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FUZZY MAXIMAL AND MINIMAL CLOPEN SETS

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ABSTRACT. The aim of the paper is to introduce the notions of fuzzy maximal and fuzzy minimal clopen sets in fuzzy topological spaces. The notions of fuzzy maximal and fuzzy minimal clopen sets are respectively independent to the notions of fuzzy maximal and fuzzy minimal open (resp. closed) sets. Further, fuzzy maximal clopen and fuzzy minimal clopen sets are discussed using fuzzy disconnectedness.

1. INTRODUCTION

Since the introduction of fuzzy sets by Zadeh [7], it has been a milestone of developing many ideas in various research fields. Chang introduced the notion of fuzzy topology in [2]. In topology and fuzzy topology, open sets have played a prominent role for many decades. The notions of minimal open, maximal open, minimal closed, maximal closed sets are introduced by Nakaoka and Oda in [4] and [5]. As a result of this, Ittanagi and Wali introduced the ideas of fuzzy minimal open (resp. closed) fuzzy maximal open (resp. closed) set in [3]. Meanwhile, Mukherjee [1] developed maximal and minimal clopen sets and compared the interrelations of these sets to disconnectedness. In [6], Swaminathan and Sivaraja discussed some properties of fuzzy minimal open and fuzzy maximal open and on that paper, it is shown that if a fuzzy topological space

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having both fuzzy minimal open and fuzzy maximal open set, then it may be fuzzy disconnected. With respect to the Theorem 2.5 of this current paper, it reveals that a fuzzy set which is both fuzzy maximal clopen and fuzzy minimal clopen set satisfies not only the disconnectedness of the fuzzy space but also a fuzzy set which is both fuzzy maximal clopen and fuzzy minimal clopen and these are the only fuzzy clopen sets in the fuzzy space. It is the odd behavior observed among the notions of fuzzy minimal open and fuzzy maximal open with respect to fuzzy minimal clopen and fuzzy maximal clopen and it is the finest of its own.

In this paper (X, τ) or X stands for fuzzy topological space. The symbols λ, μ, γ, η,... are used to denote fuzzy sets and a fuzzy point with support $x(\in X)$ and value $\alpha(0 < \alpha \leq 1)$ will be denoted by x_α. Then 0_X and 1_X will denote the fuzzy sets having values 0 and 1 respectively at each point of X. $x_\alpha \in \lambda$ will mean $\alpha \leq \lambda(x)$, where as $\lambda(x) \leq \mu(x)$, for each $x \in X$.

2. Fuzzy maximal and minimal clopen sets

To proceed main results, we recall basic definitions and results:

Definition 2.1. *(Ittanagi and Wali [3]) A proper fuzzy open set μ of X is said to be a fuzzy maximal open set if λ is a fuzzy open set such that $\mu < \lambda$, then $\lambda = \mu$ or $\lambda = 1_X$.*

Definition 2.2. *(Ittanagi and Wali [3]) A proper fuzzy open set μ of X is said to be a fuzzy minimal open set if λ is a fuzzy open set such that $\lambda < \mu$, then $\lambda = \mu$ or $\lambda = 0_X$.*

Definition 2.3. *(Ittanagi and Wali [3]) A proper fuzzy closed set γ of X is said to be a fuzzy minimal closed set if α is a fuzzy closed set such that $\alpha < \gamma$, then $\alpha = \gamma$ or $\alpha = 0_X$.*

Definition 2.4. *(Ittanagi and Wali [3]) A proper fuzzy closed set γ of X is said to be a fuzzy maximal closed set if α is a fuzzy closed set such that $\gamma < \alpha$, then $\alpha = \gamma$ or $\alpha = 1_X$.*

Theorem 2.1. *(Ittanagi and Wali [3]) If ϑ is a fuzzy maximal open set and α is a fuzzy minimal open set in a fuzzy topological space X with $\alpha \nless \vartheta$ then $\vartheta = 1_X - \alpha$.*

Definition 2.5. *A proper fuzzy clopen set α of a fuzzy topological space X is said to be fuzzy minimal clopen if β is a fuzzy clopen set such that $\beta < \alpha$, then $\beta = \alpha$ or $\beta = 0_X$.*

Example 1. *Let $X = \{a, b, c, d\}$. Then fuzzy sets $\gamma_1 = \frac{0}{a} + \frac{1}{b} + \frac{1}{c} + \frac{0}{d}$; $\gamma_2 = \frac{0}{a} + \frac{0}{b} + \frac{0}{c} + \frac{1}{d}$; $\gamma_3 = \frac{0}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ and $\gamma_4 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{0}{d}$ are defined as follows: Consider the fuzzy topology $\tau = \{0_X, \gamma_1, \gamma_2, \gamma_3, \gamma_4, 1_X\}$. Here γ_4 is fuzzy minimal clopen set but it is neither fuzzy minimal open nor fuzzy minimal closed set.*

It is clear that a fuzzy minimal open set or a fuzzy minimal closed set need not to be a fuzzy minimal clopen set. Hence the notion of fuzzy minimal clopen set is independent to the notions of fuzzy minimal open set as well as fuzzy minimal closed set. It is also easy to see that if a set α is both fuzzy minimal open and fuzzy minimal closed, then α is fuzzy minimal clopen. Again, a fuzzy clopen set is fuzzy minimal clopen if it is either fuzzy minimal open or fuzzy minimal closed.

Definition 2.6. *A proper fuzzy clopen set α of a fuzzy topological space X is said to be fuzzy maximal clopen if β is a fuzzy clopen set such that $\alpha < \beta$, then $\beta = \alpha$ or $\beta = 1_X$.*

It is evident that a fuzzy maximal open or a fuzzy maximal closed set may not be a fuzzy maximal clopen set. Therefore, it follows that the notion of fuzzy maximal clopen set is independent to the notion of fuzzy maximal open as well as fuzzy maximal closed set. It is also easy to see that if a set γ is both fuzzy maximal open and fuzzy maximal closed, then γ is fuzzy maximal clopen. In fact, a fuzzy clopen set is fuzzy maximal clopen if it is either fuzzy maximal open or fuzzy maximal closed. In [6], we observed that if a fuzzy topological space has only one proper fuzzy open set, then it is both fuzzy maximal open and fuzzy minimal open. Even we observe that it is neither fuzzy maximal clopen nor fuzzy minimal clopen. If a space has only two proper fuzzy open sets such that one is not contained in other, then both are fuzzy maximal clopen and fuzzy minimal clopen. In addition, fuzzy maximal or fuzzy minimal clopen sets can exists only in a fuzzy disconnected space. As Theorems 2.2 and 2.3 and Corollaries 2.1 and 2.2 are obvious, their proofs are omitted.

Theorem 2.2. *If α is a fuzzy minimal clopen set and β is a fuzzy clopen set in X, then either $\alpha \wedge \beta = 0_X$ or $\alpha < \beta$.*

Corollary 2.1. *If α and β are distinct fuzzy minimal clopen sets in X , then $\alpha \wedge \beta = 0_X$.*

Theorem 2.3. *If α is a fuzzy maximal clopen set and β is a fuzzy clopen set in X, then either $\alpha \vee \beta = 1_X$ or $\beta < \alpha$.*

Corollary 2.2. *If α and β are distinct fuzzy maximal clopen sets in X, then $\alpha \vee \beta = 1_X$.*

Lemma 2.1. *If α is fuzzy minimal clopen in a fuzzy topological space X, then $1_X - \alpha$ is fuzzy maximal clopen in X and conversely.*

Proof. Let α, β be any two proper fuzzy clopen sets such that $1_X - \alpha < \beta$. Now we have $1_X - \beta < \alpha$. As α is being a fuzzy minimal clopen set, we have $1_X - \beta = \alpha$ or $1_X - \beta = 0_X$ implies that $1_X - \alpha = \beta$ or $\beta = 1_X$. Hence, $1_X - \alpha$ is a fuzzy maximal clopen set. Similarly follows the converse. □

Similar to Theorem 3.1 of [6], we have the following theorem.

Theorem 2.4. *If α is fuzzy minimal clopen and β is fuzzy maximal clopen in X, then either $\alpha < \beta$ or $\alpha < 1_X - \beta$.*

Proof. Similar to the proof of Theorem 3.1 of [6]. □

Theorem 2.5. *If a fuzzy topological space X contains a fuzzy set α which is both fuzzy maximal and fuzzy minimal clopen, then:*

(i) *α and $1_X - \alpha$ are the only fuzzy sets in the space which are both fuzzy maximal and fuzzy minimal clopen and*
(ii) *α and $1_X - \alpha$ are the only proper fuzzy clopen sets in the space.*

Proof.

(i) By Lemma 2.1, $1_X - \alpha$ is both fuzzy maximal and fuzzy minimal clopen for any fuzzy maximal and fuzzy minimal clopen set α in X. Let there exist fuzzy maximal and fuzzy minimal clopen set β in X distinct from α. By Lemma 2.1, $1_X - \beta$ is also both fuzzy maximal and fuzzy minimal clopen. As α,β both being fuzzy minimal and fuzzy maximal clopen, by Corollary 2.1, and 2.2, we have $\alpha \wedge \beta = 0_X$ and $\alpha \vee \beta = 1_X$. Hence $\beta = 1_X - \alpha$. If for any $\alpha \neq \beta$, β and $1_X - \beta$ are identical to $1_X - \alpha$ and α respectively. Hence, the result follows for all possible combinations of α,

β, $1_X - \alpha$ and $1_X - \beta$.

(ii) Let γ be a proper fuzzy clopen set in X. For a fuzzy maximal clopen set α, we have $\alpha \vee \gamma = 1_X$ or $\gamma < \alpha$. For a fuzzy minimal clopen set α we have $\alpha \wedge \gamma = 0_X$ or $\alpha < \gamma$. $\alpha \vee \gamma = 1_X$ and $\alpha \wedge \gamma = 0_X$ implies that $\gamma = 1_X - \alpha$. $\alpha \vee \gamma = 1_X$ and $\alpha < \gamma$ implies that $\gamma = 1_X$.$\gamma < \alpha$ and $\alpha \wedge \gamma = 0_X$ implies that $\gamma = 0_X$.

□

Theorem 2.6. *In a fuzzy topological space X, fuzzy maximal clopen and minimal clopen sets appear in pairs.*

Proof. By Theorem 2.5, if α is both fuzzy maximal and fuzzy minimal clopen in fuzzy topological space X, then $1_X - \alpha$ is also both fuzzy maximal and fuzzy minimal clopen, also such pairs of sets in X are unique. By Lemma 2.1, if α is fuzzy maximal (resp. minimal) clopen in X, then $1_X - \alpha$ is fuzzy minimal (resp. maximal) clopen in X. □

Theorem 2.7. *If α is a fuzzy maximal open set and β is a fuzzy minimal open set of fuzzy topological space X with $\alpha \nless \beta$, then α is a fuzzy maximal clopen and β is a fuzzy minimal clopen set.*

Proof. On deploying maximality of α by Theorem 2.1, $\alpha = 1_X - \beta$. Hence, both α, β are fuzzy clopen. Since α is both fuzzy clopen and fuzzy maximal open (resp. fuzzy minimal open), it is easy to see that α is fuzzy maximal clopen. Similarly β holds. □

Theorem 2.8. *If γ is a fuzzy maximal clopen set in X, then $\gamma \vee \mu$ is not a proper fuzzy clopen set distinct from γ for any proper fuzzy open or fuzzy closed set μ in X.*

Proof. Let $\gamma \vee \mu$ be a proper fuzzy clopen set in X. As γ is a fuzzy maximal clopen set and $\gamma < \gamma \vee \mu$, either $\gamma \vee \mu = 1_X$ or $\gamma = \gamma \vee \mu$. Then $\gamma = \gamma \vee \mu$ implies that $\mu < \gamma$. □

Theorem 2.9. *If γ is a fuzzy minimal clopen set in X, then $\gamma \wedge \mu$ is not a proper fuzzy clopen set distinct from γ for any proper fuzzy open or fuzzy closed set μ in X.*

From Theorem 2.8 and Theorem 2.9 it is observed that the form of fuzzy clopen sets in fuzzy topological space consists of a fuzzy maximal or minimal clopen set.

Theorem 2.10. *If Λ is a collection of distinct fuzzy maximal clopen sets and $\alpha \in \Lambda$, then $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta \neq 0_X$. If Λ is a finite collection, then $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$ is a fuzzy minimal clopen if and only if $1_X - \alpha = \bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$.*

Proof. As $\beta \in \Lambda - \alpha$ is a fuzzy maximal clopen set , $1_X - \beta$ is a fuzzy minimal clopen set. Then by Theorem 2.4, $1_X - \beta < \alpha$. Hence we get, $1_X - \bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta < \alpha$ which implies $\alpha = 1_X$ if $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta = 0_X$.This is a contradiction to our assumption that α is a fuzzy maximal clopen set. Hence, $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta \neq 0_X$.

Evidently, $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$ is a fuzzy minimal clopen if $1_X - \alpha = \bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$. Now let $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$ be a fuzzy minimal clopen set. If Λ is a finite collection, then $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$ is a fuzzy clopen set. Since $1_X - \bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta < \alpha$ we have $1_X - \alpha < \bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$. As α is a fuzzy maximal clopen, then $1_X - \alpha$ is fuzzy minimal clopen. If $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$ is a fuzzy minimal clopen set distinct from $1_X - \alpha$ then by corollary 2.1 we have $(\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta)\wedge(1_X-\alpha) = 0_X$. This implies $\bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta < \alpha$. So we get, $1_X-\alpha < \bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta < \alpha$ which is wrong. Hence, we obtain $1_X - \alpha = \bigwedge_{\beta\in\Lambda-\{\alpha\}} \beta$. □

Theorem 2.11. *If Λ is a collection of distinct fuzzy minimal clopen sets and $\alpha \in \Lambda$, then $\bigvee_{\beta\in\Lambda-\{\alpha\}} \beta \neq 1_X$. If Λ is a finite collection,then $\bigvee_{\beta\in\Lambda-\{\alpha\}} \beta$ is a fuzzy maximal clopen if and only if $1_X - \alpha = \bigvee_{\beta\in\Lambda-\{\alpha\}} \beta$*

It is observed that if γ is fuzzy clopen in (X, τ), then $\alpha \wedge \gamma$ is fuzzy clopen in (α, τ_α). In addition, if α is fuzzy clopen in (X, τ) then a set fuzzy clopen in (α, τ_α) is also fuzzy clopen in (X, τ).

Theorem 2.12. *Let α, ϑ be fuzzy clopen sets in X such that $\alpha \wedge \vartheta \neq 0_X$ then $\alpha \wedge \vartheta$ is a fuzzy minimal clopen set in (α, τ_α) if ϑ is fuzzy minimal clopen in (X, τ).*

Proof. Similar to that of Theorem 4.11 in [6]. □

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