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ZERO FORCING GRAPH ASSOCIATED TO THE TOTAL GRAPH OF $\mathbb{Z}_n$ WITH RESPECT TO NIL IDEAL

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ABSTRACT. The total graph $T(\Gamma_N(\mathbb{Z}_n))$ of $\mathbb{Z}_n$ with respect to its nil ideal $N(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : x^2 \equiv 0(\text{mod } n)\}$ is a simple, undirected graph with vertex set $\mathbb{Z}_n$ and any two distinct vertices x and y of $T(\Gamma_N(\mathbb{Z}_n))$ are adjacent if and only if $x + y \in N(\mathbb{Z}_n)$. In this paper, we introduce a new graph structure called a *Zero forcing graph* of $T(\Gamma_N(\mathbb{Z}_n))$, denoted by $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$, as a simple, undirected graph in which all the possible zero forcing sets of minimum cardinality of $T(\Gamma_N(\mathbb{Z}_n))$ are taken as vertices and any two distinct vertices S_1 and S_1 of this graph are adjacent if and only if $S_1 \cup S_2 = \mathbb{Z}_n$.

1. INTRODUCTION

The idea of the total graph of a commutative ring R, denoted by $T(\Gamma(R))$, was first put forward by Anderson and Badawi [7] who defined it as a simple, undirected graph with vertex set R and any two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$, where $Z(R)$ denotes the set of all the zero-divisors of R.

In the year 2003, P. W. Chen [12] introduced a new class of a graph of a commutative ring R with vertex set R and two distinct vertices x and y are adjacent if and only if $xy \in N(R)$, where $N(R)$ denotes the set of all the nil elements of the ring R. This concept was further modified by Ai-Hua Li and

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Qi-Sheng Li [2] who defined it as an undirected, simple graph $\Gamma_N(R)$ having vertex set $Z_N(R)^* = \{x \in R^* \mid xy \in N(R) \text{ for some } y \in R^* = R - \{0\}\}$ and two distinct vertices x and y are adjacent if and only if $xy \in N(R)$ or $yx \in N(R)$.

The total graph $T(\Gamma_N(\mathbb{Z}_n))$ of the non-reduced commutative ring $\mathbb{Z}_n$ with respect to nil ideal, is a simple undirected graph with all the elements of $\mathbb{Z}_n$ as vertices and two distinct vertices x and y are adjacent if and only if $x + y \in N(\mathbb{Z}_n)$, where $N(\mathbb{Z}_n)$ denotes the set of all the nil elements of $\mathbb{Z}_n$, i.e. $N(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : x^2 = 0\}$.

The concept of a *zero forcing set* and a *zero forcing number* of a graph G, denoted by $Z(G)$, was first introduced by the "AIM Minimum Rank- Special Graphs Work Group" in [1]. One can find extensive literature on zero forcing numbers in [1,4-6,8-11]. We shall briefly discuss the concept in the following section. In this paper, we introduce a new class of graphs called *zero forcing graphs* associated to $T(\Gamma_N(\mathbb{Z}_n))$ as a simple, undirected graph in which all the possible zero forcing sets of $T(\Gamma_N(\mathbb{Z}_n))$ are taken as vertices and any two distinct vertices S_1 and S_2 are adjacent if and only if $S_1 \cup S_2 = Z_n$. We denote this graph by $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$.

2. Preliminaries

The zero forcing process on a simple undirected graph G is defined as follows: Given a subset S of G such that each vertex in S is colored black while each vertex in $G \setminus S$ is colored white, a black vertex with exactly one white neighbor will force its white neighbor to become black. It is an iterative process that continues until all the vertices of G turn black. The set S is said to be a *zero forcing set* while the minimum number of vertices in such a zero forcing set is said to be the *zero forcing number* of the graph G and is denoted by $Z(G)$.

A non-empty subset S of the set of all the vertices V of a graph G is said to be a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* γ of a graph G is defined to be the minimum cardinality of a dominating set in G and the corresponding dominating set is called a $\gamma-$set of G. A graph G is said to be *excellent* if for every vertex v of G, there exist a $\gamma-$set containing v. A set of vertices in a graph G is said to be *independent* if no two vertices in that set are adjacent. The maximum cardinality of an independent set of a graph G is called the *independence number* of the graph G and is denoted by

$\beta_0(G)$. The *independence domination number*, denoted by $i(G)$, is the minimum cardinality of an independent dominating set. A graph is said to be *well-covered* if every maximal independent set has the same size. Equivalently, a graph G is well-covered if $\beta_0(G) = i(G)$. The partition of the vertex set V(G) of a graph G into dominating sets is called a *domatic partition* of G. The maximum number of such partitions is called the *domatic number* of G and is denoted by $d(G)$. A graph G is said to be *domatically full* if $d(G) = \delta(G) + 1$. For any graph G, the diameter of G, denoted by $diam(G)$ is given by $diam(G) = sup\{d(x, y)$: where x and y are distinct vertices of G$\}$ and $d(x, y)$ is the length of the shortest path joining x and y. The *girth* of the graph G, denoted by $gr(G)$, is the length of the shortest cycle in G. If G contains no cycles, then $gr(G) = \infty$. The graph G is said to be *Eulerian* if and only if the degree of each of its vertices is even. A graph is said to be *planar* if it can be drawn in a plane without any two edges intersecting each other. The *clique number* of a graph G, denoted by ω(G), is the size of the largest complete subgraph of G. The *chromatic number* of a graph G, denoted by $\chi(G)$ is the least number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices are assigned the same color.

A ring R is said to be *non-reduced* if it contains at least one non-zero nil element. Otherwise it is said to be *reduced*.

3. THE BASIC STRUCTURE OF $T(\Gamma_N(\mathbb{Z}_n))$

For any non-reduced $\mathbb{Z}_n$, the total graph $T(\Gamma_N(\mathbb{Z}_n))$ of $\mathbb{Z}_n$ with respect to its nil ideal $N(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : x^2 \equiv 0 \pmod{n}\}$ is a simple, undirected graph having vertex set $\mathbb{Z}_n$ and any two distinct vertices x and y of $T(\Gamma_N(\mathbb{Z}_n))$ are adjacent if and only if $x + y \in N(\mathbb{Z}_n)$.

Theorem 3.1. *[3] Let $\mathbb{Z}_n$ be non-reduced and let n_1 be the smallest non-zero nil element of $\mathbb{Z}_n$. Then the following conditions hold:*

(1) If $|\mathbb{Z}_n|$ is odd, then $T(\Gamma_N(\mathbb{Z}_n)) = K_{\frac{n}{n_1}} \cup (\frac{n_1-1}{2})K_{\frac{n}{n_1},\frac{n}{n_1}}$.

(2) If $|\mathbb{Z}_n|$ is even, then $T(\Gamma_N(\mathbb{Z}_n)) = 2K_{\frac{n}{n_1}} \cup (\frac{n_1}{2} - 1)K_{\frac{n}{n_1},\frac{n}{n_1}}$.

4. Zero forcing set of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$

The zero forcing graph $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ of the total graph $T(\Gamma_N(\mathbb{Z}_n))$ is a simple, undirected graph in which all the possible zero forcing sets of $T(\Gamma_N(\mathbb{Z}_n))$ are taken as vertices and any two distinct vertices S_1 and S_2 are adjacent if and only if $S_1 \cup S_2 = \mathbb{Z}_n$.

Lemma 4.1. *From the proof of Theorem 3.1 [3], it is easy to observe that for any odd n, each zero forcing set of $T(\Gamma_N(\mathbb{Z}_n))$ contains $(\frac{n}{n_1} - 1)$ elements of $N(\mathbb{Z}_n)$, $(\frac{n}{n_1} - 1)$ elements of each of the cosets $i + N(\mathbb{Z}_n)$ for $i = 1, 2, ..., \frac{n_1-1}{2}$ and $(\frac{n}{n_1} - 1)$ elements of each of the cosets $(n-i)+N(\mathbb{Z}_n)$. Also for any even n, each zero forcing set of $T(\Gamma_N(\mathbb{Z}_n))$ contains $(\frac{n}{n_1}-1)$ elements of $N(\mathbb{Z}_n)$, $(\frac{n}{n_1}-1)$ elements of the coset $\frac{n_1}{2} + N(\mathbb{Z}_n)$, $(\frac{n}{n_1} - 1)$ elements of each of the cosets $i + N(\mathbb{Z}_n)$ for $i = 1, 2, ..., \frac{n_1}{2} - 1$ and $(\frac{n}{n_1} - 1)$ elements of each of the cosets $(n - i) + N(\mathbb{Z}_n)$.*

Throughout this paper, we denote $|N(\mathbb{Z}_n)|$ interchangeably by α and $\frac{n}{n_1}$ where n_1 is the smallest non-zero nil element of $\mathbb{Z}_n$. That is, $|N(\mathbb{Z}_n)| = \alpha = \frac{n}{n_1}$. Also, we use the notation β to denote the cardinality of the set $\mathbb{Z}_n \setminus N(\mathbb{Z}_n)$.

5. Graphical properties of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$

Theorem 5.1. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, $Z(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = n(1 - \frac{1}{\alpha})$.*

Proof. Let us consider the following two cases:

Case 1: When n is odd.

For any odd n, since $T(\Gamma_N(\mathbb{Z}_n)) = K_\alpha \cup (\frac{\beta}{2\alpha})K_{\alpha,\alpha}$, so the number of vertices in a zero forcing set of $T(\Gamma_N(\mathbb{Z}_n))$ of minimum cardinality $= (\alpha-1)+(2\alpha-2).(\frac{\beta}{2\alpha})$ $= \alpha - 1 + \beta - \frac{\beta}{\alpha} = \alpha - 1 + n - \alpha - (\frac{n}{\alpha} - 1) = n(1 - \frac{1}{\alpha})$.

Case 2: When n is even.

For any even n, since $T(\Gamma_N(\mathbb{Z}_n)) = 2K_\alpha \cup (\frac{\beta-\alpha}{2\alpha})K_{\alpha,\alpha}$, so the number of vertices in a zero forcing set of $T(\Gamma_N(\mathbb{Z}_n))$ of minimum cardinality $= (\alpha - 1) + (\alpha - 1) + (2\alpha - 2).(\frac{\beta-\alpha}{2\alpha}) = 2(\alpha - 1) + (\alpha - 1).(\frac{n}{\alpha} - 2) = 2(\alpha - 1) + n - 2\alpha - \frac{n}{\alpha} + 2$ $= 2\alpha - 2 + n - 2\alpha - \frac{n}{\alpha} + 2 = n(1 - \frac{1}{\alpha})$.

In both the cases, $Z(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = n(1 - \frac{1}{\alpha})$. □

Theorem 5.2. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, the number of zero forcing sets of $T(\Gamma_N(\mathbb{Z}_n))$ of minimum cardinality $=$ the total number of vertices of $=$ $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) = (\alpha)^{\frac{n}{\alpha}}$.*

Proof. Here again, let us consider the following two possible cases:

Case 1: When n is odd.

For any odd value of n, since $T(\Gamma_N(\mathbb{Z}_n)) = K_\alpha \cup (\frac{\beta}{2\alpha})K_{\alpha,\alpha}$, so the total number of zero forcing sets of $T(\Gamma_N(\mathbb{Z}_n))$ of minimum cardinality $={}^\alpha C_{\alpha-1} \times \underbrace{{}^\alpha C_{\alpha-1} \times {}^\alpha C_{\alpha-1}}_{\frac{\beta}{2\alpha}} = \alpha(\alpha^2)^{\frac{\beta}{2\alpha}} = (\alpha)^{\frac{n}{\alpha}}$.

Case 2: When n is even.

For any even value of n, since $T(\Gamma_N(\mathbb{Z}_n)) = 2K_\alpha \cup (\frac{\beta-\alpha}{2\alpha})K_{\alpha,\alpha}$, so the total number of zero forcing sets of $T(\Gamma_N(\mathbb{Z}_n))$ of minimum cardinality $={}^\alpha C_{\alpha-1} \times {}^\alpha C_{\alpha-1} \times \underbrace{{}^\alpha C_{\alpha-1} \times {}^\alpha C_{\alpha-1}}_{\frac{\beta-\alpha}{2\alpha}} = \alpha^2(\alpha^2)^{\frac{\beta-\alpha}{2\alpha}} = (\alpha)^{\frac{n}{\alpha}}$.

In both the cases, the total number of vertices of $= \mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) = (\alpha)^{\frac{n}{\alpha}}$. □

Theorem 5.3. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, $deg(u) = (\alpha-1)^{\frac{n}{\alpha}}$, $\forall$ $u \in V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))$.*

Proof. Let us consider the following two cases:

Case 1: When n is odd.

Let $u \in \mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$. This vertex will be adjacent to all those vertices $v \in \mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ that contain the elements of $\mathbb{Z}_n$ excluded from u. The number of choices for such $v = (\alpha-1) \times \underbrace{(\alpha-1)^2 \times (\alpha-1)^2 \times ... \times (\alpha-1)^2}_{\frac{\beta}{2\alpha}}$

$= (\alpha-1) \times \{(\alpha-1)^2\}^{\frac{\beta}{2\alpha}} = (\alpha-1) \times (\alpha-1)^{\frac{\beta}{\alpha}} = (\alpha-1)^{\frac{\alpha+\beta}{\alpha}} = (\alpha-1)^{\frac{n}{\alpha}}$.

Case 2: When n is even.

Here again, any vertex w of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$, is adjacent to all those vertices that contain the elements missing in w. The number of choices for such a vertex $= (\alpha-1)\times(\alpha-1)\times\underbrace{(\alpha-1)^2 \times (\alpha-1)^2 \times ... \times (\alpha-1)^2}_{\frac{\beta-\alpha}{2\alpha}} = (\alpha-1)^2\times\{(\alpha-1)^2\}^{\frac{\beta-\alpha}{2\alpha}} =$

$(\alpha-1)^2 \times (\alpha-1)^{\frac{\beta-\alpha}{\alpha}} = (\alpha-1)^{2+\frac{\beta-\alpha}{\alpha}} = (\alpha-1)^{\frac{\alpha+\beta}{\alpha}} = (\alpha-1)^{\frac{n}{\alpha}}$.

So in both cases, $dim(u) = (\alpha-1)^{\frac{n}{\alpha}}$, $\forall$ $u \in V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))$. □

Corollary 5.1. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, the total number of edges in $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) = \frac{(\alpha)^{\frac{n}{\alpha}}(\alpha-1)^{\frac{n}{\alpha}}}{2}$.*

The proof follows directly from theorem 5.2, theorem 5.3 and the Sum of Degree theorem of graphs.

Theorem 5.4. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, let S_1 and S_2 be two vertices of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ such that $S_2 = S_1 + r$, for some $r(\neq 0) \in N(\mathbb{Z}_n)$. Then S_1 is adjacent to S_2.*

Proof. Let $S_2 = S_1 + r$. Let $S_1 = \{x_1, x_2, ..., x_\psi\}$ and $S_2 = \{x_1+r, x_2+r, ..., x_\psi+r\}$, where $\psi = n(1 - \frac{1}{\alpha})$. Then

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|$$
$$\Rightarrow |S_1 \cup S_2| = n(1 - \tfrac{1}{\alpha}) + n(1 - \tfrac{1}{\alpha}) - n(1 - \tfrac{2}{\alpha})$$
$$\Rightarrow |S_1 \cup S_2| = 2n(1 - \tfrac{1}{\alpha}) - n(1 - \tfrac{1}{\alpha})$$
$$\Rightarrow |S_1 \cup S_2| = n.$$
$$\Rightarrow S_1 \cup S_2 = \mathbb{Z}_n$$

and so by definition, S_1 and S_2 are adjacent. □

Theorem 5.5. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$,*

$$\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) > \gamma(T(\Gamma_N(\mathbb{Z}_n))).$$

Proof. By Theorem 4.1 [3], $\gamma(T(\Gamma_N(\mathbb{Z}_n))) = n_1 = \frac{n}{\alpha}$.

If possible, let $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) \leq \gamma(T(\Gamma_N(\mathbb{Z}_n)))$.

$\Rightarrow n(1 - \frac{1}{\alpha}) \leq \frac{n}{\alpha}$

$\Rightarrow 1 - \frac{1}{\alpha} \leq \frac{1}{\alpha}$

$\Rightarrow \alpha \leq 2$, a contradiction.

Therefore $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) > \gamma(T(\Gamma_N(\mathbb{Z}_n)))$. □

Theorem 5.6. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$,*

(i) $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is not a cycle.

(ii) $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is not a complete bipartite graph.

Proof.

(i) Let $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ be a cycle. Then $deg(S) = 2 \, \forall \, S \in V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))$. But this is not possible for any value of n and α by theorem 5.3. Therefore $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is not a cycle.

(ii) By theorem 5.12 (ii) (towards the end of this paper), since $gr(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 3$ for $\alpha > 2$, so $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ can never be a complete bipartite graph. □

Theorem 5.7. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$,*

(i) $\beta_0(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = (\alpha)^{\frac{n}{\alpha}-1}$.

(ii) $\gamma(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = (\alpha)^{\frac{n}{\alpha}-1}$.

(iii) *The total number of $\gamma-$sets and maximum independent sets of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) = \alpha$.*

(iv) *$\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is an excellent graph.*

Proof.

(i) Let $A_{\alpha-1}$ be the collection of all those vertices of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ that contain the same $(\alpha - 1)$ nil elements of $\mathbb{Z}_n$, leaving out the nil element x_1, say. Then $|A_{\alpha-1}| = (\alpha)^{\frac{n}{\alpha}-1}$ and by definition, $A_{\alpha-1}$ is an independent set. Also, any vertex outside $A_{\alpha-1}$ containing x_1 is adjacent to at least one of the vertices in $A_{\alpha-1}$. Therefore A_α is the independent set of maximum cardinality and so $\beta_0(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = (\alpha)^{\frac{n}{\alpha}-1}$.

(ii) Since the collection $A_{\alpha-1}$ of sets as seen in (i) is also a dominating set of minimum cardinality, so $\gamma(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = (\alpha)^{\frac{n}{\alpha}-1}$.

(iii) From (i) and (ii), the total number of $\gamma-$sets and maximum independent sets of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) = \frac{(\alpha)^{\frac{n}{\alpha}}}{(\alpha)^{\frac{n}{\alpha}-1}} = \alpha$.

The result (iv) is obvious since each and every vertex of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is a part of a $\gamma-$set of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$. □

Theorem 5.8. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, the graph $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is never complete.*

Proof. Let $\alpha > 2$. Then $(\alpha - 1)^{\frac{n}{\alpha}} - 1 < (\alpha)^{\frac{n}{\alpha}} \Rightarrow (\alpha - 1)^{\frac{n}{\alpha}} < (\alpha)^{\frac{n}{\alpha}} - 1 \Rightarrow deg(S) < |V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))| - 1, \forall\, S \in V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))$. Therefore $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is not a complete graph.
Let $\alpha = 2$. Then $deg(S) = (\alpha - 1)^{\frac{n}{\alpha}} = 1 < 2^{\frac{n}{2}} - 1 = |V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))| - 1$. Therefore $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is not complete. □

Theorem 5.9. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is Eulerian if and only if n is odd.*

Proof. The graph $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is Eulerian if and only if the degree of each of its vertices is even $\Leftrightarrow (\alpha - 1)^{\frac{n}{\alpha}}$ is even $\Leftrightarrow \alpha$ is odd $\Leftrightarrow n$ is odd. □

Theorem 5.10. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, $\omega(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = \alpha$.*

Proof. For any vertex $v \in V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))$, the vertices $v + N(\mathbb{Z}_n)$ are all adjacent to each other. Since $|N(\mathbb{Z}_n)| = \alpha$, so these vertices form the complete subgraph $K_\alpha = A$(say). Let r_1 and r_2 be two non-zero nil elements of $\mathbb{Z}_n$. Let $S_1 = \{x_1, x_2, x_3..., x_\psi\} \in A$ and let $S_2 = \{x_1 + r_1, x_2 + r_1, x_3 + r_1, ..., x_{\frac{n}{\alpha}-1} + r_1, x_{\frac{n}{\alpha}}, ..., x_\psi\} \in V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) - A$, where $\psi = n(1 - \frac{1}{\alpha})$. Then $\exists$ a vertex $S_3 = \{x_1 + (r_2 - r_1), x_2 + (r_2 - r_1), ..., x_{\frac{n}{\alpha}-1} + (r_2 - r_1), x_{\frac{n}{\alpha}} + r_2..., x_\psi + r_2\} \in A$ such that S_2 is adjacent to S_3 but not to S_1. This means that no vertex outside the complete subgraph A is adjacent to all the vertices of A. Therefore K_α is the largest complete subgraph of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ and so $\omega(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = \alpha$. □

Theorem 5.11. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, $\chi(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = \alpha$.*

Proof. Since the collection $A_{\alpha-1}$ of vertices given in theorem 5.7 (i) is a maximum independent set with $|A_{\alpha-1}| = (\alpha)^{\frac{n}{\alpha}-1}$ and since $A_{\alpha-1}$ is arbitrary, so $\chi(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) \leq \frac{|V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))|}{(\alpha)^{\frac{n}{\alpha}-1}} = \frac{(\alpha)^{\frac{n}{\alpha}}}{(\alpha)^{\frac{n}{\alpha}-1}} = \alpha$. Also since for any graph X, $\chi(X) \geq \omega(X)$, so from theorem 4.3.3, $\chi(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = \alpha$. □

Corollary 5.2. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$,*

$$\omega(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = \chi(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))).$$

Theorem 5.12. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$,*
(i) $diam(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 2$. Equivalently $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is connected.
(ii) $gr(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 3$.

Proof.
(i) Let $S_1, S_2 \in V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))$. If $S_1 \cup S_2 = \mathbb{Z}_n$, then S_1 is adjacent to S_2 and so $d(S_1, S_2) = 1$. Let $S_1 = \{x_1, x_2, ..., x_\psi\}$ and $S_2 = \{x_1+r_1, x_2, ..., x_\psi\}$, where $\psi = n(1 - \frac{1}{\alpha})$ and $r_1(\neq 0) \in N(\mathbb{Z}_n)$. Since $|S_1 \cup S_2| = 2n(1 - \frac{1}{\alpha}) - \{n(1 - \frac{1}{\alpha}) - 1\}$ $= n - \frac{n}{\alpha} + 1 < n$ for any $\alpha > 2$, so S_1 is not adjacent to S_2. But there exists a vertex $S_3 = \{x_1 + (r_2 - r_1), x_2 + r_2, ..., r_\psi + r_2\} \in V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))$ for some $r_2(\neq 0) \in N(\mathbb{Z}_n)$ such that $S_1 - -S_3 - -S_2$ is a 2-path in $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$. Therefore $diam(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 2$.
(ii) By theorem 5.10, for any $\alpha > 2$, since $\omega(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = \alpha > 2$, so the graph $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ contains a triangle $\forall\ \alpha > 2$ and therefore $gr(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 3$. □

A corollary of Euler's polyhedron formula tells us that a planar graph G with vertex set $V(G)$ can have at most $3|V(G)|-6$ edges.

We shall use the contrapositive statement of this corollary to prove our next result.

Theorem 5.13. *For any non-reduced $\mathbb{Z}_n$ with $\alpha > 2$, $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is not a planar graph.*

Proof. From corollary 5.1, the number of edges of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) = \frac{(\alpha)^{\frac{n}{\alpha}}(\alpha-1)^{\frac{n}{\alpha}}}{2}$.
We have
$3|V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))|-6-\frac{(\alpha)^{\frac{n}{\alpha}}(\alpha-1)^{\frac{n}{\alpha}}}{2} = \frac{(\alpha)^{\frac{n}{\alpha}}[6-36-(\alpha)^{\frac{n}{\alpha}}(\alpha-1)^{\frac{n}{\alpha}}]}{2} = \frac{-(\alpha)^{\frac{n}{\alpha}}[30+(\alpha-1)^{\frac{n}{\alpha}}]}{2} < 0$
$\Rightarrow 3|V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))|-6 <$total number of edges of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$. So by the corollary of Euler's polyhedron formula, $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is not planar. □

Theorem 5.14. *For any non-reduced $\mathbb{Z}_n$, if $\alpha = 2$, then $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is the disjoint union of $(2)^{\frac{n}{2}-1}$ copies of K_2's.*

Proof. It follows from theorem 5.3 that for $\alpha = 2$, the degree of each vertex of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is one. So for each vertex $u \in \mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$, $\exists$ a unique vertex $v \in \mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ such that u is adjacent to v. Consequently $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is the disjoint union of $\frac{|V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))|}{2}$, $i.e.$ $(2)^{\frac{n}{2}-1}$ K_2's. □

Corollary 5.3. *For any non-reduced $\mathbb{Z}_n$, if $\alpha = 2$, then $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ is a forest.*

Example 1. *Figure 1 represents the zero forcing graph associated to the total graph of $\mathbb{Z}_8$ (where $\alpha = 2$) with respect to its nil ideal $\{0,4\}$.*

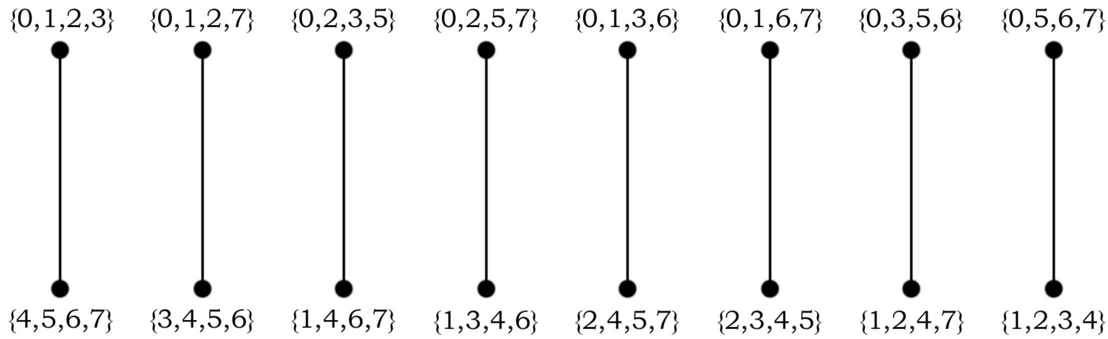

FIGURE 1. $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_8)))$

The following results follow directly from theorem 5.16.

Corollary 5.4. *For any non-reduced $\mathbb{Z}_n$ with $\alpha = 2$,*
(i) $deg(S) = 1 \ \forall \ S \in \mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$.
(ii) $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ *is never Eulerian.*
(iii) $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ *is always planar.*
(iv) $\gamma(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = Z(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 2^{\frac{n}{2}-1}$.
(v) $d(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 2$.
(vi) $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ *is domatically full.*
(vii) $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ *is excellent.*
(viii) $diam(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 1$.
(ix) $gr(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = \infty$.

Proof.
(iii) follows from corollary 5.3 since every forest is a planar graph.
(iv) For any non-reduced $\mathbb{Z}_n$ with $\alpha = 2$, since $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))) = \underbrace{K_2 \cup K_2 \cup ... \cup K_2}_{(2)^{\frac{n}{2}-1}}$, so any $\gamma-$set of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ has cardinality $\underbrace{1 + 1 + \cdots, +1}_{(2)^{\frac{n}{2}-1}} = (2)^{\frac{n}{2}-1}$.
Therefore $\gamma(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = (2)^{\frac{n}{2}-1}$. Since the same set is also a zero forcing set, so $Z(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = (2)^{\frac{n}{2}-1}$.
(v) $d(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = \frac{|V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))|}{\gamma(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))} = 2$.
(vi) Since $\delta(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 1$ and $d(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) = 2 = \delta(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))) + 1$, the result is obvious.
The proofs of (vii), (viii) and (ix) are trivial. □

Theorem 5.15. *For any non-reduced $\mathbb{Z}_n$ with $\alpha = 2$, let $S \in V(\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n))))$. Then S is a zero forcing set of minimum cardinality if and only if $x_i + N(\mathbb{Z}_n)$ form distinct cosets of $\frac{\mathbb{Z}_n}{N(\mathbb{Z}_n)}$ for each $x_i \in S$.*

Proof. By Corollary 5.4 (iv), since the minimum zero forcing sets of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$ are also $\gamma-$sets of $\mathcal{ZF}(T(\Gamma_N(\mathbb{Z}_n)))$, the result follows from theorem 4.3 [3]. □

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