphs.

1. INTRODUCTION

1.1. PRELIMINARY AND FUNDAMENTAL DEFINITIONS.

Definition 1.1. *[1, 5] **Proper edge coloring:** A Proper edge coloring of a graph G is a function that assigns the colors(called the numbers) to the edges of that graph G so that no two incident edges at any vertex receive same color.*

Definition 1.2. *[6, 10] **Equitable edge coloring :** A Proper edge coloring of a graph G is known as equitable edge coloring if* $|N(x) - N(y)| \leq 1$ *for all* $x, y \in \{1, 2, 3, .., \Delta\}$*, where,* Δ *is the maximum degree of the graph G,* $N(x)$ *and* $N(y)$ *represents the number of edges in the color classes* x *and* y *respectively.*

Definition 1.3. *[4, 8]**Path:** A Path* P_n *is defined as a walk such that there is no repetition of vertices and edges.*

Definition 1.4. *[1, 11] **Cycle:** A Cycle* C_n *is defined as a simple regular graph of degree 2. i.e All the vertices in the cycle* C_n *have same degree.*

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Definition 1.5. *[3] **Fuzzy set:** A Fuzzy Set is a pair $(U,\ m)$ where U is a set and m:U→[0,1] , a membership function. The Reference set U is called the Universe of Discourse and for each $x \epsilon U$, the value $m(x)$ is called the grade of membership of x in (U, m).The function $f = m = \mu_F$ is called the membership function of the fuzzy set $F = (U, m)$.*

Definition 1.6. *[2] **Fuzzy Graph:** A fuzzy graph $\xi = (V, \sigma, \mu)$ is an algebraic structure of non-empty set V together with a pair of functions $f : V \to [0, 1]$ and $\mu : V\times\ V \to$ [0, 1] such that for all x, y $\in V$, $\mu(x, y)\leq f(x)\wedge f(y)$ and μ is a symmetric fuzzy relation on f.*

Lemma 1.1. *[9, 10] Every simple undirected graph G may be Edge colored using a number of colors that is at most one greater than the maximum degree Δ of the graph G.*

2. Main results

Definition 2.1. ***Fuzzy Equitable edge coloring :*** *Let (a_n) be an infinite sequence of monotonic inceasing positive integers. A function $\mu : E(G) \to [0, 1]$ is called fuzzy equitable edge coloring if it is induced by the function $f : V(G) \to [0, 1]$ defined as $f(v_i) = \frac{a_i}{a_{(i+1)}}$ such that*

- (i) *$\mu(uv) \leq f(u) \wedge f(v)$,*
- (ii) *$\mu : E(G) \to \{\beta, \beta^2, \beta^3, \ldots .\beta^{\Delta}\}$ defines a proper edge coloring to the edges of the graph G,*
- (iii) *number of edges in any two color classes differ by at most one, i.e $|l_f(i) - l_f(j)| \leq 1$ for all $i, j \in \{\beta, \beta^2, \beta^3, \ldots\}$, where (a) $\beta \in [0, 1]$ is calculated from the vertex labels, (b) $l_f(i)$ and $l_f(j)$ denote the number of edges in the color classes i and j respectively.*

Definition 2.2. ***Fuzzy Equitable Edge chromatic number:*** *The Fuzzy equitable edge chromatic number is defined as the minimum number of colors needed for the fuzzy equitable edge coloring of this graph. It is denoted by $\chi^{'}_{fe}$.*

Remark 2.1.

(1) *Through this coloring there are two advantages.*
  (a) *The ordinary undirected (crisp) graph G is transformed into a fuzzy graph.*

(b) *The graph G is fuzzy edge colorable such that the color assigned to each and every edge of G act as the fuzzy membership value.*

(2) *The function $\mu : E(G) \to [0,1]$ is defined through a value β calculated as follows. (a) $\beta = \wedge_{i=1}^{p} f(v_i)$ or (b) $\beta = \prod_{i=1}^{p} f(v_i)$. The function $\mu : E(G) \to [0,1]$ is defined by using the above value of β so that it must satisfy the conditions in Definition 1.1.1*

2.1. **CONSTRUCTIVE ALGORITHM.** The One point union of the Cycle and the path graph(OCPG) is constructed as follows

Step 1: Consider a Cycle $C_n$with n vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and a Path P_m with m vertices $u_0, u_1, u_2, \ldots, u_{m-1}$.

Step 2: Merge the vertices v_0 and u_0.Thus we get a graph called OCP(n,m) graph.Here the vertex u_0 and v_0 are same. Let it be say w.

Step 3: Observations: This graph has $m+n-1$ vertices and $m+n-1$ edges. Maximum degree $\Delta = 3$. The degree of the all the vertices except w, u_{m-1} are equal to 3. Degree of the vertex w is 3 and Degree of the vertex u_{m-1} is 1. So sum of the degree of all the vertices of this graph $= (m+n-3)2+3+1 = 2m+2n-2 = 2(m+n-1)$.

Example 1.

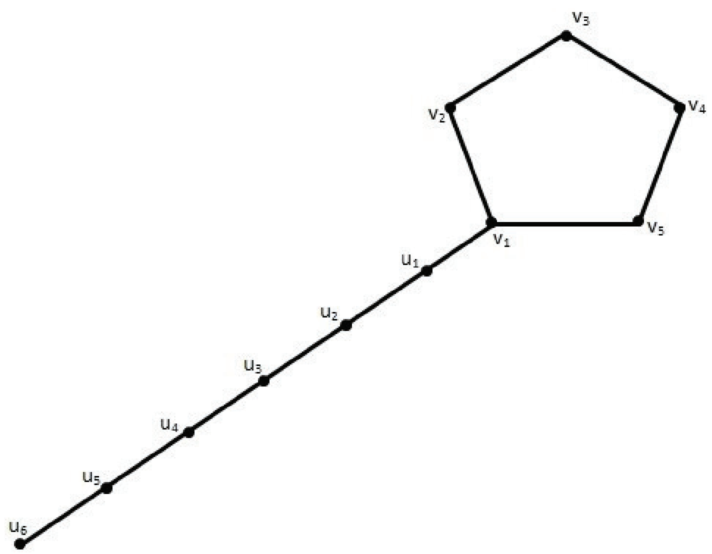

FIGURE 1. One Point Union of Path and Cycle Graph O.C.P(5,6)

Theorem 2.1. *The OCP(n,m) graph admits the fuzzy equitable edge coloring and its fuzzy equitable edge chromatic number is 3.*

Proof. To show that the graph G admits fuzzy equitable edge coloring, we first define a function $f : V(G) \rightarrow$ [0.1] by

$$f(v) = \begin{cases} \frac{a_i}{a_{i+1}} & for\ v = v_i \\ \frac{a_{n+i}}{a_{n+i+1}} & for\ v = u_i \\ \frac{a_n}{a_{n+1}} & for\ v = v_0 \end{cases}$$

Now let $\beta = \bigwedge_{v \epsilon V(G)} f(v)$.

By Lemma 1.1.1, we need Δ=3 colors for proper edge coloring of this graph G.

Here there are 3 cases based on the length of the cycle C_n .

Case(i): $n \equiv 0\ mod\ 3$.

Here n is a multiple of 3. Let us color the edges of the graph G by define a function $\mu : E(G) \rightarrow \{\beta, \beta^2, \beta^3\}\ as$:

$$\mu(uv) = \begin{cases} \beta & \text{for} \quad u = v_i \quad \text{and } v = v_{i+1} \text{ mod } n, i \equiv 1 \text{ mod } 3 \text{ and } i < n \\ \beta^2 & \text{for} \quad u = v_i \quad \text{and } v = v_{i+1} \text{ mod } n, i \equiv 2 \text{ mod } 3 \text{ and } i < n \\ \beta^3 & \text{for} \quad u = v_i \quad \text{and } v = v_{i+1} \text{ mod } n, i \equiv 0 \text{ mod } 3 \text{ and } i < n \\ \beta & \text{for} \quad u = v_0 \quad \text{and } v = u_1 \\ \beta & \text{for} \quad u = v_i \quad \text{and } v = v_{i+1} \text{ mod } m, i \equiv 0 \text{ mod } 3 \text{ and } i < m \\ \beta^2 & \text{for} \quad u = v_i \quad \text{and } v = v_{i+1} \text{ mod } m, i \equiv 1 \text{ mod } 3 \text{ and } i < m \\ \beta^3 & \text{for} \quad u = v_i \quad \text{and } v = v_{i+1} \text{ mod } m, i \equiv 2 \text{ mod } 3 \text{ and } i < m \end{cases}$$

Hence from the above mapping we see that $|l_f(i) - l_f(j)| \leq 1\ for\ all\ i,j \in \{\beta, \beta^2, \beta^3, \ldots \beta^{\Delta}\}$ and hence the graph G with $n \equiv 0 \mod 3$ admits the fuzzy equitable edge coloring with its fuzzy equitable edge chromatic number $\chi'_{fe}(G) = 3$.

Case(ii): $n \equiv 1 \mod 3$.

Here $n = 3k + 1$, k is some positive integer.

Let us color the edges of the graph G by define a function $\mu : E(G) \rightarrow \{\beta, \beta^2, \beta^3\}\ as$

$$\mu(uv) = \begin{cases} \beta & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod n, i \equiv 1 \bmod 3 \text{ and } i < n \\ \beta^2 & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod n, i \equiv 2 \bmod 3 \text{ and } i < n \\ \beta^3 & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod n, i \equiv 0 \bmod 3 \text{ and } i < n \\ \beta^2 & \text{for} \quad u = v_0 \text{ and } v = u_1 \\ \beta^3 & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod m, i \equiv 0 \bmod 3 \text{ and } i < m \\ \beta^1 & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod m, i \equiv 1 \bmod 3 \text{ and } i < m \\ \beta^2 & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod m, i \equiv 2 \bmod 3 \text{ and } i < m \end{cases}$$

Hence from the above mapping we see that $|l_f(i) - l_f(j)| \leq 1\ for\ all\ i, j \in \{\beta, \beta^2, \beta^3, \ldots .\beta^{\Delta}\}$ and hence the graph G with $n \equiv 1 \bmod 3$ admits the fuzzy equitable edge coloring with its fuzzy equitable edge chromatic number $\chi'_{fe}(G) = 3$.

Case (iii): $n \equiv 2 \bmod 3$.

Here $n = 3k + 2$, k is some positive integer.

Let us color the edges of the graph G by define a function $\mu : E(G) \rightarrow \{\beta, \beta^2, \beta^3\}\ as$:

$$\mu(uv) = \begin{cases} \beta & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod n, i \equiv 1 \bmod 3 \text{ and } i < n \\ \beta^2 & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod n, i \equiv 2 \bmod 3 \text{ and } i < n \\ \beta^3 & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod n, i \equiv 0 \bmod 3 \text{ and } i < n \\ \beta^2 & \text{for} \quad u = v_0 \text{ and } v = u_1 \\ \beta^2 & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod m, i \equiv 0 \bmod 3 \text{ and } i < m \\ \beta^3 & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod m, i \equiv 1 \bmod 3 \text{ and } i < m \\ \beta & \text{for} \quad u = v_i \text{ and } v = v_{i+1} \bmod m, i \equiv 2 \bmod 3 \text{ and } i < m \end{cases}$$

Hence from the above mapping we see that $|l_f(i) - l_f(j)| \leq 1$ for all $i, j \in \{\beta, \beta^2, \beta^3, \ldots \beta^{\Delta}\}$ and hence the graph G with $n \equiv 2 \bmod 3$ admits the fuzzy equitable edge coloring with its fuzzy equitable edge chromatic number $\chi'_{\mathrm{fe}}(\mathrm{G}) = 3$.

Therefore the OCP(n, m) graph admits fuzzy equitable edge coloring with its fuzzy equitable edge chromatic number is $\chi'_{fe}(G) = 3$.

Also the OCP(n, m) graph is transformed into a fuzzy graph with the above edge coloring, the color assigned to each edge is its membership value.$\chi'_{fe}(G) = 3$. $\square$

2.2. **CONSTRUCTIVE ALGORITHM. (Double Wheel Graph)**

Step 1: Draw two cycles C_n such that one lies inside the other cycle. Let the vertices of these Cycles be $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$

Step 2: Introduce a central vertex say v_0.

Step 3: Join the vertices $u_i$and v_i with the central vertex $v_0,\ i = 1, 2, 3, \ldots,\ n,$ Thus we get a graph called Double wheel graph [7].

Step 4: Here we observe that there are 2n+1 vertices, 4n edges. Maximum degree of this graph is $\Delta = 2n$. Degree of the central vertex v_0 is $2n$, degree of the vertices $u_i\ and\ v_i\ ,\ i = 1, 2, 3, \ldots, n$ are equal to 3. This graph is denoted by $DW(n)$.

Example 2.

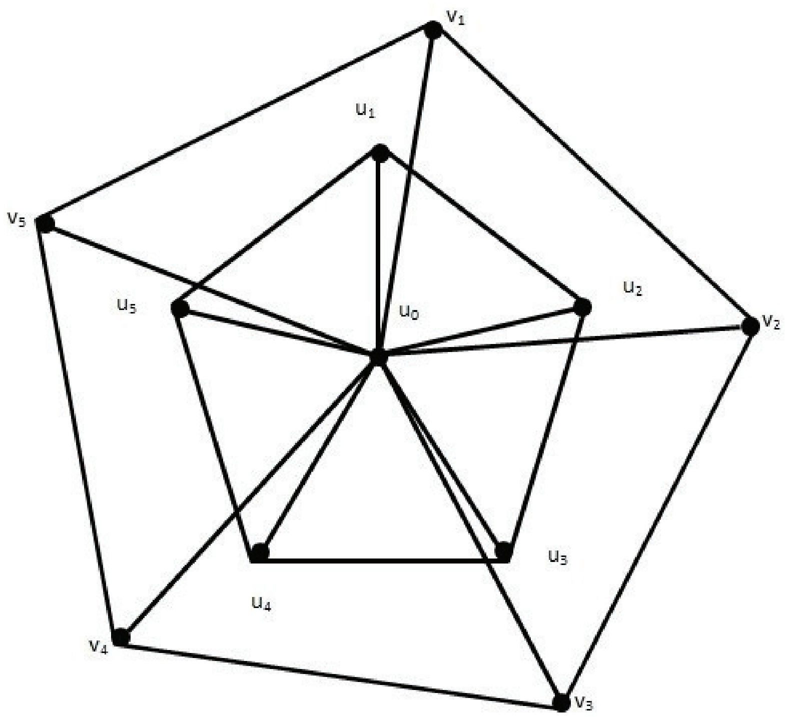

FIGURE 2. Double Wheel Graph DW(n)

Theorem 2.2. *The Double-Wheel graph admits the fuzzy equitable edge coloring and its Fuzzy Equitable edge chromatic number is* $\chi'_{fe}\ (DW(n)) = 2n$.

Proof. Let G be a Double wheel graph [7]. To prove that the graph G admits fuzzy equitable edge coloring, we first define a function $f : V(G) \to [0.1]$ by

$$f(v) = \begin{cases} \frac{a_i}{a_{i+1}} \; for \; v = v_i \; , i = 1, 2, 3, \ldots, n \\ \frac{a_{n+i}}{a_{n+i+1}} \; for \; v = u_i, i = 1, 2, 3, \ldots, n \\ \frac{a_{2n+1}}{a_{2n+2}} \; for \; v = v_0 \end{cases}.$$

Now let $\beta = \prod_{v \epsilon V(G)} f(v)$. By Lemma 1.1.1, we need $\Delta = 2n$ colors for proper edge coloring of this graph G. So to color the edges of G properly, let us define a function $\mu : E(G) \to \{ \beta, \beta^2, \beta^3, \ldots . \beta^{2n}\}$ by

$$\mu(u\, v) = \begin{cases} \beta^{2i-1} \; for \; u = \; v_0 \; and \; v = v_i, i = 1, 2, 3, \ldots, \; n \\ \beta^{2i} \; for \; \; u = \; v_0 \; and \; v = u_i, i = 1, 2, 3, \ldots, \; n \\ \beta^{2i-1} \; for \; u = u_i \; and \; v = u_{i+1}, \; i = 1, 2, 3, \ldots, \; n-1 \\ \beta^{2i} \; for \; u = v_i \; and \; v = v_{i+1}, \; i = 1, 2, 3, \ldots, \; n-1 \\ \beta^{2n-1} \; for \; u = \; u_n \; and \; v = u_{n+1} \\ \beta^{2n} \; for \; u = \; v_n \; and \; v = v_{n+1} \end{cases}.$$

Therefore from this mapping we find that $|l_f(i) - l_f(j)| \leq 1$ for all $i, j \in \{\beta, \beta^2, \beta^3, \ldots, \beta^{\Delta}\}$ and hence the graph G admits the fuzzy equitable edge coloring. Also the graph G is transformed into a Fuzzy graph with the above edge coloring, the color assigned to each edge is its membership value and so $\chi^{'}_{fe}(G) = 2n$. □

CONCLUSION

In this work we discussed about the Fuzzy equitable edge coloring of some simple graphs related to cycle graphs.

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