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ST-COLORING OF JOIN AND DISJOINT UNION OF GRAPHS

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ABSTRACT. A Strong T-coloring (ST-coloring) of a graph $G = (V, E)$ is a function $c : V(G) \to Z^+ \bigcup\{0\}$ such that for all $u \neq w$ in $V(G)$, if $(u, w) \in E(G)$ then $\mid c(u) - c(w) \mid \notin T$ and $\mid c(u) - c(w) \mid \neq \mid c(x) - c(y) \mid$ for any two distinct edges (u, w) and (x, y) in $E(G)$. For a ST-coloring c of the graph G, $c_{ST} - span, sp^c_{ST}(G)$ is the maximum value of $\mid c(u) - c(w) \mid$ over all the vertices of G and the minimum of $sp^c_{ST}(G)$ is denoted by $sp_{ST}(G)$, where the minimum is taken over all ST-coloring c of G. Considering the edges, the $c_{ST} - edgespan, esp^c_{ST}(G)$ is the maximum value $\mid c(u) - c(w) \mid$ over all the edges(u, w) and the minimum of $esp^c_{ST}(G)$ is defined as $esp_{ST}(G)$, where the minimum is taken over all ST-coloring c of G. In this paper, we establish some results related to ST-chromatic number, span and edge span of join and disjoint union of Graphs.

1. INTRODUCTION

Graph coloring is considered as one of the emerging topics in the field of graph theory, which has been extensively studied by the researchers. It naturally arises in Channel assignment problem [6] in the field of telecommunication.T-coloring of graph is one of the vertex colorings similar to the other types of graph colorings. It was W. K. Hale [3] who first introduced T-coloring by setting up an interrelation between graph coloring and the channel assignment problem in 1980. Extensive literature on T-colorings are found in [1, 2, 4–7, 9, 10] and the

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references therein. In 2019, Roselin et al. [8] introduced a particular type of T-coloring, named as ST-coloring. In this paper, we consider strong T-colorings on some binary operations viz., join and disjoint Union of Graphs.

The rest paper is organised as follows: Section 2 contains basic preliminaries. In section 3 and section 4 respective results on ST-coloring of disjoint union and join of graphs are presented. Finally, in section 5 a conclusion is drawn.

2. Some Preliminaries

Definition 2.1. *([8], Definition 2.1) A Strong T-coloring of* $G = (V, E)$ *is a function* $c : V(G) \to Z^+ \bigcup\{0\}$ *such that for all* $u \neq w$ *in* $V(G)$*,*

(i) $(u, w) \in E(G)$ *then* $\mid c(u) - c(w) \mid \notin T$ *and*
(ii) $\mid c(u) - c(w) \mid \neq \mid c(x) - c(y) \mid$ *for any two distinct edges* (u, w) *and* (x, y) *in* $E(G)$

Definition 2.2. *[8] For a* $ST-coloring$ *c, the* $c_{ST}-span, sp^c_{ST}(G)$ *is the maximum value* $\mid c(u) - c(w) \mid$ *over all the vertices and the minimum of* $sp^c_{ST}(G)$ *is known as* $sp_{ST}(G)$*, where the minimum is taken over all ST-coloring* c *of G.*

Definition 2.3. *[8] For a* $ST - coloring$ *c, The* $c_{ST} - edgespan, esp^c_{ST}(G)$ *is the maximum value* $\mid c(u) - c(w) \mid$ *over all the edges*(u, w) *and the minimum of* $esp^c_{ST}(G)$ *is known as* $esp_{ST}(G)$*, where the minimum is taken over all* $ST - coloring$ c *of G.*

Theorem 2.1. *([8], Theorem 3.2) Let,* H *be a subgraph of a graph G. For each finite set* T *of positive integers containing zero: (i)* $sp_{ST}(H) \leq sp_{ST}(G)$*; (ii)* $esp_{ST}(H) \leq esp_{ST}(G)$*.*

Theorem 2.2. *([8], Theorem 3.1) For all graphs G, (i)* $sp_T(G) \leq sp_{ST}(G)$*; (ii)* $esp_T(G) \leq esp_{ST}(G)$*.*

Theorem 2.3. *([8], Observation 1) For all graphs* G*,* $\chi_{ST}(G) \geq \chi(G) = \chi_T(G)$*.*

3. ST-coloring on vertex disjoint Union of two graphs

Theorem 3.1. *For any T-sets of positive integers containing zero and and for any two vertex disjoint graphs* G_1 *and* G_2

(i) $\chi_{ST}(G_1 + G_2) \geq \max\{\chi_{ST}(G_1), \chi_{ST}(G_2)\}$.
(ii) $sp_{ST}(G_1 + G_2) = \max\{sp_{ST}(G_1), sp_{ST}(G_2)\}$.
(iii) $esp_{ST}(G_1 + G_2) = \max\{esp_{ST}(G_1), esp_{ST}(G_2)\}$.

Proof. Let, the vertex sets of G_1 and G_2 be $V_1 = \{v_1, v_2, v_3, ...v_p\}$ and $V_2 = \{v_{p+1}, v_{p+2}, v_{p+3}...., v_n\}$ respectively.

(i) Since, the disjoint union of two graphs G_1 and G_2, $G_1 + G_2$ contains isomorphic subgraphs to both G_1 and G_2. Hence, by theorem (2.3), $\chi_{ST}(G_1+G_2) \geq \chi_{ST}(G_1)$ and $\chi_{ST}(G_1 + G_2) \geq \chi_{ST}(G_2)$. Thus,

$$\chi_{ST}(G_1 + G_2) \geq \max\{\chi_{ST}(G_1), \chi_{ST}(G_2)\}.$$

(ii) Let, T be a set of positive integers containing zero with k as its largest element. Since the disjoint union of two graphs G_1 and G_2, $G_1 + G_2$ contains isomorphic subgraphs to both G_1 and G_2. Hence, by invoking theorem (2.1), $sp_{ST}(G_1) \leq sp_{ST}(G_1 + G_2)$ and $sp_{ST}(G_2) \leq sp_{ST}(G_1 + G_2)$. Thus,

$$sp_{ST}(G_1 + G_2) \geq \max\{sp_{ST}(G_1), sp_{ST}(G_2)\}. \tag{3.1}$$

Let, f and g are two ST-colorings of G_1 and G_2 respectively, such that $sp^f_{ST}(G_1) = sp_{ST}(G_1)$ and $sp^g_{ST}(G_2) = sp_{ST}(G_2)$ defined as $f(v_i) = (k + 2)^{i+1}$ and $g(v_i) = (k + 2)^i$. Let, c be a coloring on $(G_1 + G_2)$ defined as

$$c(v_i) = \begin{cases} f(v_i), & if\ v_i\ \epsilon\ V(G_1) \\ g(v_i), & if v_i\ \epsilon\ V(G_2) \end{cases}.$$

If (v_i, v_j) is an edge in $(G_1 + G_2)$, then either both v_i and v_j are vertices of G_1 or both v_i and v_j are vertices of G_2. Then,

$$\begin{aligned} &\mid c(v_i) - c(v_j) \mid = \mid f(v_i) - f(v_j) \mid \leq sp_{ST}(G_1) \quad \text{or} \\ &\mid c(v_i) - c(v_j) \mid = \mid g(v_i) - g(v_j) \mid \leq sp_{ST}(G_2). \end{aligned} \tag{3.2}$$

Now, we are to show,

$$\mid c(v_i) - c(v_j) \mid \neq \mid c(v_l) - c(v_m) \mid, \tag{3.3}$$

where, $(v_i, v_j), (v_l, v_m)$ are two distinct edges of $(G_1 + G_2)$.

If (v_i, v_j) and (v_l, v_m) both are the edges of G_1 or G_2, as f and g are ST-colorings, hence equation (3.3) holds. Let, $(v_i, v_j)\epsilon E(G_1)$ and $(v_l, v_m)\epsilon E(G_2)$. Then, i, j, l, m are all distinct as the edges (v_i, v_j) and (v_l, v_m) are non adjacent and distinct. Without loss of generality, let us consider that m is the smallest and i is the largest integer. Then, we have either $m < j < l < i$ or $m < l < j < i$. Let us consider, $m < j < l < i$.

If possible equation (3.3) is not true. Then,

$$\begin{aligned} &\mid c(v_i) - c(v_j) \mid = \mid c(v_l) - c(v_m) \mid \\ \Rightarrow &\mid f(v_i) - f(v_j) \mid = \mid g(v_l) - g(v_m) \mid \\ \Rightarrow &\mid (k+2)^{i+1} - (k+2)^{j+1} \mid = \mid (k+2)^l - (k+2)^m \mid \\ \Rightarrow &(k+2)^{l-m} + (k+2)^{j-m+1} - (k+2)^{i-m+1} = 1 \end{aligned}$$

which is not true for any positive integer k, as $i - m + 1 > j - m + 1 > l - m$. Hence,

$$\mid c(v_i) - c(v_j) \mid \neq \mid c(v_l) - c(v_m) \mid$$

as $(v_i, v_j), (v_l, v_m)$ are two distinct edges.

The proof for $m < l < j < i$ is analogous to the proof for $m < j < l < i$.

Hence, c is a ST-coloring in $G_1 + G_2$.

Therefore, from equation (3.2)

$$\mid c(v_i) - c(v_j) \mid \leq \max\{sp_{ST}(G_1), sp_{ST}(G_2)\}$$

$$\Rightarrow sp_{ST}(G_1 + G_2) \leq sp^c_{ST}(G_1 + G_2) \leq \max\{sp_{ST}(G_1), sp_{ST}(G_2)\}. \tag{3.4}$$

Hence, from equations (3.1) and (3.4) we have

$$sp_{ST}(G_1 + G_2) = \max\{sp_{ST}(G_1), sp_{ST}(G_2)\}$$

(iii) By considering the edges, and proceeding in a similar manner as in case (ii), case (iii) follows. □

4. ST-CHROMATIC NUMBER OF JOIN OF TWO GRAPHS

Theorem 4.1. *Let G_1 and G_2 be any two disjoint graphs with p and q numbers of vertices respectively, then the ST- chromatic number of their join $(G_1 \vee G_2)$ is $p + q$, i.e., $\chi_{ST}(G_1 \vee G_2) = p + q$*

Proof. Let, T be a set of positive integers containing zero and k be the largest element in T. Let, $V(G_1) = \{u_1, u_2, u_3, \ldots, u_p\}$ and $V(G_2) = \{v_{p+1}, v_{p+2}, v_{p+3}, \ldots, v_{p+q}\}$ then, $V(G_1 \vee G_2) = \{u_1, u_2, u_3, \ldots, u_p, v_{p+1}, v_{p+2}, v_{p+3}, \ldots, v_{p+q}\}$.
Let, c be a coloring defined on $G_1 \vee G_2$ as:

$$c(v_i) = (K+2)^i.$$

Here, we shall prove that

$$\mid c(v_i) - c(v_j) \mid \neq \mid c(v_l) - c(v_m) \mid, \tag{4.1}$$

where, $(v_i, v_j), (v_l, v_m) \epsilon E(G_1 \vee G_2)$.

If $(v_i, v_j), (v_l, v_m)$ are adjacent then clearly equation (4.1) holds. Hence, assume that $(v_i, v_j), (v_l, v_m)$ be two non adjacent edges. Hence, i, j, l, m are distinct positive integers. Without loss of generality, let i is the largest and m is the smallest integer. Then, we have either $m < j < l < i$ or $m < l < j < i$.

Let us consider, $m < j < l < i$.

If possible equation (4.1) is not true. Then,

$$\begin{aligned} &\mid c(v_i) - c(v_j) \mid \doteq \mid c(v_l) - c(v_m) \mid \\ \Rightarrow \qquad & (k+2)^{l-m} + (k+2)^{j-m} - (k+2)^{i-m} = 1, \end{aligned}$$

which is not true for any k as $i - m > l - m > j - m$. Hence,

$$\mid c(v_i) - c(v_j) \mid \neq \mid c(v_l) - c(v_m) \mid,$$

as $(v_i, v_j), (v_l, v_m)$ are two distinct edges.

The proof for $m < l < j < i$ is analogous to the proof for $m < j < l < i$. Hence, c is a ST-coloring in $G_1 \vee G_2$.

In $G_1 \vee G_2$, each of u_i's are adjacent to all v_j's. Then for all $j = 1$ to q,

$$\begin{aligned} &\mid c(u_1) - c(v_j) \mid \neq \mid c(u_2) - c(v_j) \mid \neq \ldots.. \neq \mid c(u_p) - c(v_j) \mid \\ \Rightarrow \qquad & c(u_1) \neq c(u_2) \neq c(u_3) \neq \ldots. \neq c(u_p) \end{aligned}$$

$\Rightarrow$ all the vertices of G_1 in $G_1 \vee G_2$ will have distinct colors. Hence $'p'$ no's of colors will be required to color the vertices of G_1 in $G_1 \vee G_2$ for ST-coloring.

Similarly, for all $i = 1$ to p

$$\begin{aligned} &\mid c(u_i) - c(v_1) \mid \neq \mid c(u_i) - c(v_2) \mid \neq \ldots.. \neq \mid c(u_i) - c(v_q) \mid \\ \Rightarrow \qquad & c(v_1) \neq c(v_2) \neq c(v_3) \neq \ldots. \neq c(v_q) \end{aligned}$$

$\Rightarrow$ all the vertices of G_2 in $G_1 \vee G_2$ will have distinct positive integers (or colors). So, $'q'$ no's of positive integers (or colors) will be required to color the vertices of G_2 and $G_1 \vee G_2$ for ST-coloring.

Hence, Total numbers of colors required to color all the vertices in $G_1 \vee G_2$ is $p+q$. Hence, $\chi_{ST}(G_1 \vee G_2) = p+q$. □

Corollary 4.1. *Let, W_n be a wheel graph. For any set T, $\chi_{ST}(W_n) = n$.*

Proof. Let, T be any set of positive integers containing zero with the biggest element k. Let, v_n be its centre of the wheel W_n, whereas $v_1, v_2, v_3, \ldots . v_{n-1}$ are the vertices of the rim or the cycle C_{n-1} of the wheel W_n. Let c be a coloring of the wheel W_n defined as

$$c(v_i) = (k+2)^i.$$

Since $W_n = C_{n-1} \vee K_1$.By invoking the theorem (4.1), clearly c is a ST-coloring of W_n. Hence, $\chi_{ST}(W_n) = n-1+1 = n$. □

Corollary 4.2. *Let, $F_{m,n}$ be a fan graph. For any set T, $\chi_{ST}(F_{m,n}) = m+n$.*

Proof. Since, a fan graph $F_{m,n}$ is defined as the graph join of the null graph of m vertices $\overline{K_m}$ and path graph of n vertices P_n, i.e., $F_{m,n} = \overline{k_m} \vee P_n$. Hence, by the previous theorem (4.1), $\chi_{ST}(F_{m,n}) = \chi_{ST}(\overline{k_m} \vee P_n) = m+n$ □

5. Conclusion

In this paper, various results related to ST-chromatic number of join and disjoint unions of graphs are established. Some results related to ST-span and ST-edge span of disjoint union of various graphs are also derived. Further studies on different operations will conduit new additional insight to the ST-coloring of graph.

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