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AN EQUICONVERGENCE THEOREM FOR LINEAR ORDINARY DIFFERENTIAL OPERATOR

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ABSTRACT. The aim of the present paper is to prove a general equiconvergence theorem for ordinary linear differential operator of 3^{rd}-order, $Lu := u^3 + q(x)u$, which extends the results of Horvath, Joó and Komornik for the Schrödinger operators of second order.

1. FIRST SECTION: IMPORTANT

The equiconvergence theorem plays an important role in the theory of expansions. A general equiconvergence theorem was published in [1] by Horváth, Joó and Komornik for one dimensional Schrödinger operator without any restriction of the distribution of the eigenvalues (namely they drop the condition of the boundedness of v_α, see below) on the complex plane, generalizing some results of the case of orthonormal bases consisting of eigenfunctions of 2^{nd}–order operator (see e.g. [2–5]). In [6] an equiconvergence theorem was proved for the following ordinary linear differential operator of 4^{th}-order:

$$Lu := u^{(4)},$$

where $u^{(4)} = \frac{d^4u}{dt^4}$.

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The aim of the present paper is to prove a similar equiconvergence theorem for the ordinary linear differential operator:

$$Lu := u^{(3)} + qu$$

using the results reported in [3, 7, 8]. Besides, we must mentioned here to a fruitful method was developed by V. A. Il'n (cf. [8]). His method works on the theorem of equiconvergence of ordinary differential operator of second order (Schrödinger operator), also we use the results of the papers [9–19].

2. Basic definitions

In this section, we state some basic definitions and preliminaries associated with the equiconvergence theorem for completeness.

Definition 2.1. *In $H^n_{loc}(G)$, the set of all complex functions $v \in L^2_{loc}(G)$ (where $G \subseteq \mathbb{R}$ is finite or infinite interval) having the distributional derivatives in $L^2_{loc}(G)$ of order up to p.*

Definition 2.2. *Let $v > 0$ be an arbitrary number, $x \in K$ where K is an arbitrary fixed compact interval $K \subset G$ and $R \in (0,\ dis(K,\ \partial G))$, we define the function $W_R(t)$ by:*

$$W_R(t) := \begin{cases} \frac{\sin v(x-t)}{\pi(x-t)} & \text{if } |x-t| \leq R \\ 0 & \text{if } |x-t| > R, \end{cases}$$

where $dis(K,\ \partial G)$ is defined by:

$$dis(K,\ \partial G) := \inf\{|x-a|,\ x \in K \text{ and } a \in \partial G\}.$$

Definition 2.3. *Let $f \in L^2(G)$ and $0 < R_o < dis(K,\ \partial G)$, we define the function $D_{R_o} : L^2 \to R$ by:*

$$D_{R_o}[f] := \frac{2}{R_o}\int_{\frac{R_o}{2}}^{R_o} f(R).dR.$$

Definition 2.4. *Let $v > 0$ be an arbitrary number, (v_α) be the dual system of the system of eigenfunction (u_α), (i.e. $(v_\alpha) \subset L^2(G)$ and $\langle u_k,\ v_j\rangle = \delta_{kj}$) and v_α is the imaginary part of μ_α which will be defined in Section 3 of this paper. We define the function:*

$$\sigma_v(f, x) := \sum_{v_\alpha < v} \langle f,\ v_\alpha\rangle u_\alpha(x) + \frac{1}{2}\sum_{v_\alpha = v} \langle f,\ v_\alpha\rangle u_\alpha(x).$$

Definition 2.5. *Let $f \in L^2(G)$ and $x \pm R \in G$, we define the function $S_v(f,x)$ by:*

$$S_v(f,x) := \int_{x-R}^{x+R} \frac{\sin v(y-x)}{\pi(y-x)} f(y).dy.$$

Definition 2.6. *Let $K \subset G$ be a compact interval and $0 < b < dis(K,\ \partial G)$, then:*

$$K_b := \{x \in G : dis(x,\ \partial G) \leq b\}.$$

3. An Equiconvergence Theorem

The equiconvergence theorems play an important role in the theory of expansions, and they are very useful in the spectral investigation of differential operators, because many results known for the most special operators may be transferred by their applications to more general ones. The aim of present paper is to prove an equiconvergence theorem for the operator

$$Lu := u^{(3)} + qu.$$

Let G be an open interval (finite or infinite) on the real line $\mathbb{R}$, $q \in L^1_{loc}(G)$ a complex function and consider the differential operator $Lu := u^{(3)} + q(x)u$ defined on $H^3_{loc}(G)$. Given a complex number λ, the function $u : G \to \mathbb{C}$, $u \equiv 0$, is called an eigenfunction of order (-1) of operator L with the eigenvalue λ. Furthermore, a function $u : G \to \mathbb{C}$, $u \not\equiv 0$, is called an eigenfunction of order k ($k = 0, 1, 2, ...$) of the operator L with the eigenvalue λ. If the function $u^* := Lu - \lambda u$ is an eigenfunction of order $(k-1)$ with the same λ.

Let us now be given a complete and minimal system $(u_\alpha) \subset L^2(G)$ of eigenfunctions of the operator L, denoted by λ_α(resp.O_α) the eigenvalues (resp. the order) of u_α, and assume:

(1) $\sup\limits_\alpha O_\alpha < \infty$,
(2) In case $O_\alpha > 0$, then $\lambda_\alpha u_\alpha - Lu_\alpha = u_{\alpha-1}$.

In the meantime, we introduce some notations: Choose the three roots $\mu_{i,\alpha}$, $(i = 1, 2, 3)$, of λ_α such that Re$\mu_{1,\alpha} \geq$ Re$\mu_{2,\alpha} \geq$ Re$\mu_{3,\alpha}$, and put $\lambda_\alpha := \mu_{2,\alpha}$, $\rho_\alpha :=$Reμ_α, $v_\alpha := |$Im$\mu_\alpha|$, $W(t) := D_{R_o}[W_R]$, and

$$\delta(v, v_\alpha) := \begin{cases} 1 & \text{if } v > v_\alpha \\ \frac{1}{2} & \text{if } v = v_\alpha \\ 0 & \text{if } v < v_\alpha. \end{cases}$$

We shall prove the following theorem.

Theorem 3.1. *Suppose that:*

(3) $\sup_{t>0} \sum_{t \leq v_\alpha \leq t+1} 1 < \infty$,

and moreover the above two assumptions (1) and (2) together with $u^* = 0$ *are fulfilled. Then the following three statements are equivalent:*

(a) For any compact interval $K \subset G$*, then:*

$$\sup_\alpha \|v_\alpha\|_{L^2(G)} \|u_\alpha\|_{L^2(G)} < \infty.$$

(b) For any compact interval $K \subset G$ *and every* $f \in L^2(G)$*, we have:*

$$\lim_{v \to \infty} \sup_{x \in K} |S_v(f,x) - \sigma_v(f,x)| = 0.$$

(c) For any compact interval $K \subset G$ *and every* $f \in L^2(G)$*, we have:*

$$\lim_{v \to \infty} \|f - \sigma_v(f)\|_{L^2(G)} = 0.$$

Proof. $(a) \Rightarrow (b)$: Denote by ω_1, ω_2 and ω_3 the three cube roots of unity. Now define the following term:

$$v_\alpha(y) := u_\alpha(y) + \int_x^y \sum_{p=1}^{3} \frac{\omega_p}{3\mu_\alpha^2} e^{\mu_\alpha \omega_p (y-\zeta)} Q(\zeta).d\zeta,$$

where $Q(\zeta) := q(\zeta) u_\alpha(\zeta) - u_\alpha^*(\zeta)$. According to the definition of $v_\alpha(y)$ and the equation:

$$0 = \sum_{p=1}^{3} \omega_p = \sum_{p=1}^{3} \omega_p^2 \text{ and } \sum_{p=1}^{3} \omega_p^3 = 3,$$

we have

$$v_\alpha(x) = u_\alpha(x),\ v_\alpha'(x) = u_\alpha'(x) \text{ and } v_\alpha''(x) = u_\alpha(x),$$

and

$$v_\alpha^{(2)}(x) = u_\alpha^{(2)}(x) + \mu_\alpha^2 \int_x^y \sum_{p=1}^{3} \frac{\omega_p^3 e^{\mu_\alpha \omega_p (y-\zeta)}}{3\mu_\alpha^2} Q(\zeta).d\zeta.$$

By using the following assertion:

$$\frac{d}{dx} \int_a^{g(x)} f(x,t).dt = \int_a^{g(x)} f_x(x,t).dt + f(x, g(x)).g'(x),$$

we have:

$$\begin{aligned} v_\alpha^{(3)}(x) &= u_\alpha^{(3)}(x) + Q(y) + \mu_\alpha^3 \int_x^y \sum_{p=1}^{3} \frac{\omega_p^3 e^{\mu_\alpha \omega_p (y-\zeta)}}{3\mu_\alpha^2} Q(\zeta).d\zeta \\ &= u_\alpha^{(3)}(x) + Q(y) + \mu_\alpha^3 \left(v_\alpha(y) - u_\alpha(y)\right) \\ &= u_\alpha^{(3)}(x) + Q(y) + \lambda_\alpha v_\alpha(y) - \lambda_\alpha u_\alpha(y) \\ &= \lambda_\alpha v_\alpha(y) + \left[u_\alpha^{(3)}(y) + Q(y) - \lambda_\alpha u_\alpha(y)\right] \\ &= \lambda_\alpha v_\alpha(y). \end{aligned}$$

i.e.;

$$v_\alpha^{(3)}(x) = \lambda_\alpha v_\alpha(y).$$

Consequently, $v_\alpha(y)$ is a linear combination of the functions $e^{\mu_\alpha \omega_1 y}$, $e^{\mu_\alpha \omega_2 y}$ and $e^{\mu_\alpha \omega_3 y}$. From assumption (3), we have for any $0 \le t \le R \le s \le 2R$ and $R > 0$:

$$0 = \begin{vmatrix} v_\alpha(x-2s) & v_\alpha(x-t) + v_\alpha(x+t) - 2v_\alpha(x) ch\mu_\alpha t & v_\alpha(x+2s) & v_\alpha(x+3s) \\ e^{-2\mu_{1,\alpha} s} & 2(ch\mu_{1,\alpha} t) & e^{2\mu_{1,\alpha} s} & e^{3\mu_{1,\alpha} s} \\ e^{-2\mu_\alpha s} & 0 & e^{2\mu_\alpha s} & e^{3\mu_\alpha s} \\ e^{-2\mu_{3,\alpha} s} & 2(ch\mu_{3,\alpha} t - ch\mu_\alpha t) & e^{2\mu_{3,\alpha} s} & e^{3\mu_{3,\alpha} s} \end{vmatrix}.$$

Expanding this determinant according to the first row we get:

$$\begin{aligned} &\left[v_\alpha(x-t) + v_\alpha(x+t) - 2v_\alpha(x) ch\mu_\alpha t\right] d(\mu_\alpha, s) \\ &= \sum_{\substack{-2 \le k \le 3 \\ |k| \ge 2}} d_k(\mu_\alpha, s, t) v_\alpha(x+ks), \end{aligned} \tag{3.1}$$

where

$$d(\mu_\alpha, s) := \begin{vmatrix} e^{-2\mu_{1,\alpha} s} & e^{2\mu_{1,\alpha} s} & e^{3\mu_{1,\alpha} s} \\ e^{-2\mu_\alpha s} & e^{2\mu_\alpha s} & e^{3\mu_\alpha s} \\ e^{-2\mu_{3,\alpha} s} & e^{2\mu_{3,\alpha} s} & e^{3\mu_{3,\alpha} s} \end{vmatrix},$$

$$d_{-2}(\mu_\alpha, s) := \begin{vmatrix} 2(ch\mu_{1,\alpha} - ch\mu_\alpha t) & e^{2\mu_{1,\alpha} s} & e^{3\mu_{1,\alpha} s} \\ 0 & e^{2\mu_\alpha s} & e^{3\mu_\alpha s} \\ 2(ch\mu_{3,\alpha} - ch\mu_\alpha t) & e^{2\mu_{3,\alpha} s} & e^{3\mu_{3,\alpha} s} \end{vmatrix},$$

$$d_2(\mu_\alpha, s, t) := \begin{vmatrix} e^{-2\mu_{1,\alpha} s} & 2(ch\mu_{1,\alpha} t - ch\mu_\alpha t) & e^{3\mu_{1,\alpha} s} \\ e^{-2\mu_\alpha s} & 0 & e^{3\mu_\alpha s} \\ e^{-2\mu_{3,\alpha} s} & 2(ch\mu_{1,\alpha} t - ch\mu_\alpha t) & e^{3\mu_{3,\alpha} s} \end{vmatrix},$$

$$d_3(\mu_\alpha, s, t) := \begin{vmatrix} e^{-2\mu_{1,\alpha}s} & 2(ch\mu_{1,\alpha}t - ch\mu_\alpha t) & e^{2\mu_{1,\alpha}s} \\ e^{-2\mu_\alpha s} & 0 & e^{2\mu_\alpha s} \\ e^{-2\mu_{3,\alpha}s} & 2(ch\mu_{3,\alpha} - ch\mu_\alpha t) & e^{2\mu_{3,\alpha}s} \end{vmatrix}.$$

Taking into account, the definition of v_α we obtain:

$$\begin{aligned}
&[u_\alpha(x-t) + 2u_\alpha(x+t)u_\alpha(x)ch\mu_\alpha t]\, d(\mu_\alpha, s) \\
&= \sum_{\substack{-2\le k\le 3\\ |k|\ge 2}} d_k(\mu_\alpha, s, t)u_\alpha(x+ks) - d(\mu_\alpha, s)\int_x^{x+t}\sum_{p=1}^{3}\frac{\omega_p e^{\mu_\alpha\omega_p(x+t-\zeta)}}{3\mu_\alpha^2}Q(\zeta).d\zeta \\
&+ d(\mu_\alpha, s)\int_{x-t}^{x}\sum_{p=1}^{3}\frac{\omega_p e^{\mu_\alpha\omega_p(x+t-\zeta)}}{3\mu_\alpha^2}Q(\zeta).d\zeta \\
&+ \sum_{\substack{-2\le k\le 3\\ |k|\ge 2}} d_k(\mu_\alpha, s, t)\int_x^{x+ks}\sum_{p=1}^{3}\frac{\omega_p e^{\mu_\alpha\omega_p(x+t-\zeta)}}{3\mu_\alpha^2}Q(\zeta).d\zeta.
\end{aligned} \tag{3.2}$$

Denote $Q(\mu_\alpha, s) := e^{(3\mu_{1,\alpha}+2\mu_\alpha-2\mu_{3,\alpha})s}$. Now, we estimate $d_k(\mu_\alpha, s, t)$. In this determinant the minor corresponding to the element $2(ch\mu_{1,\alpha}t - ch\mu_\alpha t)$ is in absolute value smaller than $e^{Re(3\mu_\alpha-2\mu_{3,\alpha})s}$, further the minor corresponding to the element $2(ch\mu_{3,\alpha}t - ch\mu_\alpha t)$ is in absolute value smaller than $e^{Re(3\mu_{1,\alpha}+2\mu_\alpha)s}$. This means that for the order of the terms dividing by $Q(\mu_\alpha, s)$ we obtain the following orders respectively:

$$|ch\mu_{1,\alpha}t - ch\mu_\alpha t|.e^{Re(\mu_{2,\alpha}-3\mu_{1,\alpha})s} \le |ch\mu_{1,\alpha} - ch\mu_\alpha t|.e^{-2Re\mu_{1,\alpha}s},$$

$$|ch\mu_{3,\alpha}t - ch\mu_\alpha t|.e^{Re(3\mu_{1,\alpha}+2\mu_\alpha-3\mu_{1,\alpha}-2\mu_\alpha+2\mu_{3,\alpha})s} \le |ch\mu_{3,\alpha}t - ch\mu_\alpha t|.e^{2Re\mu_{3,\alpha}s},$$

where we used $|\text{Re}\mu_\alpha| \le |\text{Re}\mu_{k,\alpha}|$, $k \ne 2$. Obviously $\text{Re}\mu_{1,\alpha} > 0$ and $\text{Re}\mu_{3,\alpha} < 0$, therefore:

$$|ch\mu_{1,\alpha}t - ch\mu_\alpha t|.e^{-Re2\mu_{1,\alpha}s} \le \begin{cases} c|\mu_\alpha|te^{-Re2\mu_{1,\alpha}s}, & 0 \le t \le \frac{1}{|\mu_\alpha|},\ |\mu_\alpha| \ne 0 \\ ce^{Re\mu_{1,\alpha}(t-2s)}, & t > \frac{1}{|\mu_\alpha|}, \end{cases}$$

and

$$|ch\mu_{3,\alpha}t - ch\mu_\alpha t|.e^{Re2\mu_{3,\alpha}s} \le \begin{cases} c|\mu_\alpha|te^{-|Re2\mu_{3,\alpha}|s}, & 0 \le t \le \frac{1}{|\mu_\alpha|} \\ ce^{|Re\mu_{3,\alpha}|(t-2s)}, & t > \frac{1}{|\mu_\alpha|}, \end{cases}$$

Using these estimates, we obtain:

$$|d_k(\mu_\alpha s, t)| \le c|Q(\mu_\alpha s)|.\left(e^{-Re2\mu_{1,\alpha}s} + e^{Re2\mu_{3,\alpha}s}\right).|\mu_\alpha|t,\ 0 \le t \le \frac{1}{|\mu_\alpha|}, \tag{3.3}$$

$$|d_k(\mu_\alpha s, t)| \leq c|Q(\mu_\alpha s)|.\left(e^{Re 2\mu_{1,\alpha}(t-2s)} + e^{|Re 2\mu_{3,\alpha}|(t-2s)}\right), \ t > \frac{1}{|\mu_\alpha|}. \tag{3.4}$$

Now, we have that:

$$\left|\int_R^{2R} \frac{d(\mu_\alpha s)}{Q(\mu_\alpha s)}.ds\right| > \frac{R}{2}, \text{ if } R_o \geq R \geq \frac{R_o}{2} \text{ and } |\mu_\alpha| \geq A(R_o) \geq 2. \tag{3.5}$$

Next, we also estimate $|\langle \mu_\alpha, W\rangle - \delta(v, v_\alpha)u_\alpha(x)|$. We have:

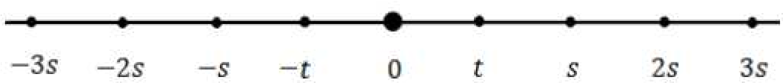
$$\begin{aligned} &\langle \mu_\alpha, W\rangle - \delta(v, v_\alpha)u_\alpha(x) \\ &= D_{R_o}\left(\int_0^R \frac{\sin vt}{\pi t} \times \left[u_\alpha(x+t) + u_\alpha(x-t) - 2u_\alpha(x) ch\mu_\alpha t\right].dt\right) \\ &+ D_{R_o}\left(\int_0^R \left[\frac{2\sin vt ch\mu_\alpha t}{\pi t} - \delta(v, v_\alpha)\right].dt\right) u_\alpha(x). \end{aligned} \tag{3.6}$$

From (3.2), we obtain:

$$\begin{aligned} &\left[u_\alpha(x-t) + u_\alpha(x+t) - 2u_\alpha(x) ch\mu_\alpha t\right] d(\mu_\alpha, s) \\ &= \sum_{\substack{-2\leq k\leq 3 \\ |k|\geq 2}} d_k(\mu_\alpha, s, t) u_\alpha(x+ks) + \int_{x-2s}^{x+3s} D(\mu_\alpha, s, t, x-\zeta) Q(\zeta).d\zeta, \end{aligned} \tag{3.7}$$

where

$$D(\mu_\alpha, s, t, x-\zeta) = \begin{cases} -d(\mu_\alpha, s)\sum_{p=1}^{3} \frac{\omega_p e^{\mu_\alpha \omega_p (x+t-\zeta)}}{3\mu_\alpha^2} \\ ...+\sum\limits_{-2\leq k\leq 3} d_k(\mu_\alpha, s, t)\sum_{p=1}^{3} \frac{\omega_p e^{\mu_\alpha \omega_p (x+ks-\zeta)}}{3\mu_\alpha^2} & \text{, if } 0 \leq \zeta - x \leq t \\ \sum\limits_{-2\leq k\leq 3} d_k(\mu_\alpha, s, t)\sum_{p=1}^{3} \frac{\omega_p e^{\mu_\alpha \omega_p (x+ks-\zeta)}}{3\mu_\alpha^2} & \text{, if } t \leq \zeta - x \leq 2s,\ 2s \leq \zeta - x \leq 3s \\ d(\mu_\alpha, s)\sum_{p=1}^{3} \frac{\omega_p e^{\mu_\alpha \omega_p (x-t-\zeta)}}{3\mu_\alpha^2} \\ ...-d_{-2}(\mu_\alpha, s, t)\sum_{p=1}^{3} \frac{\omega_p e^{\mu_\alpha \omega_p (x+ks-\zeta)}}{3\mu_\alpha^2} & \text{, if } -t \leq \zeta - x \leq 0 \\ -d_{-2}(\mu_\alpha, s, t)\sum_{p=1}^{3} \frac{\omega_p e^{\mu_\alpha \omega_p (x+ks-\zeta)}}{3\mu_\alpha^2} & \text{, if } -2s \leq \zeta - x \leq -t \\ -d_{-2}(\mu_\alpha, s, t)\sum_{p=1}^{3} \frac{\omega_p e^{\mu_\alpha \omega_p (x+ks-\zeta)}}{3\mu_\alpha^2} & \text{, if } -3s \leq \zeta - x \leq -2s. \end{cases}$$



Here, we give the proof of the estimate:

$$|D(\mu_\alpha, s, t, x-\zeta)| \leq c|Q(\mu_\alpha, s)||\mu_\alpha|^{-2}\min\{1,\ |\mu_\alpha|t\}e^{3\rho_\alpha s}. \tag{3.8}$$

Begin the proof of (3.8) with:

1 $x \leq \zeta \leq x+t$. Then D can be expressed in the following form:

$$D = \sum_{p=1}^{3} \frac{\omega_p}{3\mu_\alpha^2} \begin{vmatrix} 0 & e^{\mu_\alpha\omega_p(t+x-\zeta)} & e^{\mu_\alpha\omega_p(2s+x-\zeta)} & e^{\mu_\alpha\omega_p(3s+x-\zeta)} \\ e^{-\mu_{1,\alpha}s} & 2(ch\mu_{1,\alpha}t - ch\mu_\alpha t) & e^{\mu_{1,\alpha}s} & e^{3\mu_{1,\alpha}s} \\ e^{-2\mu_\alpha s} & 0 & e^{2\mu_\alpha s} & e^{3\mu_\alpha s} \\ e^{-\mu_{3,\alpha}s} & 2(ch\mu_{3,\alpha}t - ch\mu_\alpha t) & e^{2\mu_{3,\alpha}s} & e^{3\mu_{3,\alpha}s} \end{vmatrix}.$$

Let again $\mu_q := \mu_{q,\alpha} := \mu_\alpha\omega_p$. We have the following cases in expanding the determinants:

(i) $e^{\mu_q(t+x-\zeta)}.e^{-k\mu_q s}$ multiplied by a product where there is no t, μ_q and $e^{-k\mu_r s}$ for any r.

If $|\mu_\alpha|t \leq 1$, then:

$$e^{\mu_q(t+x-\zeta)} = O\left(|\mu_\alpha|t\right).e^{-k\mu_q s}\prod_{\substack{j\neq -k\\ r(j)\neq q}} e^{j\mu_{r(j)}s} = O(Q(\mu_\alpha, s)).$$

If $|\mu_\alpha|t \geq 1$, then in case $q \geq 2$, we have $e^{\alpha_q(t+x-\zeta)} = O(e^{\rho_\alpha R})$, and if $q = 2$, then there exists $j \geq 2$ with $r(j) = 2$, hence $e^{\mu_q(t+x-\zeta)}e^{-k\mu_q s}e^{j\mu_{r(j)}s} = O\left(e^{-k\mu_{r(j)}s}e^{j\mu_q s}\right)$, which shows that the whole product is $O(Q(\mu_\alpha, s))$.

(ii) $\left[e^{\mu_q(t+x-\zeta)}e^{k\mu_q s} - e^{\mu_{q(ks+x-\zeta)}s}.2(ch\mu_q t - ch\mu_\alpha t)\right].\prod\limits_{\substack{j\neq -k\\ r(j)}} e^{j\mu_{r(j)}s}$.

If $q = 1$, then this is $O\left(e^{k\mu_q s}.e^{\rho_\alpha R}.\prod\limits_{\substack{j\neq -k\\ r(j)\neq q}} e^{j\mu_{r(j)}}s\right) = O\left(e^{\rho_\alpha R}Q(\mu_\alpha, s)\right)$ in case $|\mu_\alpha|t \geq 1$, and $O\left(|\mu_\alpha|te^{k\mu_q s}.e^{\rho_\alpha R}.\prod\limits_{\substack{j\neq -k\\ r(j)\neq q}} e^{j\mu_{r(j)}s}\right) = O\left(|\mu_\alpha|tQ(\mu_\alpha, s)\right)$ in case $|\mu_\alpha|t \leq 1$.

If $q = 2$, then $e^{\mu_q(t+x-\zeta)}e^{k\mu_q s} = O\left(e^{\rho_\alpha R}e^{k\mu_q s}\right)$ if $|\mu_\alpha|t \geq 1$, and $O\left(|\mu_\alpha|te^{k\mu_q s}\right)$ if $|\mu_\alpha|t \leq 1$.

If $q = 3$, we estimate the expression in brackets by $O\left(e^{\mu_q(ks-2R)}\right)$ if $|\mu_\alpha|t \geq 1$, and $O\left(e^{\mu_q(ks-2R)}|\mu_\alpha|t\right)$ if $|\mu_\alpha|t \leq 1$. Since k is paired with μ_q which has negative real part, there exists $j_o \leq -2$ and $r(j_o) \leq 2$ such that the factor $e^{j_o\mu_{r(j)}s}$ occurs in the product stated in (ii). Now, we have $e^{\mu_q(ks-2R)}e^{j_o\mu_{r(j)}s} = O\left(e^{j_o\mu_{r(j)}s}.e^{2\rho_\alpha R}\right)$. Indeed, if $r(j_o) = 1$, then Re$\mu_{r_{(j_o)}} > 0$, Reμ_q 0, hence $e^{\mu_q((k-j_o)s-2R)} = O(1) = O\left(e^{(k-j_o)\mu_{r(j_o)}}\right)$ and if $r(j_o) = 2$, then $e^{\mu_q((k-j_o)s-2R)}e^{\mu_{r(j_o)}(j_o-k)s} = e^{(\mu_q-\mu_\alpha)((k-j_o)s-2R)}e^{-2R\mu_\alpha} = O\left(e^{-2R\mu_\alpha}\right) = O\left(e^{2\rho_\alpha R}\right)$. Consequently, we can estimate the product stated in (ii) by:

$$O\Bigg(\min\{1,\ |\mu_\alpha|t\}.e^{2\rho_\alpha R}.e^{j_o\mu_q s+k\mu_{r(j_o)}s}\prod_{\substack{j\neq k,j_o\\ r(j_o)\neq q}} e^{j\mu_{r(j)}s}\Bigg)$$
$$=O\left(\min\{1,\ |\mu_\alpha|t\}.e^{2\rho_\alpha R}.Q(\mu_\alpha, s)\right).$$

(iii) $e^{\mu_q(ks+x-\zeta)}e^{\mu_q ls} - e^{\mu_q(ls+x-\zeta)}e^{\mu_q ks} = 0$ does not give new members.

(iv) $e^{\mu_q(ks+x-\zeta)}e^{-\mu_q ls}.2(ch\mu_r t - ch\mu_\alpha t)\prod_{\substack{j\neq k,-l\\ r(j)\neq q,r}} e^{j\mu_{r(j)}s}$.

If ($q = 1$ and $r = 3$) resp. ($q = 3$ and $r = 1$), then $e^{\mu_q((k-l)s+x-\zeta)}(ch\mu_r t - ch\mu_\alpha t) = O\left(e^{\mu_q ks}e^{-\mu_r ls}\min\{1,\ |\mu_\alpha|t\}\right)$, resp. $O\left(e^{-\mu_q ls}e^{\mu_r ks}\min\{1,\ |\mu_\alpha|t\}\right)$, hence the whole product is estimated by: $O\left(\min\{1,\ |\mu_\alpha|t\}Q(\mu_\alpha, s)\right)$.

If ($q = 2$ and $r = 3$) resp. ($q = 2$ and $r = 1$), then we get: $e^{\mu_q((k-l)s+x-\zeta)}(ch\mu_r t - ch\mu_\alpha t) = O\left(e^{\mu_q ks} - e^{-\mu_r ls}e^{2\rho_\alpha R}.\min\{1,\ |\mu_\alpha|t\}\right)$, resp. $O\left(e^{-\mu_q ls}e^{\mu_r ks}e^{2\rho_\alpha R}\min\{1,\ |\mu_\alpha|t\}\right)$.

Let now $q = 1$ and $r = 1$, then there exists $j \geq 2$ with $r(j) = 3$ and then:

$$e^{\mu_q((k-l)s+x-\zeta)}(ch\mu_r t - ch\mu_\alpha t)e^{j\mu_{r(j)}s} = O\left(\max\{1,\ |\mu_\alpha|t\}e^{\mu_q ks}e^{\mu_r js}e^{-l\mu_{r(j)}s}\right)$$

Hence the whole product is $O\left(\max\{1,\ |\mu_\alpha|t\}Q(\mu_\alpha, s)\right)$.

Finally let $q = 3$ and $r = 3$, then there exists $j \leq -2$ with $r(j) \leq 2$, hence:

$$e^{\mu_q((k-l)s+x-\zeta)}(ch\mu_r t - ch\mu_\alpha t)e^{j\mu_{r(j)}s}$$
$$=O\left(\min\{1,\ |\mu_\alpha|t\}.e^{-\mu_q ls}e^{j\mu_r s}e^{k\mu_{r(j)}s}e^{2\rho_\alpha R}\right),$$

because after dividing with the exponential we get:

$$e^{\mu_q((k-l)s+x-\zeta)}e^{\mu_\alpha(-js-t)}e^{(j-k)\mu_{r(j)}s}$$
$$=e^{(\mu_q-\mu_{r(j)})(ks+x-\zeta)}.e^{(\mu_r-\mu_{r(j)})(-js-t)}.e^{\mu_{r(j)}(x-\zeta-t)}$$
$$=O(1).O(1).O\left(e^{2\rho_\alpha R}\right).$$

Thus the whole product (iv) is $O\left(\min\{1,\ |\mu_\alpha|t\}.e^{2\rho_\alpha R}Q(\mu_\alpha,s)\right)$.

$\boxed{\mathbf{2}}$ $x+t\le\zeta\le x+2s$ or $x+2s\le\zeta\le x+3s$.

If $x+t\le\zeta\le x+2s$, then in the determinants defining D we substitute $e^{\mu_\alpha\omega_p(t+x-\zeta)}$ by zero, and if $x+2s\le\zeta\le x+3s$, then we substitute $e^{\mu_\alpha\omega_p(t+x-\zeta)}$ and $e^{\mu_\alpha\omega_p(2s+x-\zeta)}$ by zeros. We only with the products which are new with respect to the case $\boxed{\mathbf{1}}$.

(i) $e^{\mu_q(ks+x-\zeta)}2(ch\mu_q t-ch\mu_\alpha t)\prod\limits_{\substack{j\neq k\\ r(j)\neq q}}e^{j\mu_{r(j)}s}$.

If $q=1$, then $e^{\mu_q(ks+x-\zeta)}(ch\mu_q t-ch\mu_\alpha t)=O\left(e^{\mu_q ks}\min\{1,\ |\mu_\alpha|t\}\right)$, hence the whole product is $O\left(\min\{1,\ |\mu_\alpha|t\}Q(\mu_\alpha s)\right)$.

If $q=3$, then there exists $j=-2$ with $r(j)\le 2$, then:

$$\begin{aligned}&e^{\mu_q(ks+x-\zeta)}e^{j\mu_{r(j)}s}(ch\mu_q t-ch\mu_\alpha t)\\ &=O\left(\min\{1,\ |\mu_\alpha|t\}e^{\mu_q(ks+x-\zeta)}e^{j\mu_{r(j)}s}\right)\\ &=O\left(\min\{1,\ |\mu_\alpha|t\}Q(\mu_\alpha s)e^{4\rho_\alpha s}\right).\end{aligned}$$

Hence the whole product is: $O\left(\min\{1,\ |\mu_\alpha|t\}Q(\mu_\alpha s)e^{4\rho_\alpha s}\right)$.

(ii) $e^{\mu_q(ks+x-\zeta)}e^{l\mu_q s}2(ch\mu_q t-ch\mu_\alpha t)\prod\limits_{\substack{j\neq k,l\\ r(j)\neq q,r}}e^{j\mu_{r(j)}s}$, where $l=2$.

Let $r=1$. Then in case Re$\mu_q\ge 0$ resp. Re$\mu_q\le 0$, we have:

$$e^{\mu_q((k+l)s+x-\zeta)}(ch\mu_r t-ch\mu_\alpha t)=O\left(\min\{1,\ |\mu_\alpha|t\}e^{k\mu_q s}.e^{l\mu_r s}\right),$$

resp. $O\left(\min\{1,\ |\mu_\alpha|t\}e^{k\mu_r s}.e^{l\mu_q s}\right)$, (here we used $ls+x-\zeta\le 0$ and $ks+x-\zeta\ge 0$). If $r=3$, then there exists $j\le -2$ with $r(j)\le 2$. Hence if Re$\mu_q\ge 0$ resp. $\mu_q\le 0$, we have:

$$\begin{aligned}&e^{\mu_q((k+l)s+x-\zeta)}(ch\mu_r t-ch\mu_\alpha t)e^{j\mu_{r(j)}s}\\ &=O\big(\min\{1,\ |\mu_\alpha|t\}e^{k\mu_q s}.e^{l\mu_r(j)s}e^{j\mu_r s}e^{3\rho_\alpha s}\big),\end{aligned}$$

resp. $O\left(\min\{1,\ |\mu_\alpha|t\}.e^{l\mu_q s+k\mu_{r(j)}s+j\mu_r s}e^{3\rho_\alpha s}\right)$, so the whole product is $O\left(\min\{1,\ |\mu_\alpha|t\}.e^{3\rho_\alpha s}Q(\mu_\alpha,s)\right)$. The estimate of D is completely proved for $x\le\zeta$. The case $x\ge\zeta$ can be dealt with similarly, so the proof of (3.8) is complete.

Now, from (3.7) we obtain:

$$\begin{aligned}&|u_\alpha(x-t)+u_\alpha(x+t)-2u_\alpha(x)ch\mu_\alpha t|\,|d(\mu_\alpha,s)|\\ &\le\sum_{\substack{-2\le k\le 3\\ |k|\ge 2}}|d_k(\mu_\alpha,s,t)|\,|u_\alpha(x+ks)|+\int_{x-2s}^{x+3s}|D(\mu_\alpha,s,t,x-\zeta)|\,|Q(\zeta)|\,d\zeta.\end{aligned}$$

Using (3.3), (3.4) and (3.8) we get:

$$\begin{aligned}
&|u_\alpha(x-t)+u_\alpha(x+t)-2u_\alpha(x)ch\mu_\alpha t|\left|\frac{d(\mu_\alpha,s)}{Q(\mu_\alpha,s)}\right| \\
&\qquad \le c\left(e^{-Re2\mu_{1,\alpha}s}+e^{Re2\mu_{3,\alpha}s}\right).\,|\mu_\alpha|\,t\|\mu_\alpha\|_{L^\infty(K_{6R})} \\
&\qquad + c\,|\mu_\alpha|^{-1}\,t.e^{|\rho_\alpha|s}\int_{x-2s}^{x+3s}|Q(\zeta)|\,d\zeta,\ \text{if } 0\le t\le \frac{1}{|\mu_\alpha|},
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&|u_\alpha(x-t)+u_\alpha(x+t)-2u_\alpha(x)ch\mu_\alpha t|\left|\frac{d(\mu_\alpha,s)}{Q(\mu_\alpha,s)}\right| \\
&\qquad \le c\left(e^{-Re2\mu_{1,\alpha}(t-2s)}+e^{|Re\mu_{3,\alpha}|(t-2s)}\right).\|\mu_\alpha\|_{L^\infty(K_{6R})} \\
&\qquad + c\,|\mu_\alpha|^{-1}\,t.e^{|\rho_\alpha|s}\int_{x-2s}^{x+3s}|Q(\zeta)|\,d\zeta,\ \text{if } t> \frac{1}{|\mu_\alpha|}.
\end{aligned} \tag{3.10}$$

If $|\mu_\alpha|>\max\left\{1,\frac{1}{R}\right\}$, then we obtain from (3.9) and (3.10):

$$\begin{aligned}
&\left|\frac{d(\mu_\alpha,s)}{Q(\mu_\alpha,s)}\right|\int_0^R\left|\frac{u_\alpha(x-t)+u_\alpha(x+t)-2u_\alpha(x)ch\mu_\alpha t}{t}\right|.dt \\
&\le c\left(e^{-Re2\mu_{1,\alpha}s}+e^{Re2\mu_{3,\alpha}s}\right).\|\mu_\alpha\|_{L^\infty(K_{6R})}+c\,|\mu_\alpha|^{-2}\,t.e^{|\rho_\alpha|s}\int_{x-2s}^{x+3s}|Q(\zeta)|\,d\zeta \\
&\quad + c\,|\mu_\alpha|\left(\frac{e^{Re\mu_{1,\alpha}(R-2s)}}{Re\mu_{1,\alpha}}+\frac{e^{Re\mu_{3,\alpha}(R-2s)}}{|Re\mu_{3,\alpha}|}\right).\|\mu_\alpha\|_{L^\infty(K_{6R})} \\
&\quad + c\,|\mu_\alpha|^{-2}\log|\mu_\alpha|.e^{|\rho_\alpha|s}\int_{x-2s}^{x+3s}|Q(\zeta)|\,d\zeta \\
&\le c\left(e^{Re\mu_{1,\alpha}(R-2s)}+e^{|Re\mu_{3,\alpha}|(R-2s)}\right).\|\mu_\alpha\|_{L^\infty(K_{6R})} \\
&\quad + c\,|\mu_\alpha|^{-2}\left(1+\log|\mu_\alpha|\right)e^{|\rho_\alpha|s}\int_{x-2s}^{x+3s}|Q(\zeta)|\,d\zeta,
\end{aligned}$$

where we used Re$\mu_{1,\alpha}\ge c_4|\mu_\alpha|$ and Re$\mu_{3,\alpha}\le -c_4|\mu_\alpha|$, ($c_4$ is a constant).

Using (3.5), we have:

$$\begin{aligned}
&\int_0^R\left|\frac{u_\alpha(x-t)+u_\alpha(x+t)-2u_\alpha(x)ch\mu_\alpha t}{t}\right|.dt \\
&\le c\left(e^{-Re\mu_{1,\alpha}s}+e^{Re\mu_{3,\alpha}s}\right).\|\mu_\alpha\|_{L^\infty(K_{6R})} \\
&+ c\,|\mu_\alpha|^{-2}\log|\mu_\alpha|e^{\rho_\alpha s}\int_{x-4R}^{x+6R}|Q(\zeta)|\,d\zeta.
\end{aligned} \tag{3.11}$$

If $|\mu_\alpha| \geq A(R_o) \geq 2$, we have from [5]:

$$\int_{x-4R}^{x+6R} |Q(\zeta)|\, d\zeta \leq c\|u_\alpha\|_{L(K_{8R})}. \tag{3.12}$$

Using (3.12) we obtain from (3.11):

$$\begin{aligned} \int_0^R \left|\frac{u_\alpha(x-t)+u_\alpha(x+t)-2u_\alpha(x)ch\mu_\alpha t}{t}\right|.dt &\leq c\left(e^{-Re\mu_{1,\alpha}s}+e^{Re\mu_{3,\alpha}s}\right) \\ \times \|\mu_\alpha\|_{L^\infty(K_{8R})} + c\frac{\log|\mu_\alpha|}{|\mu_\alpha|^2}\|\mu_\alpha\|_{L^\infty(K_{8R})}e^{2|\rho_\alpha|R}, &\text{ if } |\mu_\alpha| \geq A(R_o) \geq 2. \end{aligned} \tag{3.13}$$

We recall that $R_o \geq R \geq \frac{R_o}{2} > 0$, if we choose $R_o > 0$ such that $2R_o < \varepsilon_o$, then according to (3.6), (3.13), ([3], relation 3) and ([17], Lemma 3.2), we have for $|\mu_\alpha| \geq A(R_o) \geq 2$,

(3.14)

$$\begin{aligned} &|\langle u_\alpha, W\rangle - \delta(v, v_\alpha)u_\alpha(x)| \\ &\qquad \leq c\left(\frac{1}{1+(v-v_\alpha)^2} + e^{-Re\mu_{1,\alpha}R_o/2} + e^{Re\mu_{3,\alpha}R_o/2} + \frac{\log|\mu_\alpha|}{|\mu_\alpha|^2}\right)\|u_\alpha\|_{L^2(K)}. \end{aligned}$$

Now we prove $(a) \Rightarrow (b)$: From (3.14), assumption (3) follows

$$\begin{aligned} \sum_{\alpha=1}^{\infty} |\langle u_\alpha, W\rangle - \delta(v, v_\alpha)u_\alpha(x)\|v_\alpha\|_{L^2(G)}| &\leq c\sum_{\alpha=1}^{\infty}\frac{|\langle u_\alpha, W\rangle - \delta(v, v_\alpha)u_\alpha(x)|}{\|u_\alpha\|_{L^2(K')}} \\ &\leq c\sum_{\alpha=1}^{\infty}\left(\frac{1}{1+(v-v_\alpha)^2} + e^{-c_4|\mu_\alpha|R_o/2} + \frac{\log|\mu_\alpha|}{|\mu_\alpha|^2}\right) \leq c_5, \end{aligned}$$

where c_5 is a constant independent of v. Hence for any fixed x and v, the series:

$$F(x,y) := \sum_{\alpha=1}^{\infty}\left[\langle u_\alpha, W\rangle - \delta(v, v_\alpha)u_\alpha(x)\right]\overline{v_{\alpha(y)}}$$

is absolutely convergent in $L^2(G)$ and

$$\int_G u_\alpha(y)F(x,y)dy = \langle u_\alpha, W\rangle - \delta(v, v_\alpha)u_\alpha(x).$$

Since (u_α) is complete and minimal, therefore the Fourier expansion is unique. Hence;

$$F(x,y) = W(y) - \sum_{v_{\alpha<v}} u_\alpha(x)\overline{v_\alpha(y)} - \frac{1}{2}\sum_{v_{\alpha=v}} u_\alpha(x)\overline{v_\alpha(y)}$$

$$\sup_{v>0}\sup_{x\in K}\left\|W(y) - \sum_{v_{\alpha<v}} u_\alpha(x)\overline{v_\alpha(y)} - \frac{1}{2}\sum_{v_{\alpha=v}} u_\alpha(x)\overline{v_\alpha(y)}\right\|_{L^2(G)} = M < \infty.$$

Thus; for every $f \in L^2(G)$, we have: $\sup\limits_{v>0}\sup\limits_{x\in K}|S_v(f,x)-\sigma_v(f,x)| \leq M\|f\|_{L^2(G)}$. Now, it sufficient to show that: $\lim\limits_{v\to\infty}\sup\limits_{x\in K}|S_v(f,x)-\sigma_v(f,x)| = 0$, for any f from a dense subset of $L^2(G)$. However, this last property is satisfied for any finite linear combination f of the eigenfunction u_α (since (u_α) is a dense subspace because that f is continuously differentiable), and $\sigma_v(f,x) \equiv f(x)$, for v sufficiently large. Hence, $(a) \Rightarrow (b)$ is proved completely.

$(b) \Rightarrow (c)$: It follows from (b) that:

$$\lim_{v\to\infty}\|S_v(f)-\rho_v(f)\|_{L^2(K)} = 0.$$

On other hand it is a classical result that:

$$\lim_{v\to\infty}\|f-S_v(f)\|_{L^2(K)} = 0.$$

The statement is proved.

$(c) \Rightarrow (a)$: For every $f \in L^2(G)$, $\|\rho_v(f)\|_{L^2(G)}$, $v = 1,2,3,\cdots$, is bounded we get that for all α:

$$\|\langle f, v_\alpha\rangle u_\alpha(x)\|_{L^2(G)} \leq c\,\|f\|_{L^2(G)}\,.$$

Since $\sup\limits_{f\in L^2(G)}\frac{|\langle f,v_\alpha\rangle|}{\|f\|_{L^2(G)}} = \|v_\alpha\|_{L^2(G)}$, we obtain (a). The proof of the theorem is complete, and for more applications see [20–22]. □

4. Conclusion

Motivated by the results of the previous pieces of literature [1, 3, 7, 9], we investigated a general form of the equiconvergence theorem, using the method of V. A. Il'n [8]. It is shown that the theorem for certain situation is more general than the previous theorem. We consider the Schrödinger operator with any complex potential function $q : G \to C$ on any (finite or infinite) interval G, with arbitrary (complex) eigenvalues λ_n, see [6, 23, 24]. Finally, it is necessary to be stressed that the coefficients of the differential operators don't need to be assumed sufficiently smooth. Furthermore, there is no assumption on the distribution of the eigenvalues in the complex plane.

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