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NUMERICAL ANALYSIS OF AN UNSTEADY MHD FREE CONVECTIVE FLUID FLOW WITH UNCERTAIN PARAMETERS

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ABSTRACT. An axisymmetric unsteady MHD free convective fluid flow is considered in fuzzy environment and consequently the initial and boundary conditions along with the parameters involved there are uncertain. Finite difference method (FDM) is used to solve the fuzzy forms of the non-dimensional governing equations along with the fuzzified dimensionless boundary conditions. The values of the uncertain parameters as well as the initial and the boundary conditions are considered as triangular fuzzy number (TFN) and α -cut technique is applied to find the solutions by taking the value of α as 1, for which solutions with membership grade 1 may be obtained. Computer programming code has been developed in Python (an object oriented computer programming language). Velocity, temperature and concentration profiles are observed under the effects of various involved uncertain parameters.

1. INTRODUCTION

Combined heat and mass transfer in fluid saturated porous medium has a lot of industrial applications [1]. Most of the research works dealing with porous media have used the Darcy law. However, in case of fluid flows with high velocity, the Darcy law is not applicable, as it does not account for inertial effects in the porous medium. The most relevant way to deal with high-velocity transport in porous media is the Darcy-Forchheimer drag force model. This adds a drag

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force of second order to the momentum conservation equation. An influential study on Forchheimer inertial effects in porous media convection is presented by Vafai and Tien [3].

The real life phenomena are uncertain and imprecise. Uncertainty occurs in the fluid flows also as various uncertain and vague conditions present there. Consideration of exact values of the parameters as well as the initial and boundary conditions in a fluid flow problem may cause errors in the solutions. Fuzzy set theory (FST) may be used to overcome these uncertainties. With the help of FST, we may find a region in which the solutions of the problems lie.

Kaleva [6] employed the concept of FST on differential equations, which helps in great means in the study of the fuzzy initial value problem (FIVP)s and fuzzy boundary value problem (FBVP)s. Seikkala [10] promoted the concept of FIVP. Then, another several researchers have started to do works on FBVP [4, 5]. Nayak and Chakraverty [9] have calculated non-probabilistic solution of moving plate problem having uncertain parameters.

2. MATHEMATICAL FORMULATION

An axisymmetric unsteady free convective boundary layer flow in a Darcy-Forchheimer fluid saturated porous medium is considered. The flow past a vertical cone with a transverse magnetic field with intensity B_0 . The model is considered in a cartesian (x', y') coordinate system with the following assumptions.

- (i) At time $t' \leq 0$, the cone surface and the surrounding fluid which are at rest possess the same temperature T'_{∞} and concentration level C'_{∞} everywhere in the fluid.
- (ii) As time begins, heat supplied from the cone surface to the fluid and concentration level near the cone surface are hiked at a rate of $q_w(x') = x'^m$ and $q_w^*(x') = x'^n$ respectively. They are managed at the same level.
- (iii) The concentration C' of the diffusing species in the binary mixture is very less in comparison to the other chemical species which are present and hence the Soret and Dufour effects can be ignored.
- (iv) As the viscosity and the thermal conductivity of the fluid are dependent on temperature, they are taken as variable using [2,8].
- (v) The semi vertical angle of the cone is α and r' is the local radius of the cone.

(vi) The x'-direction is measured along the cone surface from the leading edge O, and the y'-direction is normal to the cone generator. The cone apex is located at the origin.

Following the above assumptions with the Boussinesq's approximations, we have the flow governing equations as follows.

$$\frac{\partial(u'r')}{\partial x'} + \frac{\partial(v'r')}{\partial y'} = 0$$

$$\begin{aligned} \frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} = \nu \frac{\partial^2 u'}{\partial y'^2} + \frac{1}{\rho} \frac{\partial \mu}{\partial y'} \frac{\partial u'}{\partial y'} - \frac{\sigma B_0^2 u'}{\rho} + g\beta \cos \alpha (T' - T'_\infty) \\ + g\beta^* \cos \alpha (C' - C'_\infty) - \frac{\nu u'}{K} - \frac{b}{K} u'^2 \end{aligned}$$

$$\frac{\partial T'}{\partial t'} + u' \frac{\partial T'}{\partial x'} + v' \frac{\partial T'}{\partial y'} = \frac{\lambda}{\rho C_p} \frac{\partial^2 T'}{\partial y'^2} + \frac{1}{\rho C_p} \frac{\partial \lambda}{\partial y'} \frac{\partial T'}{\partial y'}$$

$$\frac{\partial C'}{\partial t'} + u' \frac{\partial C'}{\partial x'} + v' \frac{\partial C'}{\partial y'} = D \frac{\partial^2 C'}{\partial y'^2} + \frac{\partial D}{\partial y'} \frac{\partial C}{\partial y'} - Ch(C' - C'_{\infty}).$$

Following are the initial and boundary conditions at the surface and far from the cone surface:

$$\begin{split} t' &\leq 0: u' = 0, v' = 0, T' = T'_{\infty}, C' = C'_{\infty}, \forall x', y' \\ t' &> 0: \left\{ \begin{aligned} y' &= 0: u' = 0, v' = 0, \frac{\partial T'}{\partial y'} = -\frac{q_w(x')}{\lambda_{\infty}}, \frac{\partial C'}{\partial y'} = -\frac{q_w(x')}{D} \\ x' &= 0: u' = 0, T' = T'_{\infty}, C' = C'_{\infty} \\ y' &\to \infty: u' \to 0, T' \to T'_{\infty}, C' \to C'_{\infty} \end{aligned} \right\}. \end{split}$$

The above governing equations are highly coupled, parabolic and non-linear. An analytical solution is intractable and in order to attain a numerical solution, the model should be made dimensionless. For this, the following transformations are introduced:

$$x = \frac{x'}{l}, y = \frac{y'}{l}Gr^{\frac{1}{4}}, r = \frac{r'}{l}, where \ r' = x'\sin\alpha, v = \frac{v'l}{\nu_{\infty}}Gr^{-\frac{1}{4}} \\ u = \frac{u'l}{\nu_{\infty}}Gr^{-\frac{1}{2}}, t = \frac{\nu_{\infty}t'}{l^2}Gr^{\frac{1}{2}}, Gr = \frac{g\beta\cos\alpha[q_w(l)/\lambda_{\infty}]l^4}{\nu_{\infty}^2}, \\ \theta = \frac{T'-T'_{\infty}}{[q_w(l)/\lambda_{\infty}]}, \phi = \frac{C'-C'_{\infty}}{[q_w^*(l)l/D]}, \mu = -\frac{\mu_{\infty}\theta_r}{\theta-\theta_r}, \lambda = -\frac{\lambda_{\infty}\theta_c}{\theta-\theta_c} \end{cases} \right\},$$

where l is the reference length.

Applying the above non-dimensional parameters, the equations are transformed into the following forms:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{\theta_r}{\theta - \theta_r} \frac{\partial^2 u}{\partial y^2} + \frac{\theta_r}{(\theta - \theta_r)^2} \frac{\partial u}{\partial y} \frac{\partial \theta}{\partial y} - Mu + \theta + Nc\phi \\ &+ \frac{\theta_r}{(\theta - \theta_r)Gr^{\frac{1}{2}}Da} u - \frac{Fs}{Da} u^2 \\ Pr\left(\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y}\right) &= -\frac{\theta_c}{\theta - \theta_c} \frac{\partial^2 \theta}{\partial y^2} + \frac{\theta_c}{(\theta - \theta_c)^2} \left(\frac{\partial \theta}{\partial y}\right)^2 \\ Sc\left(\frac{\partial \phi}{\partial t} + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y}\right) &= -\frac{\theta_r}{\theta - \theta_r} \frac{\partial^2 \phi}{\partial y^2} + \frac{\theta_r}{(\theta - \theta_r)^2} \frac{\partial \phi}{\partial y} \frac{\partial \theta}{\partial y} - Sc \cdot \xi\phi. \end{aligned}$$

Corresponding non-dimensionalized initial and boundary conditions are-

$$\begin{split} t &\leq 0: u = 0, v = 0, \theta = 0, \phi = 0, \forall x, y \\ t &> 0: \left\{ \begin{aligned} y &= 0: u = 0, v = 0, \frac{\partial \theta}{\partial y} = -x^m, \frac{\partial \phi}{\partial y} = -x^n \\ x &= 0: u = 0, \theta = 0, \phi = 0 \\ y &\to \infty: u \to 0, \theta \to 0, \phi \to 0 \end{aligned} \right\}. \end{split}$$

3. FUZZIFICATION OF THE EQUATIONS

According to Dutta *et al.* [7], the fuzzy forms of the governing equations along with the boundary conditions are identical with the crisp forms. Here we use $\tilde{}$ sign to characterize the fuzzified expressions. So, the fuzzified governing equations are:

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{u}}{\partial y} = -\frac{\tilde{\theta}_r}{\tilde{\theta} - \tilde{\theta}_r} \frac{\partial^2 \tilde{u}}{\partial y^2} + \frac{\tilde{\theta}_r}{(\tilde{\theta} - \tilde{\theta}_r)^2} \frac{\partial \tilde{u}}{\partial y} \frac{\partial \tilde{\theta}}{\partial y} - \tilde{M}\tilde{u} + \tilde{\theta} + \tilde{N}c\tilde{\phi}$$

$$+ \frac{\tilde{\theta}_r}{(\tilde{\theta} - \tilde{\theta}_r)\tilde{G}r^{\frac{1}{2}}\tilde{D}a} \tilde{u} - \frac{\tilde{F}s}{\tilde{D}a}\tilde{u}^2$$

$$(3.1)$$

(3.2)
$$\tilde{Pr}\left(\frac{\partial\tilde{\theta}}{\partial t} + \tilde{u}\frac{\partial\tilde{\theta}}{\partial x} + \tilde{v}\frac{\partial\tilde{\theta}}{\partial y}\right) = -\frac{\tilde{\theta}_c}{\tilde{\theta} - \tilde{\theta}_c}\frac{\partial^2\tilde{\theta}}{\partial y^2} + \frac{\tilde{\theta}_c}{(\tilde{\theta} - \tilde{\theta}_c)^2}\left(\frac{\partial\tilde{\theta}}{\partial y}\right)^2$$

$$(3.3) \qquad \tilde{S}c\left(\frac{\partial\tilde{\phi}}{\partial t} + \tilde{u}\frac{\partial\tilde{\phi}}{\partial x} + \tilde{v}\frac{\partial\tilde{\phi}}{\partial y}\right) = -\frac{\tilde{\theta}_r}{\tilde{\theta} - \tilde{\theta}_r}\frac{\partial^2\tilde{\phi}}{\partial y^2} + \frac{\tilde{\theta}_r}{(\tilde{\theta} - \tilde{\theta}_r)^2}\frac{\partial\tilde{\phi}}{\partial y}\frac{\partial\tilde{\theta}}{\partial y} - \tilde{S}c \cdot \tilde{\xi}\tilde{\phi}.$$

Corresponding fuzzified initial and boundary conditions are-

$$\begin{split} t &\leq 0: \tilde{u} = 0, \tilde{v} = 0, \tilde{\theta} = 0, \tilde{\phi} = 0, \forall x, y \\ t &> 0: \left\{ \begin{aligned} y &= 0: \tilde{u} = 0, \tilde{v} = 0, \frac{\partial \tilde{\theta}}{\partial y} = -x^{\tilde{m}}, \frac{\partial \tilde{\phi}}{\partial y} = -x^{\tilde{n}} \\ x &= 0: \tilde{u} = 0, \tilde{\theta} = 0, \tilde{\phi} = 0 \\ y &\to \infty: \tilde{u} \to 0, \tilde{\theta} \to 0, \tilde{\phi} \to 0 \end{aligned} \right\} \end{split}$$

4. METHOD OF SOLUTION

Here, to solve the fuzzy forms of the governing equations along with the initial and boundary conditions, we have applied the FDM. By dropping the \tilde{s} sign from Eq.(3.1)- Eq.(3.3) and applying the finite difference scheme there, the following expressions are obtained:

$$\begin{aligned} u(i,j,k) &= \frac{1}{-\frac{1}{\Delta t} + \frac{u(i,j+1,k) - u(i,j,k)}{\Delta x} - \frac{v(i,j,k)}{\Delta y} - \frac{2\theta_r}{\Delta^2 y\{\theta(i,j,k) - \theta_r\}}}{\left[-\frac{u(i+1,j,k)}{\Delta t} + M - \frac{\theta_r}{\theta_r} + M - \frac{\theta_r}{\theta_r} + \frac{Fs}{Da}u(i,j,k) + M - \frac{\theta_r}{\theta_r} + \frac{1}{\theta_r} + \frac{\theta_r}{\theta_r} + \frac{\theta_r}{\theta_r}$$

$$\theta(i,j,k) = \frac{1}{-\frac{1}{\Delta t} - \frac{u(i,j,k)}{\Delta x} - \frac{v(i,j,k)}{\Delta y} - \frac{2\theta_c}{Pr \bigtriangleup^2 y\{\theta(i,j,k) - \theta_c\}}} \left[-\frac{\theta(i+1,j,k)}{\Delta t} - u(i,j,k) - u(i,j,k) - \frac{\theta(i,j+1,k)}{\Delta x} - \frac{v(i,j,k)\theta(i,j,k+1)}{\Delta y} - \frac{\theta_c}{Pr\{\theta(i,j,k) - \theta_c\}} - \frac{\theta_c}{Pr\{\theta(i,j,k) - \theta_c\}} \right]$$

$$(4.2) \qquad \frac{\theta(i,j,k+1) + \theta(i,j,k-1)}{\bigtriangleup^2 y} + \frac{\theta_c}{Pr(\theta - \theta_c)^2} \left\{ \frac{\theta(i,j,k+1) - \theta(i,j,k)}{\Delta y} \right\}^2 \right]$$

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$$\phi(i,j,k) = \frac{1}{-\frac{1}{\Delta t} - \frac{u(i,j,k)}{\Delta x} - \frac{v(i,j,k)}{\Delta y} - \frac{2\theta_r}{Sc\Delta^2 y\{\theta(i,j,k) - \theta_r\}} + \xi} \left[-\frac{\phi(i+1,j,k)}{\Delta t} - u(i,j,k)\frac{\phi(i,j+1,k)}{\Delta x} - \frac{v(i,j,k)\phi(i,j,k+1)}{\Delta y} - \frac{\theta_r}{Sc\{\theta(i,j,k) - \theta_r\}} - \frac{\theta_r}{Sc\{\theta(i,j,k) - \theta_r\}} - \frac{\phi(i,j,k+1) + \phi(i,j,k-1)}{\Delta^2 y} + \frac{\theta_r}{Sc(\theta(i,j,k) - \theta_r)^2} - \frac{\theta_r}{Sc(\theta(i,j,k) - \theta_r)$$

Corresponding boundary conditions are

 $\forall i, j \text{ and } N \text{ is the total number of sub-divisions in the interval } [0, \infty).$

In the computer programming code developed for solving the model, α -cut technique is also used. The distribution of the velocity, temperature and species concentration against the various involved parameters are investigated by taking $\alpha = 1$ as it gives the solutions with membership grade 1 and they are shown graphically.

5. RESULTS AND DISCUSSION

The system of the fuzzified algebraic Eqs. (4.1), (4.2) and (4.3) along with Eq. (4.4) is solved numerically using an iterative scheme based on Gauss Seidal iterative method. The parameter values are taken as $\theta_r = [4, 5, 6], \theta_c = [4, 5, 6], M = [0.1, 0.2, 0.3], Nc = [1, 2, 3], Gr = [1.5, 2, 2.5], Da = [0.1, 0.2, 0.3], Fs = [0.1, 0.2, 0.3], Pr = [6, 7, 8], Sc = [0.20, 0.22, 0.24], \xi = [0.1, 0.15, 0.2], m = [0.4, 0.5, 0.6] and n = [0.4, 0.5, 0.6], unless otherwise stated. A lot of numerical results have been achieved throughout the study, but in order to get a physical acumen into the problem, a representative set is presented graphically in Figures 1-6.$

Figure 1 and Figure 2 indicate the effect of the variable viscosity parameter θ_r on velocity u and species concentration ϕ respectively. It is observed that θ_r has inverse effect on u. As the values of θ_r rises, the resistance force between different layers of the fluid also rises and as a result velocity of the fluid diminishes. Concentration of the species ϕ also decreases with the increasing values of θ_r .



The variation of the temperature of the fluid θ is directly proportional to the variable thermal conductivity parameter θ_c (Figure 3). It is expected that the temperature within the fluid rises as a result of increase of thermal conductivity. So, θ enhances with θ_c .



Figure 4 shows the effect of Darcy number Da on u. Increasing Da enhances the permeability and simultaneously depreciates the Darcian resistance as less



solid fibers are present in the region. The flow is therefore accelerated for higher Da. The Forchheimer drag force is a second-order retarding force. Increasing Forchheimer number Fs results a strong increase in Forchheimer drag force which decelerates the flow, that is, lowers velocities (Figure 5).

Schmidt number Sc and chemical reaction parameter ξ has inverse effects on ϕ . As Sc advances, the thickness of the concentration boundary layer decreases due to which the concentration gradient steps up. As a result ϕ diminishes (Figure 6).

6. CONCLUSION

An unsteady free convection boundary layer flow past a vertical cone which is driven by buoyancy force has been investigated numerically using fuzzy concept. The following conclusions are drawn from the investigation.

- (i) The effects of the various uncertain parameters involved in a fluid flow model are distinct.
- (ii) Variable viscosity parameter has adverse effect on both the velocity and the species concentration distribution. On the other hand, temperature enhances with the variable thermal conductivity parameter.
- (iii) Velocity is inversely proportional to Forchheimer number whereas directly proportional to Darcy number.
- (iv) Species concentration diminishes for increasing values of Schmidt number.

u'/v'	velocity components along	μ	dynamic viscosity
	the x and y directions		
ρ	density	$ u = \frac{\mu}{ ho} $	kinematic viscosity
σ	electrical conductivity	K	permeability of porous medium
g	gravitational acceleration	β	coefficient of thermal expansion
β^*	coefficient of volumetric expansion	T'	temperature
b	Forchheimer geometrical constant	λ	thermal conductivity
C_p	specific heat at constant pressure	D	molecular diffusivity
Ch	coefficiet of chemical reaction	θ_r	variable viscosity parameter
θ_c	variable thermal conductivity parameter	$M = \frac{\sigma B_0^2 l^2}{\mu_\infty} G r^{-\frac{1}{2}}$	magnetic field parameter
$Nc = \frac{\beta^* (C'_w - C'_\infty)}{\beta (T'_w - T'_\infty)}$	Buoyancy ratio parameter	$Da = \frac{K}{l^2}$	Darcy number
$Fs = \frac{b}{l}$	Forchheimer number	$Pr = \frac{\nu_{\infty}\rho C_p}{\lambda_{\infty}}$	Prandtl number
$Sc = \frac{\nu}{D}$	Schmidt number	$\xi = \frac{Chl^2}{\nu_{\infty}Gr^{\frac{1}{2}}}$	chemical reaction parameter

TABLE 1. Nomenclature

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