ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.11, 9463–9480
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.11.50

FUZZY CONJUNCTIVE GRAMMAR AND FUZZY SYNCHRONIZED ALTERNATING PUSHDOWN AUTOMATA

R. PATHRAKUMAR[1] AND M. RAJASEKAR

ABSTRACT. Inspired by synchronized alternating pushdown automata (SAPDA) in crisp case, we study the fuzzy synchronized alternating pushdown automata (FSAPDA), which accept the fuzzy conjunctive language (FCL) generated by fuzzy conjunctive grammar (FCG).

1. INTRODUCTION

Conjunctive grammars (CG) was introduced by Okhotin [23]. Conjunctive grammars can be naturally considered as an extension of context-free grammars equipped with an explicit intersection operation. The generative capacity of conjunctive grammars covers some important non context-free language constructs, such as $\{a^n b^n c^n / n \geq 0\}, \{a^m b^n c^m d^n / m, n \geq 0\}$ and $\{wcw / w \in \{a, b\}^*\}$,where the latter is known to be not in the intersection closure of context-free languages [28]. The language of all computations of any given Turing machine is also known to be conjunctive, which has certain implications on undecidability and descriptional complexity [23].

Alternating automata models were first introduced by Chandra, et.al [7]. In these models, computations alternate between existential and universal modes of acceptance. Thus, for a word to be accepted it must meet both disjunctive

---

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 05C78.

Key words and phrases. Fuzzy context-free grammar, Fuzzy conjunctive grammar, Fuzzy pushdown automata, fuzzy synchronized alternating pushdown automata.

and conjunctive conditions. In the case of alternating finite state automata and alternating Turing machines, the alternating models have been shown to be equivalent in expressive power to their non-alternating counterparts [7].

Alternating pushdown automata (APDA) were also considered in [7] and were further explored in [17]. Like conjunctive grammars, APDA add the power of intersection over context-free languages. Therefore, in contrast with finite automata and Turing machines, here alternation increases the expressiveness of the model. In fact, APDA accept exactly the exponential-time languages, [17], and are thus too strong to be a counterpart for CG.

A synchronized version of alternating finite state automata was introduced in [33] and further explored and refined in [13]. Synchronized alternating pushdown automata (SAPDA) was introduced by Aizikowitz and Kaminski [1]. The correspondence between conjunctive grammars and synchronized alternating pushdown automata shown by Aizikowitz and Kaminski in [4].

The mathematical formulation of fuzzy automata was proposed by Wee in 1967 [31]. Thereafter, there were a considerable number of authors having contributed to this field such as Mordeson, Malik [22], and others [6, 14, 19, 21, 24–26, 29, 32, 34]. Notably, fuzzy automata have been used to wast water treatment, color sequence of eye detection, fire-flame detection, VHDL-framework modeling [8, 11, 27, 30]. In recent years their application have been further extended to include parallel processing, image generation and compression, type theory for object-oriented languages, DNA computing, etc.

Fuzzy grammars and fuzzy languages were first discussed by Lee and Zadeh [15]. Many researchers have then contributed in developing the fuzzy theory of grammars from different angles [5, 9, 10, 19, 20]. Lee and Zadeh have proved that the context sensitive fuzzy grammar is always recursive. They have also obtained the Chomsky and the Greibach normal form for a given fuzzy context free grammar. Gerla [9] established that, a fuzzy language is recursive if and only if it is generated by a fuzzy grammar. Generalization of fuzzy grammar namely L-grammars was studied mainly by Li and Pedrycz [18]. Lan and Zhiwen [16] have discussed a relationship between a fuzzy grammar and a fuzzy finite automaton. Malik and Mordeson [16] have introduced a max-min fuzzy language and find a fuzzy automaton that generates this language. Asveld [2,3] generalized fuzzy context free grammar and used them to model grammatical errors occurs in robust recognizing and parsing algorithm.

The remainder of this paper is organized as follows: In Setion 2 we study the fuzzy conjunctive grammars (FCG) and their languages. In Setion 3 we study the fuzzy synchronized alternating pushdown automata (FSAPDA) and their languages. In Setion 4 our main aim is to study the equivalence between FSAPDA and FCG: theoretically we present this fact from FCG to FSAPDA and from FSAPDA to FCG.

2. Fuzzy Conjunctive Grammar

Fuzzy conjunctive grammars can be naturally considered as an extension of fuzzy context-free grammars equipped with an explicit intersection operation.

Definition 2.1. *A fuzzy conjunctive grammar is a quadruple $G = (V, \Sigma, P, S)$, where*

(1) V *is a finite set of non-terminal symbol(Variables)*
(2) Σ *is a finite set of terminal symbols disjoint from V*
(3) $S \in V$ *is the designated start symbols*
(4) P *is a finite set of production rules of the form*

$$A \xrightarrow{r} (\alpha_1/r_1\& \dots \&\alpha_n/r_n),$$

where $A \in V$, $\alpha_i \in (V \cup \Sigma)^*$, *for* $i = 1, 2, \dots, n$ *and* $r = \min\{r_1, \dots, r_n\}$, $r \in (0, 1]$. *If* $n = 1$ *we write it as* $A \xrightarrow{r} \alpha_1$ *and call it an fuzzy context-free grammar.*

Definition 2.2. *Let {G=(V, Σ, P, S)} be a fuzzy conjunctive grammar. Then the set*

$$\mu(L, G) = \{(w, \rho)|w \in \Sigma^*, \rho = \max\{r|S \stackrel{r}{\Longrightarrow} (w/r_1\& \dots \&w/r_n)\}\}$$

is called fuzzy conjunctive language generated by G with membership grade ρ, where "max" is taken over all derivation chains of S to w and $r \in (0, 1]$.

Example 1. *The following fuzzy conjunctive grammar $G = (V, \Sigma, P, S)$ generates the fuzzy non context-free language $\{(a^n b^n c^n, r)|n = 1, 2, \dots\}$, where*

(1) $V = \{S, A, B, C, D\}$,
(2) $\Sigma = \{a, b, c\}$, *and*

(3) *The production* P *defined as*

$$\begin{aligned} S &\xrightarrow{0.4} (A/0.7\&C/0.4),\\ A &\xrightarrow{0.1} aA,\\ A &\xrightarrow{0.2} B,\\ B &\xrightarrow{0.3} bBc,\\ B &\xrightarrow{0.4} \epsilon,\\ C &\xrightarrow{0.5} Cc,\\ C &\xrightarrow{0.6} D,\\ D &\xrightarrow{0.7} aDb,\\ D &\xrightarrow{0.8} \epsilon. \end{aligned}$$

The word $w = aabbcc$ *is generated by FCG through the following chain of derivations:*

$$\begin{aligned} S &\overset{0.4}{\Longrightarrow^*} (A/0.7\&C/0.4)\\ &\overset{0.1}{\Longrightarrow^*} (aA/0.1\&Cc/0.5)\\ &\overset{0.1}{\Longrightarrow^*} (aaA/0.1\&Ccc/0.5)\\ &\overset{0.2}{\Longrightarrow^*} (aaB/0.2\&Dcc/0.6)\\ &\overset{0.3}{\Longrightarrow^*} (aabBc/0.3\&aDbcc/0.7)\\ &\overset{0.3}{\Longrightarrow^*} (aabbBcc/0.3\&aaDbbcc/0.7)\\ &\overset{0.4}{\Longrightarrow^*} (aabbcc/0.4\&aabbcc/0.8)\\ S&\overset{r=0.1}{\Longrightarrow} aabbcc, \end{aligned}$$

where $r = \min\{0.4, 0.1, 0.1, 0.2, 0.3, 0.3, 0.4\} = 0.1$ *and* $r \in (0, 1]$.

Hence, $(aabbcc, 0.1) \in \mu(L, G)$*, it means that, the word* $aabbcc$ *generated by* G *with membership grade* 0.1*. Therefore,* $\mu(L, G) = \{(a^n b^n c^n, 0.1) | n = 1, 2, \dots\}$.

3. Fuzzy Synchronized Alternating Pushdown Automata

A fuzzy synchronized alternating pushdown automata can be naturally considered as an extension of fuzzy pushdown automata equipped with an explicit intersection operation.

Definition 3.1. *A fuzzy synchronized alternating pushdown automata is a septuple $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where*

(1) *Q is a finite set of states,*
(2) *Σ is a input alphabet,*
(3) *Γ is the stack alphabet,*
(4) *$q_0 \in Q$ is the initial state,*
(5) *$Z_0 \in \Gamma$ is the initial stack symbol,*
(6) *$F \subseteq Q$ is a final state,*
(7) *δ is mapping from $Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma$ to finite fuzzy subset of $((q_1, \alpha_1)/r_1 \wedge \cdots \wedge (q_n, \alpha_n)/r_n)/r$, where $r = \min\{r_1, \ldots, r_n\}$ and $r \in (0, 1]$;*
If $n = 1$, then FSAPDA is called a standard FPDA.

Definition 3.2. *Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ be an FSAPDA. We define the relation yields in one step on the configurations of M, denoted by $\overset{r}{\vdash}$, as follows.*

(1) *If $\delta(q, \sigma, Z) = (q_1, \gamma_1)/r$, then $(q, \sigma\omega, Z\gamma) \overset{r}{\vdash}{}^* (q_1, \omega, \gamma_1\gamma)$,*
(2) *If $\delta(q, \sigma, Z) = ((q_1, \gamma_1)/r_1 \wedge \cdots \wedge (q_n, \gamma_n)/r_n)/r$, then $(q, \sigma\omega, Z\gamma) \overset{r}{\vdash}{}^* (((q_1, w, \gamma_1)/r_1 \wedge \cdots \wedge (q_n, \omega, \gamma_n)/r_n)\gamma)$, $n \geq 2$, and*
(3) *$(((q, \omega, \epsilon)/r_1 \wedge \cdots \wedge (q, w, \epsilon)/r_n)\gamma) \overset{r}{\vdash}{}^* (q, \omega, \gamma)$,*

where $\sigma \in \Sigma \cup \{\epsilon\}$, $\gamma \in \Gamma^*$, $r = \min\{r_1, \ldots, r_n\}$, and $r \in (0, 1]$.

Definition 3.3. *Let $M = (Q, \Sigma, \Gamma, \delta, q_0, X_0, F)$ be an FAPDA, then the language accepted by M, by final state is defined as*

$$\mu(L, M) = \{(w, \rho) | w \in \Sigma^*, \rho = \max\{r | (q_0, w, Z_0) \overset{r}{\vdash}{}^* ((q, \epsilon, \gamma_1)/r_1 \wedge \ldots \wedge(q, \epsilon, \gamma_n)/r_n), q \in F\}\},$$

and another way defined by empty stack as

$$\mu(L, M) = \{(w, \rho) | w \in \Sigma^*, \rho = \max\{r | (q_0, w, Z_0) \overset{r}{\vdash}{}^* ((q, \epsilon, \epsilon)/r_1 \wedge \ldots \wedge(q, \epsilon, \epsilon)/r_n), q \in Q\}\},$$

for any input string $w \in \Sigma^*$.

Example 2. *The FSAPDA* $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ *defined below accepts the fuzzy non-context free language* $\{(|w|_a = |w|_b = |w|_c, r)|w \in \Sigma\}$*, where*

(1) $Q = \{q_0, q_1, q_2\}$,
(2) $\Sigma = \{a, b, c\}$,
(3) $\Gamma = \{a, b, c, Z_0\}$, *and*
(4) δ *is defined as*

$$\begin{aligned}
\delta(q_0, \epsilon, Z_0) &= \{((q_1, Z_0)/0.4 \wedge (q_2, Z_0)/0.7)/0.4\},\\
\delta(q_1, \sigma, Z_0) &= (q_1, \sigma Z_0)/0.1,\ \sigma \in \{a, b\}\\
\delta(q_2, \sigma, Z_0) &= (q_2, \sigma Z_0)/0.2,\ \sigma \in \{b, c\}\\
\delta(q_1, \sigma, \sigma) &= (q_1, \sigma\sigma)/0.3,\ \sigma \in \{a, b\}\\
\delta(q_2, \sigma, \sigma) &= (q_2, \sigma\sigma)/0.4,\ \sigma \in \{b, c\}\\
\delta(q_1, \sigma', \sigma'') &= (q_1, \epsilon)/0.1,\ (\sigma', \sigma'') \in \{(a, b), (b, a)\}\\
\delta(q_2, \sigma', \sigma'') &= (q_2, \epsilon)/0.2,\ (\sigma', \sigma'') \in \{(b, c), (c, b)\}\\
\delta(q_1, c, X) &= (q_1, X)/0.5,\ X \in \{Z_0, a, b\}\\
\delta(q_2, a, X) &= (q_2, X)/0.6,\ X \in \{Z_0, b, c\}\\
\delta(q_1, \epsilon, Z_0) &= (q_0, \epsilon)/0.7,\\
\delta(q_2, \epsilon, Z_0) &= (q_0, \epsilon)/0.4.
\end{aligned}$$

The word abc *is accepted by FSAPDA* M *through the following chain of moves:*

$$\begin{aligned}
(q_0, abc, Z_0) &\overset{0.4}{\vdash^*} (q_1, abc, Z_0)/0.4 \wedge (q_2, abc, Z_0)/0.7\\
&\overset{0.1}{\vdash^*} (q_1, bc, aZ_0)/0.1 \wedge (q_2, bc, Z_0)/0.6\\
&\overset{0.1}{\vdash^*} (q_1, c, Z_0)/0.1 \wedge (q_2, c, bZ_0)/0.2\\
&\overset{0.2}{\vdash^*} (q_1, \epsilon, Z_0)/0.5 \wedge (q_2, \epsilon, Z_0)/0.2\\
&\overset{0.4}{\vdash^*} (q_0, \epsilon, \epsilon)/0.7 \wedge (q_0, \epsilon, \epsilon)/0.4\\
&\overset{r=0.1}{\vdash} (q_0, \epsilon, \epsilon),
\end{aligned}$$

where $r = \min\{0.4, 0.1, 0.1, 0.2, 0.4\} = 0.1$. *Hence,* $(abc, 0.1) \in \mu(L, M)$*, it means that, the word* abc *accepted by* M *with membership grade* 0.1*. Therefore* $\mu(L, M) = \{(|w|_a = |w|_b = |w|_c, 0.1) \mid w \in \Sigma\}$.

4. Equivalence of Fuzzy Conjunctive Grammar and Fuzzy Synchronized Alternating Pushdown Automata

Here, we study equivalence between FSAPDA and FCG: theoretically, we present this fact from FCG to FSAPDA and from FSAPDA to FCG.

4.1. **Construction.** Let $G = (V, \Sigma, P, S)$ be a FCG. Construct the single-state FSAPDA $M = (\{q\}, \Sigma, V \cup \Sigma, \delta, q, S)$ that accepts $\mu(L, G)$ by empty stack, where transition function δ is defined by three rules

(1) For each variable A,

$$\delta(q, \epsilon, A) = \{(q, \alpha)/r, \text{ if } A \xrightarrow{r} \alpha, \text{ is a production of } P\},$$

(2) For each variable A,

$$\delta(q, \epsilon, A) = \{((q, \alpha_1)/r_1 \wedge \cdots \wedge (q, \alpha_n)/r_n)/r, \text{ if } A \xrightarrow{r} (\alpha_1/r_1\& \ldots \&\alpha_n/r_n), \text{ is a production of } P\}, n \geq 2$$

(3) For each terminal σ,

$$\delta(q, \sigma, \sigma) = (q, \epsilon)/1,$$

where $A \in V$, $q \in Q$, $\sigma \in \Sigma$, and $\alpha, \alpha_1 \ldots, \alpha_n \in (V \cup \Sigma)^*$, $r = \min\{r_1, \ldots, r_n\}$, $r \in (0, 1]$.

Theorem 4.1. *Let* $\gamma \in (V \cup \Sigma)^*$*,* $w \in \Sigma^*$*,* $q \in Q$*. Then* $\gamma \overset{r}{\Longrightarrow}{}^* w$*, if, and only if,*

$$(q, w, \gamma) \overset{r}{\vdash}{}^* (q, \epsilon, \epsilon), \text{ where } r \in (0, 1].$$

Proof. Proof of the "only if" part. The proof is by induction on the number of proper conjunctive rules applied in the derivation.

Basis: Assume that no proper fuzzy conjunctive rule was applied in the derivation $\gamma \overset{r}{\Longrightarrow}{}^* w$. Let $\gamma = Y_1 Y_2 \ldots Y_m$ and let $w = w_1 w_2 \ldots w_m$, where $Y_j \overset{r_j}{\Longrightarrow}{}^* w_j$, $j = 1, 2, \ldots, m$. Then

$$(q, w_j, Y_j) \overset{r_j}{\vdash}{}^* (q, \epsilon, \epsilon), \; j = 1, 2, \ldots, m. \tag{4.1}$$

If $Y_j \in \Sigma$ and $w_j = Y_j$, then (4.1) follows by the transition $\delta(q, Y_j, Y_j) = (q, \epsilon)/1$. Now, by (4.1),

$$\begin{aligned}
(q, w, \gamma) = (q, w_1w_2 \dots w_m, Y_1Y_2 \dots Y_m) &\overset{r_1}{\vdash^*} (q, w_2w_3 \dots w_m, Y_2Y_3 \dots Y_m) \\
&\overset{r_2}{\vdash^*} (q, w_3w_4 \dots w_m, Y_3Y_4 \dots Y_m) \\
&\ \vdots \\
&\overset{r_{m-1}}{\vdash^*} (q, w_m, Y_m) \\
&\overset{r_m}{\vdash^*} (q, \epsilon, \epsilon) \\
(q, w, \gamma) &\overset{r}{\vdash} (q, \epsilon, \epsilon),
\end{aligned}$$

where $r = \min\{r_1, \dots, r_m\}$ and $r \in (0, 1]$.

Induction step : Assume the result is true for derivations with up to k applications of proper fuzzy conjunctive rules. Let $\gamma \overset{r}{\Longrightarrow}{}^* w$ be a derivation with k+1 applications of proper fuzzy conjunctive rules.
Let $w = uvt$, where $u, v, t \in \Sigma^*$. There is no proper fuzzy conjunctive rules was applied in the derivation $\gamma \overset{r_u}{\Longrightarrow}{}^* uA\alpha$. That is, $\gamma = \gamma' A\alpha$, where $\gamma' \overset{r_u}{\Longrightarrow}{}^* u$. Each of the derivation $\alpha_i \overset{r_{v_i}}{\Longrightarrow}{}^* v$, $i = 1, 2, \dots, n$ and $\alpha \overset{r_t}{\Longrightarrow}{}^* t$ has at most k applications of proper conjunctive rules. Therefore, the derivation is of the form

$$\begin{aligned}
\gamma = \gamma' A\alpha &\overset{r_u}{\Longrightarrow}{}^* uA\alpha \\
&\overset{r_u}{\Longrightarrow} u(\alpha_1/r_{v_1} \& \dots \& \alpha_n/r_{v_n})\alpha \\
&\overset{r_v}{\Longrightarrow}{}^* u(v/r_{v_1} \& \dots \& v/r_{v_n})\alpha \\
&\overset{r_v}{\Longrightarrow} uv\alpha, \text{ where } r_v = \min\{r_{v_1}, \dots, r_{v_n}\} \\
&\overset{r_t}{\Longrightarrow}{}^* uvt \\
\gamma &\overset{r}{\Longrightarrow} w,
\end{aligned}$$

where $r = \min\{r_u, r_v, r_t\}$ and $r \in (0, 1]$.
The derivation $\gamma' \overset{r_u}{\Longrightarrow}{}^* u$ implies that, $(q, u, \gamma') \overset{r_u}{\vdash^*} (q, \epsilon, \epsilon)$. Each of the derivation $\alpha_i \overset{r_{v_i}}{\Longrightarrow}{}^* v$, $i = 1, 2, \dots, n$, has atmost k application of proper fuzzy conjunctive rule. Thus, by induction hypothesis, $(q, v, \alpha_i) \overset{r_{v_i}}{\vdash^*} (q, \epsilon, \epsilon)$, $i = 1, 2, \dots, m$. Similarly, the derivation $\alpha \overset{r_t}{\Longrightarrow}{}^* t$ implies that, $(q, t, \alpha) \overset{r_t}{\vdash^*} (q, \epsilon, \epsilon)$. Combining the

above computations of M we obtain

$$\begin{aligned}
(q, uvt, \gamma) = (q, uvt, \gamma' A\alpha) &\overset{r_u}{\vdash^*} (q, vt, A\alpha) \\
&\overset{r_u}{\vdash} ((q, vt, \alpha_1)/r_{v_1} \wedge \cdots \wedge (q, vt, \alpha_n)/r_{v_n})\alpha \\
&\overset{r_v}{\vdash^*} ((q, t, \epsilon)/r_{v_1} \wedge \cdots \wedge (q, t, \epsilon)/r_{v_n})\alpha \\
&\overset{r_v}{\vdash} (q, t, \alpha), \text{ where } r_v = \min\{r_{v_1}, \ldots, r_{v_n}\} \\
&\overset{r_t}{\vdash^*} (q, \epsilon, \epsilon) \\
(q, uvt, \gamma) &\overset{r}{\vdash} (q, \epsilon, \epsilon),
\end{aligned}$$

where $r = \min\{r_u, r_v, r_t\}$ and $r \in (0, 1]$.

Proof of the “if" part. The proof is by induction on the number of proper conjunctive transitions applied in the computation.

Basis: Assume that no proper fuzzy conjunctive transition was applied in the computation $(q, w, \gamma) \overset{r}{\vdash^*} (q, \epsilon, \epsilon)$. Let $\gamma = Y_1 Y_2 \ldots Y_m$ and let $w = w_1 w_2 \ldots w_m$. After reading w_j, $j = 1, 2, \ldots m - 1$ and Y_{j+1} is exposed as the top symbol in the stack for the first time. It immediately follows from the definition of w_j, that $(q, w_j, Y_j) \overset{r_j}{\vdash^*} (q, \epsilon, \epsilon)$, $j = 1, 2, \ldots, m$. Therefore,

$$Y_j \overset{r_j}{\Longrightarrow^*} w_j, \ j = 1, 2, \ldots, m \tag{4.2}$$

If $Y_j \in \Sigma$, then by the definition of δ, $w_j = Y_j$ and (4.2) is $Y_j \overset{1}{\Longrightarrow}_{(0)} Y_j$. Now, by (4.2),

$$\begin{aligned}
\gamma = Y_1 Y_2 \ldots Y_m &\overset{r_1}{\Longrightarrow^*} w_1 Y_2 Y_3 \ldots Y_m \\
&\overset{r_2}{\Longrightarrow^*} w_1 w_2 Y_3 Y_4 \ldots Y_m \\
&\vdots \\
&\overset{r_m}{\Longrightarrow^*} w_1 w_2 \ldots w_m. \\
\gamma &\overset{r}{\Longrightarrow} w,
\end{aligned}$$

where $r = \min\{r_1, \ldots, r_m\}$ and $r \in (0, 1]$.

Induction step: Assume the result is true for computation with up to k proper fuzzy conjunctive transition. Let $(q, w, \gamma) \overset{r}{\vdash^*} (q, \epsilon, \epsilon)$, be a computation with $k + 1$ proper conjunctive transition.

Let $w = uvt$, where $u, v, t \in \Sigma^*$. There is no proper fuzzy conjunctive transition was applied in the computation $(q, w, \gamma) = (q, uvt, \gamma) \overset{r_u}{\vdash^*} (q, vt, A\alpha)$. That is, $\gamma = \gamma' A\alpha$, where $(q, u, \gamma') \overset{r_u}{\vdash^*} (q, \epsilon, \epsilon)$. Each of the computations $(q, v, \alpha_i) \overset{r_{v_i}}{\vdash^*} (q, \epsilon, \epsilon)$, $i = 1, 2, \ldots, n$ and $(q, t, \alpha) \overset{r_t}{\vdash^*} (q, \epsilon, \epsilon)$. Therefore, the computation is of the form

$$\begin{aligned}
(q, w, \gamma) = (q, uvt, \gamma) = (q, uvt, \gamma' A\alpha) &\overset{r_u}{\vdash^*} (q, vt, A\alpha) \\
&\overset{r_u}{\vdash} ((q, vt, \alpha_1)/r_{v_1} \wedge \cdots \wedge (q, vt, \alpha_n)/r_{v_n})\alpha \\
&\overset{r_v}{\vdash^*} ((q, t, \epsilon)/r_{v_1} \wedge \cdots \wedge (q, t, \epsilon)/r_{v_n})\alpha, \\
&\quad \text{where } r_v = \min\{r_{v_1}, \ldots, r_{v_n}\} \\
&\overset{r_v}{\vdash} (q, t, \alpha) \\
&\overset{r_t}{\vdash^*} (q, \epsilon, \epsilon) \\
(q, uvt, \gamma) &\overset{r}{\vdash} (q, \epsilon, \epsilon),
\end{aligned}$$

where $r = \min\{r_u, r_v, r_t\}$ and $r = (0, 1]$.

The computation $(q, u, \gamma') \overset{r_u}{\vdash^*} (q, \epsilon, \epsilon)$ implies that $\gamma' \overset{r_u}{\Longrightarrow^*} u$. Each of the computations $(q, v, \alpha_i) \overset{r_{v_i}}{\vdash^*} (q, \epsilon, \epsilon)$, $i = 1, 2, \ldots, n$ has at most k applications of proper fuzzy conjunctive transition. Thus, by induction hypothesis, $\alpha_i \overset{r_{v_i}}{\Longrightarrow^*} v$, $i = 1, 2, \ldots, n$. Similarly the computation $(q, t, \alpha) \overset{r_t}{\vdash^*} (q, \epsilon, \epsilon)$ implies that $\alpha \overset{r_t}{\Longrightarrow^*} t$. Combining the above derivations of G we obtain

$$\begin{aligned}
\gamma = \gamma' A\alpha &\overset{r_u}{\Longrightarrow^*} uA\alpha \\
&\overset{r_u}{\Longrightarrow} u(\alpha_1/r_{v_1} \& \ldots \& \alpha_n/r_{v_n})\alpha, \\
&\overset{r_v}{\Longrightarrow^*} u(v \& \ldots \& v)\alpha, \text{ where } r_v = \min\{r_{v_1}, \ldots, r_{v_n}\} \\
&\overset{r_v}{\Longrightarrow} uv\alpha \\
&\overset{r_t}{\Longrightarrow^*} uvt \\
\gamma &\overset{r}{\Longrightarrow} w,
\end{aligned}$$

where $r = \min\{r_u, r_v, r_t\}$ and $r \in (0, 1]$. The proof is complete. □

4.2. **Construction.** Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ be an FSAPDA. Construct the FCG $G = (V, \Sigma, P, S)$, where

- $V = Q \times \Gamma \times Q \cup \{S\}, S \notin Q \times \Gamma \times Q$, and

- P consists of the following rules:

(1) $S \xrightarrow{1} [q_0, Z_0, p]$, for all $p \in Q$,

(2) $[q, Y, p] \xrightarrow{r} \sigma[q_1, Y_1, q_2] \ldots [q_m, Y_m, p]$, for all $\sigma \in \Sigma \cup \{\epsilon\}$, all $p \in Q$, all ordinary transitions $\delta(q, \sigma, Y) = (q_1, Y_1 \ldots Y_m)/r$, and all choices of $q_2, \ldots, q_m \in Q$ and $Y, Y_1 \ldots, Y_m \in \Gamma$,

(3) $[q, Y, p] \xrightarrow{r} ([q_1, Y_1, p]/r_1 \& \ldots \& [q_n, Y_n, p]/r_n)$, for all $p \in Q$ and all proper conjunctive transitions $\delta(q, \epsilon, Y) = ((q_1, Y_1)/r_1 \wedge \cdots \wedge (q_n, Y_n)/r_n)/r$, where $r = \min\{r_1, \ldots, r_n\}$ and $r \in (0, 1]$.

Lemma 4.1. *Let* $m \geq 1$*,* $p, q, q_1, \ldots, q_m \in Q$*,* $Y, Y_1, \ldots, Y_m \in \Gamma^*$*, and* $u \in \Sigma^*$*.*

$$\text{If, } [q, Y, p] \stackrel{r_u}{\Longrightarrow}{}^* u[q_1, Y_1, q_2] \ldots [q_m, Y_m, p], \text{ then}$$

$$(q, u, Y) \stackrel{r_u}{\vdash}{}^* (q_1, \epsilon, Y_1 \ldots Y_m), \text{ where } r_u \in (0, 1].$$

Lemma 4.2. *Let* $m \geq 1$*,* $q, q_1 \in Q$*,* $Y, Y_1, \ldots, Y_m \in \Gamma^*$*,* $u \in \Sigma^*$*.*

$$\text{If } (q, u, Y) \stackrel{r_u}{\vdash}{}^* (q_1, \epsilon, Y_1 \ldots Y_m),$$

then for all $p, q_1, q_2, q_3, \ldots, q_m \in Q$*,*

$$[q, Y, p] \stackrel{r_u}{\Longrightarrow}{}^* u[q_1, Y_1, q_2] \ldots [q_m, Y_m, p], \text{ where } r_u \in (0, 1].$$

Theorem 4.2. *Let* $p, q \in Q$*,* $Y \in \Gamma^*$*, and* $w \in \Sigma^*$*, then*

$$[q, Y, p] \quad \stackrel{r}{\Longrightarrow}{}^* \quad w \tag{4.3}$$

if and only if

$$(q, w, Y) \quad \stackrel{r}{\vdash}{}^* \quad (p, \epsilon, \epsilon), \text{ where } r \in (0, 1]. \tag{4.4}$$

Proof. Proof of the "only if"part. The proof is by induction on the number of proper fuzzy conjunctive rules applied in the derivation.

Basis: If no proper fuzzy conjunctive rule was applied in the derivation (4.3), then (4.4) is by the "only if"part of Theorem 4.1 [12].

Induction Step: Assume the implication holds for derivations with up to k applications of proper fuzzy conjunctive rules. Let $[q, Y, p] \stackrel{r}{\Longrightarrow}{}^* w$ be a derivation with $k+1$ applications of proper fuzzy conjunctive rule. Let $w = uvt_1t_2\dots t_m$, where $u, v, t_1, \dots, t_m \in \Sigma^*$. Each of the derivations

$$\begin{aligned} & [p_i, X_i, q_1] \overset{r_{v_i}}{\Longrightarrow}{}^* v, \text{ for each } i = 1, 2, \dots, n; \\ & [q_j, Y_j, q_{j+1}] \overset{r_{t_j}}{\Longrightarrow}{}^* t_j, \text{ for each } j = 1, 2, \dots, m-1; \\ \text{and} \quad & [q_m, Y_m, p] \overset{r_{tm}}{\Longrightarrow}{}^* t_m, \end{aligned}$$

has at most k applications of proper conjunctive rules. Hence, the derivation is of the form

$$\begin{aligned} [q, Y, p] & \overset{r_u}{\Longrightarrow}{}^* u[q', X, q_1][q_1, Y_1, q_2]\dots[q_m, Y_m, p] \\ & \overset{r_u}{\Longrightarrow} u([p_1, X_1, q_1]/r_{v_1} \& \dots \& [p_n, X_n, q_1]/r_{v_n})[q_1, Y_1, q_2]\dots[q_m, Y_m, p] \\ & \overset{r_v}{\Longrightarrow}{}^* u(v\& \dots \& v)[q_1, Y_1, q_2]\dots[q_m, Y_m, p], \\ & \quad \text{where } r_v = \min\{r_{v_1}, r_{v_2}, \dots, r_{v_n}\} \\ & \overset{r_v}{\Longrightarrow} uv[q_1, Y_1, q_2]\dots[q_m, Y_m, p] \\ & \overset{r_{t_1}}{\Longrightarrow}{}^* uvt_1[q_2, Y_2, q_3]\dots[q_m, Y_m, p] \\ & \vdots \\ & \overset{r_t}{\Longrightarrow}{}^* uvt_1\dots t_m, \text{ where } r_t = \min\{r_{t_1}, r_{t_2}, \dots, r_{t_m}\} \\ & \overset{r}{\Longrightarrow}{}^* w, \end{aligned}$$

where $r = \min\{r_u, r_v, r_t\}$ and $r \in (0, 1]$.

There is no proper fuzzy conjunctive rule was applied in the derivation

$$[q, Y, p] \overset{r_u}{\Longrightarrow}{}^* u[q', X, q_1][q_1, Y_1, q_2]\dots[q_m, Y_m, p].$$

Therefore, by Lemma 4.1 ,

$$(q, u, Y) \overset{r_u}{\vdash}{}^* (q', \epsilon, XY_1\dots Y_m). \tag{4.5}$$

Since, by

$$\overset{r_u}{\Longrightarrow} u([p_1, X_1, q_1]/r_{v_1} \& \dots \& [p_n, X_n, q_1]/r_{v_n})[q_1, Y_1, q_2]\dots[q_m, Y_m, p],$$

thus

$$[q', X, q_1] \xrightarrow{r_v} ([p_1, X_1, q_1]/r_{v_1} \& \dots \& [p_n, X_n, q_1]/r_{v_n}) \in P,$$

by the Construction 4.1 ,

$$\delta(q', \epsilon, X) = ((p_1, X_1)/r_{v_1} \wedge \cdots \wedge (p_n, X_n)/r_{v_n})/r_v, \tag{4.6}$$

Therefore, by the induction hypothesis, for each of the derivations

$$[p_i, X_i, q_1] \overset{r_{v_i}}{\Longrightarrow^*} v, \text{ implies that}$$

$$(p_i, v, X_i) \overset{r_{v_i}}{\vdash^*} (q_1, \epsilon, \epsilon), \text{ for each } i = 1, 2, \ldots, n. \tag{4.7}$$

Similarly, $[q_j, Y_j, q_{j+1}] \overset{r_{t_j}}{\Longrightarrow^*} t_j$, implies that

$$(q_j, t_j, Y_j) \overset{r_{t_j}}{\vdash^*} (q_{j+1}, \epsilon, \epsilon), \text{ for each } j = 1, 2, \ldots, m-1; \tag{4.8}$$

and $[q_m, Y_m, p] \overset{r_{t_m}}{\Longrightarrow^*} t_m$, implies that

$$(q_m, t_m, Y_m) \overset{r_{t_m}}{\vdash^*} (p, \epsilon, \epsilon). \tag{4.9}$$

Therefore, the computation is of the form

$$\begin{aligned}
(q, w, Y) = (q, uvt_1 \ldots t_m, Y) &\overset{r_u}{\vdash^*} (q', vt_1 \ldots t_m, XY_1 \ldots Y_m), \text{ since } (4.5)\\
&\overset{r_u}{\vdash} ((p_1, vt_1 \ldots t_m, X_1)/r_{v_1} \wedge \ldots\\
&\quad \wedge (p_n, vt_1 \ldots t_m, X_n)/r_{v_n})(Y_1 \ldots Y_m), \text{ since } (4.6)\\
&\overset{r_v}{\vdash^*} ((q_1, t_1 \ldots t_m, \epsilon)/r_{v_1} \wedge \ldots\\
&\quad \wedge (q_1, t_1 \ldots t_m, \epsilon)/r_{v_n})(Y_1 \ldots Y_m), \text{ since } (4.7)\\
&\overset{r_v}{\vdash} (q_1, t_1 \ldots t_m, Y_1 \ldots Y_m), \quad \text{where } r_v = \min\{r_{v_1}, r_{v_2}, \ldots, r_{v_n}\},\\
&\overset{r_{t_1}}{\vdash^*} (q_2, t_2 \ldots t_m, Y_2 \ldots Y_m), \text{ since } (4.8)\\
&\vdots\\
&\overset{r_{t_{m-1}}}{\vdash^*} (q_m, t_m, Y_m), \text{ since } (4.8)\\
&\overset{r_{t_m}}{\vdash^*} (p, \epsilon, \epsilon), \text{ since } (4.9)\\
&\overset{r_t}{\vdash} (p, \epsilon, \epsilon), \text{ where } r_t = \min\{r_{t_1}, r_{t_2}, \ldots, r_{t_m}\}\\
(q, w, Y) \quad &\overset{r}{\vdash^*} (p, \epsilon, \epsilon),
\end{aligned}$$

$r = \min\{r_u, r_v, r_t\}$ and $r \in (0, 1]$.

Proof of the "if"part of proposition. The proof is by induction on the number of proper fuzzy conjunctive transitions applied in the computation.

Basis: If no proper fuzzy conjunctive transition was applied in the computation $(q, w, Y) \overset{r}{\vdash^*} (p, \epsilon, \epsilon)$, then (4.3) is by the "if"part of Theorem 4.1 [12].

Induction step: Assume the implication holds for computations up to k proper fuzzy conjunctive transitions. Let $(q, w, Y) \overset{r}{\vdash^*} (p, \epsilon, \epsilon)$ be a computation with $k + 1$ proper fuzzy conjunctive transitions. Let $w = uvt_1t_2 \ldots t_m$, where $u, v, t_1, t_2, \ldots, t_m \in \Sigma^*$. Each of the computations

$$(p_i, v, X_i) \overset{r_{v_i}}{\vdash^*} (q_1, \epsilon, \epsilon), \text{ for each } i = 1, 2, \ldots, n;$$

$$(q_j, t_j, Y_j) \overset{r_{t_j}}{\vdash^*} (q_{j+1}, \epsilon, \epsilon), \text{ for each } j = 1, 2, \ldots, m-1, \text{and}$$

$$(q_m, t_m, Y_m) \overset{r_{t_m}}{\vdash^*} (p, \epsilon, \epsilon),$$

has at most k applications of proper conjunctive transitions. Hence, the computation is of the form

$$\begin{aligned}
(q, w, Y) = (q, uvt_1 \ldots t_m, Y) &\overset{r_u}{\vdash^*} (q', vt_1 \ldots t_m, XY_1 \ldots Y_m) \\
&\overset{r_u}{\vdash} ((p_1, vt_1 \ldots t_m, X_1)/r_{v_1} \wedge \ldots \\
&\quad \wedge (p_n, vt_1 \ldots t_m, X_n)/r_{v_n})(Y_1 \ldots Y_m) \\
&\overset{r_v}{\vdash^*} ((q_1, t_1 \ldots t_m, \epsilon)/r_{v_1} \wedge \ldots \\
&\quad \wedge (q_1, t_1 \ldots t_m, \epsilon)/r_{v_n})(Y_1 \ldots Y_m) \\
&\overset{r_v}{\vdash} (q_1, t_1 \ldots t_m, Y_1 \ldots Y_m), \text{ where } r_v = \min\{r_{v_1}, r_{v_2}, \ldots, r_{v_n}\} \\
&\overset{r_{t_1}}{\vdash^*} (q_2, t_2 \ldots t_m, Y_2 \ldots Y_m) \\
&\vdots \\
&\overset{r_{t_{m-1}}}{\vdash^*} (q_m, t_m, Y_m) \\
&\overset{r_{t_m}}{\vdash^*} (p, \epsilon, \epsilon) \\
&\overset{r_t}{\vdash} (p, \epsilon, \epsilon), \text{ where } r_t = \min\{r_{t_1}, r_{t_2}, \ldots, r_{t_m}\} \\
(q, w, Y) &\overset{r}{\vdash^*} (p, \epsilon, \epsilon)
\end{aligned}$$

$r = \min\{r_u, r_v, r_t\}$ and $r \in (0, 1]$.

There is no proper fuzzy conjunctive rule was applied in the computation

$$(q, uvt_1 \dots t_m, Y) \overset{r_u}{\vdash^*} (q', vt_1t_2 \dots t_m, XY_1 \dots Y_m).$$

Therefore, by Lemma 4.2,

$$[q, Y, p] \overset{r_u}{\Longrightarrow^*} u[q', X, q_1][q_1, Y_1, q_2] \dots [q_m, Y_m, p]. \tag{4.10}$$

Since, by

$$((p_1, vt_1 \dots t_m, X_1)/r_{v_1} \wedge \dots \wedge (p_n, vt_1 \dots t_m, X_n)/r_{v_n})(Y_1 \dots Y_m),$$

thus

$$\delta(q', \epsilon, X) = ((p_1, X_1)/r_{v_1} \wedge \dots \wedge (p_n, X_n)/r_{v_n})/r_v,$$

by the Construction 4.2,

$$[q', X, q_1] \xrightarrow{r_v} ([p_1, X_1, q_1]/r_{v_1} \& \dots \& [p_n, X_n, q_1]/r_{v_n}) \in P \tag{4.11}$$

Therefore, by the basis of induction, for each of the computations

$$(p_i, v, X_i) \overset{r_{v_i}}{\vdash^*} (q_1, \epsilon, \epsilon), \text{ implies that}$$

$$[p_i, X_i, q_1] \overset{r_{v_i}}{\Longrightarrow^*} v, \text{ for each } i = 1, 2, \dots, n \tag{4.12}$$

Similarly, $(q_j, t_j, Y_j) \overset{r_{t_j}}{\vdash^*} (q_{j+1}, \epsilon, \epsilon)$, implies that

$$[q_j, Y_j, q_{j+1}] \overset{r_{t_j}}{\Longrightarrow^*} t_j, \text{ for each } j = 1, 2, \dots, m-1 \tag{4.13}$$

and $(q_m, t_m, Y_m) \overset{r_{t_m}}{\vdash^*} (p, \epsilon, \epsilon)$, implies that

$$[q_m, Y_m, p] \overset{r_{t_m}}{\Longrightarrow^*} t_m. \tag{4.14}$$

Therefore, the derivation is of the form

$$
\begin{aligned}
[q, Y, p] &\overset{r_u}{\Longrightarrow}{}^* u[q', X, q_1][q_1, Y_1, q_2] \dots [q_m, Y_m, p], \text{ since } (4.10)\\
&\overset{r_u}{\Longrightarrow} u([p_1, X_1, q_1]/r_{v_1} \& \dots \& [p_n, X_n, q_1]/r_{v_n})\\
&\quad [q_1, Y_1, q_2] \dots [q_m, Y_m, p], \text{ since } (4.11)\\
&\overset{r_v}{\Longrightarrow}{}^* u(v \& \dots \& v)[q_1, Y_1, q_2] \dots [q_m, Y_m, p], \text{ since } (4.12),\\
&\overset{r_v}{\Longrightarrow} uv[q_1, Y_1, q_2] \dots [q_m, Y_m, p]\\
&\vdots\\
&\overset{r_{t_{m-1}}}{\Longrightarrow}{}^* uvt_1 \dots t_{m-1}[q_m, Y_m, p], \text{ since } (4.13)\\
&\overset{r_t}{\Longrightarrow}{}^* uvt_1 \dots t_m, \text{ since } (4.14),\\
&\overset{r}{\Longrightarrow} w, \text{ since } w = uvt_1 \dots t_m,
\end{aligned}
$$

where $r = \min\{r_u, r_v, r_t\}$ and $r \in (0, 1]$. And the proof is complete. □

Theorem 4.3. *A language is accepted by an FSAPDA if and only if it is generated by a FCG.*

Proof.

If. Let G be a FCG and let M be the FSAPDA as above. Let $(w, r) \in \mu(L, G)$, i.e., $S \overset{r}{\Longrightarrow}{}^* w$. By Theorem (4.1) with γ being S, this is if and only if $(q_0, w, S) \overset{r}{\vdash}{}^* (q, \epsilon, \epsilon)$, i.e., $(w, r) \in \mu(L, M)$. Therefore, $\mu(L, G) = \mu(L, M)$.

Only if. Let M be an FSAPDA and let G be the FCG as above. Let $(w, r) \in \mu(L, M)$, i.e., $(q_0, w, Z_0) \overset{r}{\vdash}{}^* (p, \epsilon, \epsilon)$. By Theorem (4.2), this is if and only if $[q_0, Z_0, p] \overset{r}{\Longrightarrow}{}^* w$. Therefore, by the Construction of G, $S \overset{r}{\Longrightarrow}{}^* w$, implying $\mu(w, r) \in \mu(L, M)$, and vice versa. It follows that $\mu(L, M) = \mu(L, G)$. □

References

[1] A. M. Kaminski: *Conjunctive grammars and alternating pushdown automata*, In: Hodges, W., de Queiroz, R. (Eds.) The 15th Workshop on Logic, Language, Information and Computation-WoLLIC'2008, Volume 5110 of Lecture Notes in Artificial Intelligence, Springer, Berlin/Heidelberg (2008), 30-41.

[2] P. R. J. Asveld: *Fuzzy context-free languages-Part 1: Generalized fuzzy context-free grammars*, Theoretical Computer Science, **347** (2005), 167-190.

[3] P. R. J. ASVELD: *Fuzzy Context-free Languages-Part 2: Recognition and parsing algorithms*, Theoretical Computer Science, **347** (2005), 191-213.

[4] T. AIZIKOWITZ, M. KAMINSKI: *Conjunctive grammars and alternating pushdown automata*, Acta Informatica, **50** (2013), 175-197.

[5] I. BUCURESCU, A. PASCU: *On the Languages Generated by Context-Sensitive Fuzzy Grammars*, Intern. J. Computer Math., **9** (1981), 277-285.

[6] R. BELOHLAVEK: *Determinism and fuzzy automata*, Information Science, **143**(2002), 205-209.

[7] CHANDRA, ET.AL, : *Alternation*, J. ACM **28**(1), (1981), 114-133.

[8] O. CHUNG-MING, W. YU-HUI: *On The Color Sequence Fuzzy Automata of Eye Detection*, Proceedings of the Seventh International Conference on Machine Learning and Cybernetics, Kunming, 12-15 July (2008), 3586-3590.

[9] G. GERLA: *Fuzzy Grammars and Recursively Enumerable Fuzzy Languages*, Information sciences, **60** (1992), 137-143.

[10] H. HENG KIM, ET.AL, : *L-Fuzzy Grammars*, Information sciences, **8** (1975), 123-140.

[11] S. J. HAM, ET.AL, : *Fire-Flame Detection based on Fuzzy Finite Automation*, 2010 International Conference on Pattern Recognition, (2010), 3919-3922.

[12] H. XING: *Fuzzy pushdown automata*, Fuzzy Sets and Systems, **158** (2007) 1437-1449.

[13] O. KUPFERMAN, SH. ZUHOVITZKY: *An improved algorithm for the membership problem for extended regular expressions*, In: Diks, K., Rytter,W. (Eds.) Mathematical Foundations of Computer Science, **2420** (2002), 446-458.

[14] H. V. KUMBHOJKAR, S. R. CHAUDHARI: *On covering of products of fuzzy finite state machines*, Fuzzy Sets and Systems, **125** (2002), 215-222.

[15] E. T. LEE, L. A. ZADEH: *Note on Fuzzy Languages*, Information Sciences, **1**(4) (1969), 421-434.

[16] S. LAN, M. ZHIWEN: *Closure of finite-state automaton languages*, Fuzzy Sets and Systems, **75** (1995), 393-397.

[17] R. E. LADNER, R. J. LIPTON: *Alternating pushdown and stack automata*, SIAM J. Comput., **13**(1) (1984), 135-155.

[18] Y. LI, W. PEDRYCZ: *Minimization of lattice finite automata and its application to the decomposition of lattice languages*, Fuzzy Sets and Systems, **158** (2007), 1423-1436.

[19] M. MIZUMOTO, ET.AL, : *Some Considerations on Fuzzy Automata*, Journal of Computer and System Sciences, **3** (1969), 409-422.

[20] M. MIZUMOTO, ET.AL, : *N-Fold Fuzzy Grammars*, Information Sciences, **5** (1973), 25-43.

[21] D. S. MALIK, ET.AL, : *On subsystems of a fuzzy finite state machine*, Fuzzy Sets and Systems, **68** (1994), 83-92.

[22] J. N. MORDESION, D. S. MALIK: *Fuzzy Automata and Languages Theory and Applications*, Chapman & Hall/CRC Press (2002).

[23] A. OKHOTIN: *Conjunctive Grammars*, Journal of Automata, Languages and Combinatorics, **6**(4) (2001), 519-535.

[24] A. PAZ: *Fuzzy Star Functions, Probabilistic Automata and Their Approximation by Nonprobabilistic Automata*, Journal of Computer and System Sciences, **1** (1967), 371-390.

[25] D. QIU: *Characterizations of fuzzy finite automata*, Fuzzy Sets and Systems, **141** (2004), 391-414.

[26] E. SANTOS: *Fuzzy Automata and Languages*, Information Sciences, **10** (1976), 193-197.

[27] D. TODINCA, D. BUTOIANU: *VHDL framework for modeling fuzzy automata*, 14th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing, (2012), 171-178.

[28] D. WOTSCHKE: *The Boolean Closures of Deterministic and Nondeterministic Context-free Languages*, Lecture Notes in Computer Science, Vol. 1, Springer, Berlin, (1973), 113-121.

[29] J. WANG, M. YIN, W. GU: *Fuzzy multiset finite automata and their languages*, Soft. Comput., **17** (2013), 381-390.

[30] J. WAISSMAN, ET.AL, : *Wastewater Treatment Process Supervision by Means of a Fuzzy Automaton Model*, Proceedings of the 15th IEEE international Symposium on Intelligent Control (ISIC 2000) Rio. Patias. GREECE, 17-19 July (2000), 163-168.

[31] W. G. WEE: *On generalizations of adaptive algorithm and application of the fuzzy sets concept to pattern classification*, Ph.D. Thesis, Purdue University, 1967.

[32] Y. CAO, Y. EZAWA: *Nondeterministic fuzzy automata*, Information Sciences, **191** (2012), 86-97.

[33] H. YAMAMOTO: *An automata based recognition algorithm for semi-extended regular expressions*, In: Nielsen, M., Rovan, B. (Eds.) Mathematical Foundations of Computer Science-MFCS 2000, 25th International Symposium, Volume 1893 of Lecture Notes in Computer Science, Springer, Berlin/Heidelberg (2000), 699-708.

[34] L. A. ZADEH: *Fuzzy Languages and their relation to Human and Machine Intelligence*, This is the text of a paper presented at the Conference on Man and Computer, Bordeaux, France, (Organized by Institut de la Vie, Paris), June (1970), 22-26.

DEPARTMENT OF MATHEMATICS
ANNAMALAI UNIVERSITY
CHIDAMBARAM, TAMILNADU, INDIA
Email address: mahapathra87@gmail.com

DEPARTMENT OF MATHEMATICS
FACULTY OF ENGINEERING AND TECHNOLOGY
ANNAMALAI UNIVERSITY
CHIDAMBARAM, TAMILNADU, INDIA
Email address: mdivraj1962@gmail.com