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# IRREGULAR COLORING OF SOME SPECIAL GRAPHS

R. AVUDAINAYAKI[1] AND D. YOKESH

ABSTRACT. For a graph $G$ and a proper coloring $c : V(G) \to \{1, 2, 3, \ldots, k\}$ of the vertices of $G$ for some positive integer $k$, the color code of a vertex $v$ of $G$ (with respect to $c$) is the ordered $(k + 1)$-tuple $code(v) = (a_0, a_1, a_2, \ldots, a_k)$ where $a_0$ is the color assigned to $v$ and $1 \leq i \leq k$, $a_i$ is the number of vertices of $G$ adjacent to $v$ that are colored $i$. The coloring $c$ is irregular if distinct vertices have distinct color codes and the irregular chromatic number $\chi_{ir}(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has an irregular $k$-coloring. In this paper, we obtain the values of irregular coloring for $SF(n, 1)$, friendship graph and splitting graph of star graph.

## 1. INTRODUCTION

Let $G(V, E)$ be simple connected graph. A proper coloring of a graph $G$ is a function $c : V(G) \to N$ having the property that $c(u) \neq c(v)$ for every pair $u, v$ of adjacent vertices of $G$. A $k$-coloring of $G$ uses $k$ colors. The chromatic number $\chi(G)$ of $G$ is the minimum integer $k$ for which $G$ admits a $k$-coloring. In a graph $G$, a proper coloring $c : V(G) \to \{1, 2, 3, \ldots, k\}$ of the vertices of $G$ for some positive integer $k$, the color code of a vertex $v$ of $G$ (with respect to $c$) is the ordered $(k+1)$-tuple $code(v) = (a_0, a_1, a_2, \ldots, a_k)$, where $a_0$ is the color assigned to $v$ and $1 \leq i \leq k$, $a_i$ is the number of vertices of $G$ adjacent to $v$ that are colored $i$. The coloring $c$ is irregular if distinct vertices have distinct color codes and the

[1]*corresponding author*

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irregular chromatic number $\chi_{ir}(G)$ of $G$ is the minimum positive integer $k$ for which $G$ has an irregular $k$-coloring. Irregular coloring were introduced in [4] and studied further in [5] inspired by the problem in graph theory concerns finding means to distinguish all the vertices of a connected graph. Further some more results of irregular coloring of graphs are discussed in [1,2,6]. For graph theoretic terminology we refer to Harary [3]. In this paper, we find that the irregular coloring of $SF(n,1)$ graph, friendship graph and splitting graph of star graph.

## 2. Main results

**Definition 2.1.** *An $SF(n,m)$ is a graph consisting of a cycle $C_n$, $n \geq 3$ and $n$ set of $m$ independent vertices where each set joins each of the vertices of $C_n$.*

**Theorem 2.1.** *Let $G = SF(n,1)$, where $n \geq 3$. Then $2\binom{k-1}{2} + 1 \leq n \leq 2\binom{k}{2}$ if and only if $\chi_{ir}(G) = k$.*

*Proof.* Let $V(G) = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{u_i v_i; 1 \leq i \leq n\} \cup \{u_i u_{i+1}; 1 \leq i \leq n-1\} \cup u_n u_1$. Assume that $\chi_{ir}(G) = k$. We have to prove that $2\binom{k-1}{2} + 1 \leq n \leq 2\binom{k}{2}$. Assume to the contrary that $n \geq 2\binom{k}{2} + 1$ or $n \leq 2\binom{k-1}{2}$.

**Case (i):** $n \geq 2\binom{k}{2} + 1$

Let $A_1, A_2, \dots, A_{\binom{k}{2}}, A'_1, A'_2, \dots, A'_{\binom{k}{2}}$ be the $2\binom{k}{2}$ distinct 2 element subsets of the set $\{1, 2, \dots, k\}$, where $A_l = (i,j)$ and $A'_l = (j,i)$, $1 \leq i,j \leq k$; $1 \leq l \leq \binom{k}{2}$ and by our assumption $n \geq 2\binom{k}{2} + 1$, it follows that there exists two vertices $u_i, v_j \in V(G)$ such that $code(u_i) \neq code(v_j)$, which is a contradiction. Hence $n \leq 2\binom{k}{2}$.

**Case (ii):** $n \leq 2\binom{k-1}{2}$

Let $A_1, A_2, \dots, A_{\binom{k}{2}}, A'_1, A'_2, \dots, A'_{\binom{k}{2}}$ be the $2\binom{k-1}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \dots, k-1\}$. We can define a coloring $c$ of $G$ by assigning the 2 distinct colors in $A_l$ and $A'_l$ to the $n$ vertices of $V(G)$, where $1 \leq l \leq \binom{k-1}{2}$. Since $n \leq 2\binom{k-1}{2}$. Hence $c$ is an irregular coloring with at most $k-1$ colors. Thus $\chi_{ir}(G) \leq k-1$, this is a contradiction to our assumption. Hence $n \leq 2\binom{k-1}{2} + 1$. From the above two cases, we get $2\binom{k-1}{2} + 1 \leq n \leq 2\binom{k}{2}$.

Conversely, assume that $2\binom{k-1}{2} + 1 \leq n \leq 2\binom{k}{2}$ and to prove $\chi_{ir}(G) = k$.

Let $A_1, A_2, \ldots, A_{\binom{k}{2}}, A'_1, A'_2, \ldots, A'_{\binom{k}{2}}$ be the $2\binom{k}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \ldots, k\}$. Since $n \leq 2\binom{k}{2}$, we can define a coloring $c$ of $G$ by assigning the 2 distinct colors in $A_l$ and $A'_l$ to the $2n$ vertices of $V(G)$. By the argument used in Case (ii), this coloring is irregular and uses at most $k$ colors. Thus $\chi_{ir}(G) \leq k$. On the other hand, since $n \geq 2\binom{k-1}{2} + 1$ and there are $2\binom{k-1}{2}$ distinct subsets in $\{1, 2, \ldots, k-1\}$, the argument used in Case (i) shows that there is no irregular coloring of $G$ using $k-1$ or fewer colors. Thus $\chi_{ir}(G) \geq k$ and so $\chi_{ir}(G) = k$. $\square$

**Definition 2.2.** *The friendship graph $F_n$ is one-point union of $n$ copies of cycle $C_3$.*

**Theorem 2.2.** *Let $G = F_n$ be a friendship graph. Then $\binom{k-1}{2} + 1 \leq n \leq \binom{k}{2}$ if and only if $\chi_{ir}(G) = k + 1$.*

*Proof.* Let $G = F_n$ be a friendship graph. Assume that $\chi_{ir}(G) = k + 1$. Let

$$V(G) = \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\} \cup w$$

and

$$E(G) = \{u_i v_i; 1 \leq i \leq n\} \cup \{w u_i; 1 \leq i \leq n\} \cup \{w v_i; 1 \leq i \leq n\}$$

with $deg(w) = 2n$. Assign $c(w) = k + 1$. We have to prove that $\binom{k-1}{2} + 1 \leq n \leq \binom{k}{2}$. Assume to the contrary that $n \geq \binom{k}{2} + 1$ or $n \geq \binom{k-1}{2}$.

**Case (i):** $n \geq \binom{k}{2} + 1$

Let $A_1, A_2, \ldots, A_{\binom{k}{2}}$ be the $\binom{k}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \ldots, k\}$, where $A_l = (i, j)$ $1 \leq i, j \leq k$; $1 \leq l \leq \binom{k}{2}$ and by our assumption $n \geq \binom{k}{2} + 1$, it follows that there exists two pair of vertices $(u_l, v_l)$ and $(u_m, v_m)$ such that $code(u_l) = code(u_m)$ and $code(v_l) = code(v_m)$ , which is a contradiction. Hence $n \leq \binom{k}{2}$.

**Case (ii):** $n \geq \binom{k-1}{2}$

Let $A_1, A_2, \ldots, A_{\binom{k-1}{2}}$ be the $\binom{k-1}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \ldots, k-1\}$. We can define a coloring $c$ of $G$ by assigning the 2 distinct colors in $A_l$ to the $n$ vertices of $V(G)$, where $1 \leq l \leq \binom{k-1}{2}$. Since $n \geq \binom{k-1}{2}$. Then $c$ is an irregular coloring with at most $k-1$ colors and $c(w) = 1$. Thus $\chi_{ir}(G) \leq k$, which is a contradiction to our assumption. Hence $n \geq \binom{k-1}{2} + 1$. From the above two cases we get $\binom{k-1}{2} + 1 \leq n \leq \binom{k}{2}$.

Conversely, assume that $\binom{k-1}{2} + 1 \leq n \leq \binom{k}{2}$ and to prove $\chi_{ir}(G) = k + 1$. Let $A_1, A_2, \ldots, A_{\binom{k}{2}}$ be the $\binom{k}{2}$ distinct 2 element subsets of the set $\{1, 2, 3, \ldots, k\}$.

Since $n \leq \binom{k}{2}$, we can define a coloring of $G$ by assigning the 2 distinct colors in $A_l$ to the $n$ vertices of $V(G)$. By the argument used in Case (ii), this coloring is irregular and uses at most $k$ colors. Assign $c(w) = k+1$. Thus $\chi_{ir}(G) \leq k+1$. On the other hand, Since $n \geq \binom{k-1}{2} + 1$ and there are $\binom{k-1}{2} + 1$ distinct subsets in $\{1, 2, \ldots, k-1\}$, the argument used in case (i) shows that there is no irregular coloring of $G$ using $k-1$ or fewer colors. Assign $c(w) = k+1$. Thus $\chi_{ir}(G) \geq k+1$ and hence $\chi_{ir}(G) = k+1$. □

**Definition 2.3.** *A tree containing exactly one vertex which is not a pendent vertex is called a star graph $K_{1,n}$. For a graph $G$, the splitting graph $Spl(G)$ of a graph $G$ is obtained by adding a new vertex $v'$ corresponding to each vertex $v$ of $G$ such that $N(v) = N(v')$.*

**Theorem 2.3.** *If $G$ is a splitting graph of $K_{1,n}$ then $\chi_{ir}(S(K_{1,n})) = n+1$.*

*Proof.* Let $G$ be a splitting graph of $K_{1,n}$ with vertices $V(G) = \{v, v_1, v_2, \ldots, v_n, v', v'_1, v'_2, \ldots, v'_n\}$ and $E(G) = \{vv_i; 1 \leq i \leq n\} \cup \{v_i v'; 1 \leq i \leq n\} \cup \{v'v'_i; 1 \leq i \leq n\}$. First to prove that $\chi_{ir}(S(K_{1,n})) \geq n+1$. In $G$, $N(v'_i) = N(v'_j)$ for all $1 \leq i, j \leq n$. Therefore, we need $n$ distinct colors for the vertices set $\{v_i\}$ and $\{v'_i\}$, where $1 \leq i \leq n$, since $N(v_i) \neq N(v'_i)$. But $v'$ is adjacent to all the vertices of $v'_i$, $1 \leq i \leq n$. Hence assign the color $n+1$ to $v'$. Thus $\chi_{ir}(S(K_{1,n})) \geq n+1$. Next to prove that $\chi_{ir}(S(K_{1,n})) \leq n+1$. The following $n+1$ coloring for $S(K_{1,n})$ is irregular. For $1 \leq i \leq n$, assign the color $i$ for $v_i$ and $v'_i$, $i+1$ for $v$ and $v'$.Since $deg(v_i) \neq deg(v'_i)$ and $deg(v) \neq deg(v')$. It follows that $code(v_i) \neq code(v'_i)$ and $code(v) \neq code(v')$. Hence $\chi_{ir}(S(K_{1,n})) \neq n+1$. Thus, we get $\chi_{ir}(S(K_{1,n})) = n+1$. □

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DEPARTMENT OF MATHEMATICS
SRI SAIRAM INSTITUTE OF TECHNOLOGY
CHENNAI-44, INDIA
*Email address*: avuramesh2004@gmail.com

DEPARTMENT OF MATHEMATICS
SMK FOMRA INSTITUTE OF TECHNOLOGY
CHENNAI-03, INDIA
*Email address*: yokeshzabin@yahoo.com