

A NEW SUBCLASS OF NEGATIVE UNIVALENT FUNCTIONS INVOLVING POLYLOGARITHM FUNCTIONS

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ABSTRACT. In this current work, we introduce and study some properties for the new subclass $\mathcal{N}_{\beta, \gamma, \delta, b}^n(\phi(\xi))$ of polylogarithms functions associated with the differential operator $\mathcal{D}_{\lambda, \delta}^n f(\xi)$. Also, we have obtained coefficient inequalities, integral means of inequalities, extreme points and distortion of the class.

1. INTRODUCTION

Let \mathcal{A} represent the class of functions $f(\xi)$ of the form

$$(1.1) \quad f(\xi) = \xi + \sum_{k=2}^{\infty} a_k \xi^k,$$

which are analytic in the unit disk $\mathcal{U} = \xi : |\xi| < 1$. If the functions $f(\xi)$ are given by (1.1) and $g(\xi)$ are given by

$$g(\xi) = \xi + \sum_{k=2}^{\infty} b_k \xi^k,$$

then the Hadamard product of $f(\xi)$ and $g(\xi)$ is defined by

$$(f * g)(\xi) = \xi + \sum_{k=2}^{\infty} a_k b_k \xi^k.$$

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If $f(\xi)$ and $g(\xi)$ are analytic in \mathcal{U} , we state that $f(\xi)$ is subordinate to $g(\xi)$, i.e. $f(\xi) \prec g(\xi)$ if a Schwarz function $w(\xi)$ exists, in which $w(0) = 0$ and $|w| < 1$ such that $f(\xi) = g(w(\xi))$. Moreover, if the function $g(\xi)$ is univalent in \mathcal{U} , then the above subordination equivalence holds (see [7, 8]). $f(\xi) \prec g(\xi)$ if and only if $f(0) = g(0)$, and $f(\mathcal{U}) \subset g(\mathcal{U})$.

For $f(\xi) \in \mathcal{A}$, Al-Oboudi [2] initiated the following differential operator:

$$\mathcal{D}_\delta^n f(\xi) = \xi + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k \xi^k, (n \in N_0 = N \cup \{0\}, \delta > 0 : \xi \in \mathcal{U}).$$

For $f(\xi) \in \mathcal{A}$, Ruscheweyh [9] initiated the following differential operator:

$$\mathcal{D}^\lambda f(\xi) = \frac{\xi}{(1-\xi)^{\lambda+1}} * f(\xi) = \xi + \sum_{k=2}^{\infty} \frac{(\lambda+k-1)!}{\lambda!(k-1)!} a_k \xi^k, (\lambda > -1).$$

Let $p(\xi)$ represent the class of form $p(\xi) = 1 + p_1\xi + p_2\xi^2 + \dots$, which are analytic in \mathcal{U} and satisfy the condition $\operatorname{Re}\{p(\xi)\} > 0$.

The Polylogarithms functions $\mathfrak{G}(n, \delta)$ are given by

$$\mathfrak{G}(n, \delta) = \sum_{k=1}^{\infty} \frac{\xi^k}{[1 + (k-1)\delta]^n}.$$

Note that $\mathfrak{G}(-1, 1) = \frac{\xi}{(1-\xi)^2}$ for $k = 1, 2, 3, \dots$ is Koebe function. For more information regarding polylogarithms of univalent functions see Ponnusamy and Sabapathy [7], K. Al Shaqsi and M. Daraus [4] and Ponnusamy [8]. Now we introduce a function $\mathfrak{G}^\kappa(n, \delta)$ given by

$$\mathfrak{G}(n, \delta) * \mathfrak{G}^\kappa(n, \delta) = \frac{\xi}{(1-\xi)^{\lambda+1}}, \lambda > -1, n \in \mathbb{Z},$$

and obtain the linear operator

$$(1.2) \quad \mathcal{D}_{\lambda, \delta}^n f(\xi) = \mathfrak{G}^\kappa(n, \delta) * f(\xi).$$

Presently we get the explicit form of the function

$$\mathfrak{G}^\kappa(n, \delta) = \sum_{k=1}^{\infty} [1 + (k-1)\delta]^n \frac{(\lambda+k-1)!}{\lambda!(k-1)!} \xi^k.$$

From the equation (1.2) we define

$$\mathcal{D}_{\lambda, \delta}^n f(\xi) = \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n \frac{(\lambda+k-1)!}{\lambda!(k-1)!} a_k \xi^k.$$

Note that $\mathcal{D}_{0,1}^n = \mathcal{D}^n$, $\mathcal{D}_{\lambda,\delta}^0 = \mathcal{D}^\lambda$, which give the Salagean differential operator [10] and Ruscheweyh differential operator [9] respectively. It is obvious that the operator consists of two well known operators. Also note that $\mathcal{D}_{0,\delta}^0 = f(\xi)$ and $\mathcal{D}_{1,\delta}^0 = \mathcal{D}_{0,1}^1 = \xi f'(\xi)$.

Definition 1.1. We define $\mathcal{M}_{\beta,\gamma,\delta,b}^n(\phi(\xi))$ be the class of $f(\xi) \in \mathcal{A}$ for which

$$1 + \frac{1}{b} \left(\frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right) - \beta \left| \frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right| \prec \phi(\xi),$$

where $n, \lambda \in N_0, \beta > 0, \delta > 0, b > 0, \phi \in p; \xi \in \mathcal{U}$.

Definition 1.2. We define $\phi(\xi) = \frac{1+(1-2\alpha)\xi}{(1-\xi)}$, then $\mathcal{M}_{\beta,\gamma,\delta,b}^n(\phi(\xi)) \equiv \mathcal{M}_{\beta,\gamma,\delta,b}^n(\alpha)$ be the class of $f(\xi) \in \mathcal{A}$ for which

$$1 + \frac{1}{b} \left(\frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right) - \beta \left| \frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right| > \alpha,$$

where $n, \lambda \in N_0, \beta > 0, \delta > 0, b > 0, \phi \in p, 0 \leq \alpha \leq 1; \xi \in \mathcal{U}$.

Let \mathcal{T} represent the subclass of \mathcal{A} consisting of the functions that can be expressed in the form

$$(1.3) \quad f(\xi) = \xi - \sum_{k=2}^{\infty} a_k \xi^k.$$

Now, define the subclass $\mathcal{N}_{\beta,\gamma,\delta,b}^n(\alpha) = \mathcal{M}_{\beta,\gamma,\delta,b}^n(\alpha) \cap \mathcal{T}$. Since $\mathcal{N}_{\beta,\gamma,\delta,b}^n(\alpha) \subset \mathcal{M}_{\beta,\gamma,\delta,b}^n(\alpha)$. Note that $\mathcal{N}_{0,\lambda,1,1}^n \phi(\xi) = \mathcal{K}_\lambda^n \phi(\xi)$, $\mathcal{N}_{0,\lambda,1,1}^n(\alpha) = \mathcal{R}_\lambda^n(\alpha)$ considered by K. AlShaqs and M. Darus [4], $\mathcal{N}_{0,0,1,1}^0 \phi(\xi) = \mathcal{S}^* \phi(\xi)$ considered by Ma and Minda [6], $\mathcal{N}_{0,\lambda,1,1}^0(\alpha) = \mathcal{R}_\lambda(\alpha)$ initiated and considered by Ahuja [1] and $\mathcal{N}_{0,0,1,1}^n(\alpha) = \mathcal{R}_n(\alpha)$ initiated and considered by Kadioglu [3].

2. MAIN RESULTS

Theorem 2.1. Let $f(\xi)$ be defined by (1.3). Then $f \in \mathcal{A}$ if and only if

$$(2.1) \quad \sum_{k=2}^{\infty} (k - kb\beta + b\beta - 1 + b - b\alpha)[1 + (k-1)\delta]^n \mathcal{C}(\lambda) |a_k| \leq (1-\alpha)b,$$

where $\mathcal{C}(\lambda) = \frac{(k+\lambda-1)!}{\lambda!(k-1)!}$, $0 \leq \alpha < 1, n \in N_0 = N \cup \{0\}, \delta > 0, b > 0, \lambda \geq 0; \xi \in \mathcal{U}$.

Proof. Suppose that the inequality (2.1) is true and $|\xi| < 1$. Then it shows that the values of $1 + \frac{1}{b} \left(\frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right) - \beta \left| \frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right|$ lies in a circle centered at $w = 1$ whose radius is $(1 - \alpha)b$, i.e.,

$$\left| 1 + \frac{1}{b} \left(\frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right) - \beta \left| \frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right| - \alpha + 1 \right| < 1,$$

which gives

$$\sum_{k=2}^{\infty} (k - kb\beta + b\beta - 1 + b - b\alpha)[1 + (k-1)\delta]^n \mathcal{C}(\lambda) |a_k| \leq (1 - \alpha)b.$$

Hence $f(\xi)$ satisfies the condition (2.1).

Conversely, let us assume that the function f is defined by (1.3) in the class $\mathcal{N}_{\beta,\gamma,\delta,b}^n(\alpha)$, then $\operatorname{Re} \left(1 + \frac{1}{b} \left(\frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right) - \beta \left| \frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right| \right) > \alpha$, if we choose the value of ξ on the real axis so that $1 + \frac{1}{b} \left(\frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right) - \beta \left| \frac{\xi(\mathcal{D}_{\lambda,\delta}^n f(\xi))'}{\mathcal{D}_{\lambda,\delta}^n f(\xi)} - 1 \right|$ is real and let $\xi \rightarrow 1^-$ through real values, we obtain the result

$$\sum_{k=2}^{\infty} (k - kb\beta + b\beta - 1 + b - b\alpha)[1 + (k-1)\delta]^n \mathcal{C}(\lambda) |a_k| \leq (1 - \alpha)b.$$

Hence the result is sharp. □

Theorem 2.2. *Let*

$$f_1(\xi) = \xi \text{ and } f_k(\xi) = \xi - \sum_{k=2}^{\infty} \eta_k \frac{(1-\alpha)b}{\psi(\lambda)} \xi^k, k = 2, 3, \dots,$$

where

$$\psi(\lambda) = \sum_{k=2}^{\infty} (k - kb\beta + b\beta - 1 + b - b\alpha)[1 + (k-1)\delta]^n \mathcal{C}(\lambda).$$

Then $f \in \mathcal{N}_{\beta,\lambda,\delta,b}^n(\alpha)$ if and only if it can be expressed in the form $f(\xi) = \sum_{k=1}^{\infty} \eta_k f_k(\xi)$, where $\eta_k > 0$ and $\sum_{k=1}^{\infty} \eta_k = 1$.

Proof. Let

$$\begin{aligned} f(\xi) &= \sum_{k=1}^{\infty} \eta_k f_k(\xi) = \xi - \sum_{k=2}^{\infty} \eta_k \frac{(1-\alpha)b}{\psi(\lambda)} \xi^k \\ &= \sum_{k=2}^{\infty} \eta_k \frac{(1-\alpha)b}{\psi(\lambda)} (\psi(\lambda) - \xi^k) = (1-\alpha)b \sum_{k=1}^{\infty} \eta_k \\ &= (1-\alpha)b(1 - \eta_1) < (1-\alpha)b \end{aligned}$$

which shows that $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$.

Conversely, suppose that $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$. Since $|a_k| \leq \frac{(1-\alpha)b}{\psi(\lambda)}$, $k = 2, 3, \dots$. Let $\eta_k \leq \frac{\psi(\lambda)}{(1-\alpha)b}$, $\eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k$. Then we obtain $f(\xi) = \sum_{k=1}^{\infty} \eta_k f_k(\xi)$. \square

Theorem 2.3. Let $f(\xi) = \xi - \sum_{k=2}^{\infty} |a_k| \xi^k$, $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$, then for $|\xi| = r$, we have:

$$\begin{aligned} r - \frac{(1-\alpha)b}{(b+1-\beta b-\alpha b)(1+\delta)^n(\lambda+1)} r^2 \\ \leq |f(\xi)| \leq r + \frac{(1-\alpha)b}{(b+1-\beta b-\alpha b)(1+\delta)^n(\lambda+1)} r^2 \end{aligned}$$

and

$$\begin{aligned} 1 - \frac{2(1-\alpha)b}{(b+1-\beta b-\alpha b)(1+\delta)^n(\lambda+1)} r \\ \leq |f'(\xi)| \leq 1 + \frac{2(1-\alpha)b}{(b+1-\beta b-\alpha b)(1+\delta)^n(\lambda+1)} r. \end{aligned}$$

Theorem 2.4. The class $\mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$ is convex.

Proof. Let the function $f_j(\xi) = \xi + \sum_{k=2}^{\infty} a_{k,j} \xi^k$, $a_{k,j} \geq 0$, $j = 1, 2$ lies in the class $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$. It is sufficient to prove that $h(\xi) = (\gamma+1)f_1(\xi) - \gamma f_2(\xi)$, $0 \leq \xi \leq 1$, the class $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$. Since $h(\xi) = \xi - \sum_{k=2}^{\infty} [(1+\gamma)a_{k,1} - \gamma a_{k,2}] \xi^k$, which implies that:

$$\begin{aligned} \sum_{k=2}^{\infty} (k - kb\beta + b\beta - 1 + b - b\alpha) [1 + (k-1)\delta]^n \mathcal{C}(\lambda) (1+\gamma) a_{k,1} \\ + (k - kb\beta + b\beta - 1 + b - b\alpha) [1 + (k-1)\delta]^n \mathcal{C}(\lambda) \gamma a_{k,2} \\ \leq (1+\gamma)(1-\alpha)b + \gamma(1-\alpha)b \\ \leq (1-\alpha)b \end{aligned}$$

therefore $h \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$. Hence $\mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$ is convex. \square

Theorem 2.5. Let $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$, then f is close-to-convex of order σ ($0 \leq \sigma < 1$) in the disc $|\xi| < r_1$, where $r_1 := \left(\frac{(1-\sigma)[(k-kb\beta+b\beta-1+b-b\alpha)[1+(k-1)\delta]^n \mathcal{C}(\lambda)]}{(k)(1-\alpha)b} \right)^{\frac{1}{k-1}}$.

Theorem 2.6. Let $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$, then f is starlike of order σ ($0 \leq \sigma < 1$) in the disc $|\xi| < r_2$, where $r_2 := \inf \left(\frac{(1-\sigma)[(k-kb\beta+b\beta-1+b-b\alpha)[1+(k-1)\delta]^n \mathcal{C}(\lambda)]}{(k-\sigma)(1-\alpha)b} \right)^{\frac{1}{k-1}}$, ($k \geq 2$).

Theorem 2.7. Let $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$, then f is convex of order σ ($0 \leq \sigma < 1$) in the disc $|\xi| < r_3$, where $r_3 := \inf \left(\frac{(1-\sigma)[(k-kb\beta+b\beta-1+b-b\alpha)[1+(k-1)\delta]^nC(\lambda)]}{k(k-\sigma)(1-\alpha)b} \right)^{\frac{1}{k-1}}$, ($k \geq 2$).

Putting $\beta = 0, \delta = 1$ in the Theorem 2.3 which analogue the results of M. Thirucheran, A. Anand and T. Stalin [5] we obtain the corollary:

Corollary 2.1. Let $f(\xi) = \xi - \sum_{k=2}^{\infty} |a_k| \xi^k$, $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$, then for $|\xi| = r$ we have

$$r - \frac{(1-\alpha)b}{(b+1-\alpha b)(2)^n(\lambda+1)} r^2 \leq |f(\xi)| \leq r + \frac{(1-\alpha)b}{(b+1-\alpha b)(2)^n(\lambda+1)} r^2$$

and

$$1 - \frac{2(1-\alpha)b}{(b+1-\alpha b)(2)^n(\lambda+1)} r \leq |f'(\xi)| \leq 1 + \frac{2(1-\alpha)b}{(b+1-\alpha b)(2)^n(\lambda+1)} r.$$

Putting $\beta = 0, \delta = 1, b = 1$ in the Theorem 2.3 which analogue the results of K. AlShaqs and M. Darus [4] we have:

Corollary 2.2. Let $f(\xi) = \xi - \sum_{k=2}^{\infty} |a_k| \xi^k$, $f \in \mathcal{N}_{\beta, \lambda, \delta, b}^n(\alpha)$, then for $|\xi| = r$ we have

$$r - \frac{(1-\alpha)}{(2-\alpha)(2)^n(\lambda+1)} r^2 \leq |f(\xi)| \leq r + \frac{(1-\alpha)}{(2-\alpha)(2)^n(\lambda+1)} r^2$$

and

$$1 - \frac{2(1-\alpha)}{(2-\alpha)(2)^n(\lambda+1)} r \leq |f'(\xi)| \leq 1 + \frac{2(1-\alpha)}{(2-\alpha)(2)^n(\lambda+1)} r.$$

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