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OBTAIN NEW SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING HADAMARD PRODUCT OF LINEAR OPERATOR WITH POLYLOGARITHM FUNCTIONS

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ABSTRACT. In this article, we explore some standard properties for the new subclass $\mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$ of analytic function which is associated with the differential operator $\mathcal{G}_{\delta}^{n,\lambda}f(\zeta)$ and also we obtain Briot-Bouquet differential subordination, coefficient inequalities, integral means of inequalities, extreme points and distortion of the class of polylogarithms functions.

1. INTRODUCTION

Let $f(\zeta)$ be the form of analytic functions of class $\mathcal{A}$,

$$f(\zeta)=\zeta+\sum_{k=2}^{\infty}a_k\zeta^k, \tag{1.1}$$

which are analytic function in $\mathcal{U}=\zeta:\ \ |\zeta|<1$,where $\mathcal{U}$ is the unit disc. Let the function $f(\zeta)$ is given by (1.1) and $g(\zeta)$ is given by

$$g(\zeta)=\zeta+\sum_{k=2}^{\infty}b_k\zeta^k,$$

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then the Hadamard product of $f(\zeta)$ and $g(\zeta)$ is expressed by

$$(f*g)(\zeta)=\zeta+\sum_{k=2}^{\infty}a_kb_k\zeta^k.$$

For $f\in\mathcal{A}$, Al-Oboudi [2] introduced the following differential operator:

$$\mathcal{D}_\delta^n f(\zeta)=\zeta+\sum_{k=2}^{\infty}[1+(k-1)\delta]^n a_k\zeta^k, \qquad (n\in N_0=N\cup\{0\},\delta>0:\zeta\in\mathcal{U}).$$

For $f\in\mathcal{A}$, Ruscheweyh [9] introduced the following differential operator:

$$\mathcal{R}^\lambda f(\zeta)=\frac{\zeta}{(1-\zeta)^{\lambda+1}}*f(\zeta)=\zeta+\sum_{k=2}^{\infty}\frac{(\lambda+k-1)!}{\lambda!(k-1)!}a_k\zeta^k, \quad (\lambda>-1).$$

Now consider the Polylogarithm function $I(n,\delta)$ given by

$$I(n,\delta)=\sum_{k=1}^{\infty}\frac{\zeta^k}{[1+(k-1)\delta]^n}.$$

Note that $I(-1,1)=\frac{\zeta}{(1-\zeta)^2}$ for $k=1,2,3,\ldots.$ is Koebe function. For more details about polylogarithms in the theory of univalent functions see Ponnusamy and Sabapathy [7], K. Al Shaqsi and M. Daraus [4], and Ponnusamy [8].

Now we introduce a function $I^\kappa(n,\delta)$ given by

$$I(n,\delta)*I^\kappa(n,\delta)=\frac{\zeta}{(1-\zeta)^{\lambda+1}}, \qquad (\lambda>-1,n\in Z)$$

and obtain the linear operator

$$\mathcal{G}_\delta^{n,\lambda}f(\zeta)=I^\kappa(n,\delta)*f(\zeta). \tag{1.2}$$

Now we find the explicit form of the function

$$I^\kappa(n,\delta)=\sum_{k=1}^{\infty}[1+(k-1)\delta]^n\frac{(\lambda+k-1)!}{\lambda!(k-1)!}\zeta^k.$$

From equation (1.2), we define

$$\mathcal{G}_\delta^{n,\lambda}f(\zeta)=\sum_{k=2}^{\infty}[1+(k-1)\delta]^n\frac{(\lambda+k-1)!}{\lambda!(k-1)!}a_k\zeta^k.$$

Note that $\mathcal{G}_1^{n,0}=\mathcal{D}^n,\mathcal{G}_1^{0,\lambda}=\mathcal{D}^\lambda$ which give the Salagean differential operator [10] and Ruscheweyh differential operator [9] respectively. It is obvious that the operator $\mathcal{G}_\delta^{n,\lambda}$ includes two well known operators. Also note that $\mathcal{G}_1^{0,0}=f(\zeta)$ and $\mathcal{G}_1^{0,1}=\mathcal{G}_1^{1,0}=\zeta f'(\zeta)$.

Let p be the class of functions of the form $p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + ...$, analytic in $\mathcal{U}$, which satisfy $Re\{p(\zeta)\} > 0$.

Definition 1.1. *We define $\mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$ be the class of functions $f \in \mathcal{A}$ for which*

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{G}_\delta^{n,\lambda} f(\zeta))'}{\mathcal{G}_\delta^{n,\lambda} f(\zeta)} - 1\right) \prec \phi(\zeta),$$

where $n, \lambda \in N_0, \delta > 0, b > 0, \phi \in p; \zeta \in \mathcal{U}$.

Definition 1.2. *For $\phi(\zeta) = \frac{1+(1-2\beta)\zeta}{(1-\zeta)}$, we define $\mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta)) \equiv \mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$ be the class of functions $f \in \mathcal{A}$ for which*

$$1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{G}_\delta^{n,\lambda} f(\zeta))'}{\mathcal{G}_\delta^{n,\lambda} f(\zeta)} - 1\right) > \beta,$$

where $n, \lambda \in N_0, \delta > 0, b > 0, 0 \leq \beta \leq 1;$ all $\zeta \in \mathcal{U}$.

If $f(\zeta)$ and $g(\zeta)$ are analytic in $\mathcal{U}$, we state that $f(\zeta)$ is subordinate to $g(\zeta)$, i.e., $f(\zeta) \prec g(\zeta)$, if there exists a Schwarz function $w(\zeta)$, with $w(0) = 0$ and $|w| < 1$ such that $f(\zeta) = g(w(\zeta))$. Furthermore, if the function $g(\zeta)$ is univalent in $\mathcal{U}$, then the above subordination equivalence holds (see [7, 8]). $f(\zeta) \prec g(\zeta)$ if and only if $f(0) = g(0)$, and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Note that $\mathcal{P}_{1,1}^{n,\lambda}\phi(\zeta) = \mathcal{K}_\lambda^n\phi(\zeta), \mathcal{P}_{1,1}^{n,\lambda}(\beta) = \mathcal{R}_\lambda^n(\beta)$ was studied by K. AlShaqsi and M. Darus [4], $\mathcal{P}_{1,1}^{0,0}\phi(\zeta) = \mathcal{S}^*\phi(\zeta)$ was studied by Ma and Minda [6], $\mathcal{P}_{1,1}^{0,\lambda}(\beta) = \mathcal{R}_\lambda(\beta)$ was introduced and studied by Ahuja [1] and $\mathcal{P}_{1,1}^{n,0}(\beta) = \mathcal{R}_n(\beta)$ was introduced and studied by Kadioglu [3].

2. Main results

To construct Briot-Bouquet differential subordination theorem, we have to follow the next lemma:

Lemma 2.1. *Let β, v be the complex numbers. Let $\phi \in p$ be convex univalent in $\mathcal{U}$ with $\phi(0) = 1$ and $Re[\beta\phi(\zeta) + v] > 0, \zeta \in \mathcal{U}$.*

Theorem 2.1. *Let $n, \lambda \in N_0, \delta > 0$ and $\phi \in p$, then $\mathcal{P}_{\delta,b}^{n,\lambda+1}(\phi(\zeta)) \subset \mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$.*

Proof. Let $f \in \mathcal{P}_{\delta,b}^{n,\lambda+1}(\phi(\zeta))$ and $p(\zeta) = 1 + \frac{1}{b}\left(\frac{\zeta(\mathcal{G}_\delta^{n,\lambda} f(\zeta))'}{\mathcal{G}_\delta^{n,\lambda} f(\zeta)} - 1\right)$, where $p(\zeta)$ is analytic in $\mathcal{U}$ with $p(0) = 1$, then

$$\zeta(\mathcal{G}_\delta^{n,\lambda} f(\zeta))' = (\lambda + 1)\mathcal{G}_\delta^{n,\lambda+1} f(\zeta) - (\lambda)\mathcal{G}_\delta^{n,\lambda} f(\zeta).$$

Hence

$$1+\frac{1}{b}\frac{\zeta(\mathcal{G}_\delta^{n,\lambda+1}f(\zeta))'}{\mathcal{G}_\delta^{n,\lambda+1}f(\zeta)}=p(\zeta)+\frac{\zeta p'(\zeta)}{bp(\zeta)+1-b+\lambda}. \tag{2.1}$$

Applying Lemma 2.1 in (2.1) we get $p\prec\phi$, that is $f\in\mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$. This completes the theorem. □

Remark 2.1. *If we put $b=1$ in the above theorem, we obtain the result of K. AlShaqsi and M. Darus [4].*

Theorem 2.2. *Let $f\in\mathcal{P}_{\delta,b}^{n,\lambda+1}(\phi(\zeta))$ and let c be real number such that $c>-1$, then $\mathcal{F}$ defined by $\mathcal{F}=\frac{c+1}{\zeta^c}\int_0^\zeta t^{c-1}f(t)dt$ belongs to the class $f\in\mathcal{P}_{\delta,b}^{n,\lambda}(\phi(\zeta))$.*

Proof. Let $f\in\mathcal{P}_{\delta,b}^{n,\lambda+1}(\phi(\zeta))$, then $q(\zeta)=1+\frac{1}{b}\frac{\zeta(\mathcal{G}_\delta^{n,\lambda+1}\mathcal{F}(\zeta))'}{\mathcal{G}_\delta^{n,\lambda+1}\mathcal{F}(\zeta)}\prec\phi(\zeta)$, where $q(\zeta)$ is analytic in $\mathcal{U}$ with $q(0)=1$, then

$$\zeta(\mathcal{G}_\delta^{n,\lambda}\mathcal{F})'=(\lambda+1)\mathcal{G}_\delta^{n,\lambda+1}\mathcal{F}-(\lambda)\mathcal{G}_\delta^{n,\lambda}\mathcal{F}.$$

Let $q(\zeta)=1+\frac{1}{b}\left(\frac{\zeta(\mathcal{G}_\delta^{n,\lambda+1}\mathcal{F}(\zeta))'}{\mathcal{G}_\delta^{n,\lambda+1}\mathcal{F}(\zeta)}-1\right)$, then we get

$$1+\frac{1}{b}\frac{\zeta(\mathcal{G}_\delta^{n,\lambda+1}\mathcal{F}(\zeta))'}{\mathcal{G}_\delta^{n,\lambda+1}\mathcal{F}(\zeta)}=q(\zeta)+\frac{\zeta q'(\zeta)}{bq(\zeta)+1-b+c}. \tag{2.2}$$

Applying Lemma 2.1 in (2.2) we get $q\prec\phi$. Hence the theorem is proved. □

Theorem 2.3. *Let $f(\zeta)$ be defined by* (1.1). *Then $f\in\mathcal{A}$ if and only if*

$$\sum_{k=2}^{\infty}(\beta b-b+1-k)[1+(k-1)\delta]^n\mathcal{C}(\lambda)\left|a_k\right|\leq(1-\beta)b, \tag{2.3}$$

where $\mathcal{C}(\lambda)=\frac{(k+\lambda-1)!}{\lambda!(k-1)!}, 0\leq\beta<1, n\in N_0=N\cup\{0\}, \delta>0, b>0, \lambda\geq 0; \zeta\in\mathcal{U}$.

Proof. Suppose that the inequality (2.3) is true and $|\zeta|<1$. Then it is sufficient to show that $\left|1+\frac{1}{b}\left(\frac{\zeta(\mathcal{G}^{n,\lambda},_\delta f(\zeta))'}{\mathcal{G}_\delta^{n,\lambda}f(\zeta)}-1\right)-\beta+1\right|<1$, which gives $\sum_{k=2}^{\infty}(\beta b-b+1-k)[1+(k-1)\delta]^n\mathcal{C}(\lambda)\left|a_k\right|\leq(1-\beta)b$. It is clear that the values of (2.3) lies in a circle centered at $w=1$ whose radius is $(1-\beta)b$. Hence the condition (2.3) holds.

Conversely, let us consider that the function f defined by (1.1) is in the class $\mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$, then $Re\left(1+\frac{1}{b}\left(\frac{\zeta(\mathcal{R}_{\lambda,\delta}^n f(\zeta))'}{\mathcal{R}_{\lambda,\delta}^n f(\zeta)}-1\right)\right)>\beta$, by the value of ζ on the real axis. Let $\zeta\to 1^-$ through real values, we obtain the result

$$\sum_{k=2}^{\infty}(\beta b - b + 1 - k)[1 + (k-1)\delta]^n \mathcal{C}(\lambda)\,|a_k| \leq (1-\beta)b.$$

Hence the result is sharp for the function $f(\zeta) = \zeta + \frac{(1-\beta)b}{(\beta b - b + 1 - k)[1+(k-1)\delta]^n \mathcal{C}(\lambda)}$. □

Corollary 2.1. *Let* $f \in \mathcal{P}^{n,\lambda}_{\delta,b}(\beta)$*, then we have* $a_k \leq \frac{(1-\beta)b}{(\beta b - b + 1 - k)[1+(k-1)\delta]^n \mathcal{C}(\lambda)}$.

Put $b = 1$, then the class $\mathcal{P}^{n;\lambda}_{\delta,b}(\phi)$ analogous to the class $\mathcal{K}^{n}_{\lambda,\delta,}(\beta)$ introduced by M. Thirucheran, A. Anand and T. Stalin [5].

Corollary 2.2. *Let* $f \in \mathcal{P}^{n,\lambda}_{\delta,b}(\beta)$*, then we have* $a_k \leq \frac{(1-\beta)}{(\beta - k)[1+(k-1)\delta]^n \mathcal{C}(\lambda)}$.

Put $b = 1, \delta = 1$ we obtain the corollary which analogous to the result of K. AlShaqsi and M. Darus [4].

Corollary 2.3. *Let* $f \in \mathcal{P}^{n,\lambda}_{\delta,b}(\beta)$*, then we have* $a_k \leq \frac{(1-\beta)}{(\beta - k)(k)^n \mathcal{C}(\lambda)}$.

Theorem 2.4. *Let* $f \in \mathcal{P}^{n,\lambda}_{\delta,b}(\beta)$ *and suppose that* $f(\zeta) = \zeta + \frac{(1-\beta)b\epsilon_k}{\phi(\lambda,\delta)}\zeta^k$*,* $k = 2, 3, ..., |\epsilon_k| = 1$*, where* $\phi(\lambda,\delta) = (\beta b - b + 1 - k)[1 + (k-1)\delta]^n \mathcal{C}(\lambda)$*. If there exists* $w(\zeta)$ *given by* $w(\zeta)^{k-1} = \frac{\phi(\lambda,\delta)}{(1-\beta)b\epsilon_k}\sum_{k=2}^{\infty} a_k \zeta^{k-1}$*, then for* $\zeta = re^{i\theta}, 0 < r < 1$*,* $\int_0^{2\pi} |f(\zeta)|^\mu \, d\theta \leq \int_0^{2\pi} |g(\zeta)|^\mu \, d\theta, \mu > 0$.

Proof. To complete the theorem we have to prove that

$$\int_0^{2\pi} \left| 1 + \sum_{k=2}^{\infty} a_k \zeta^{k-1} \right|^\mu d\theta \leq \int_0^{2\pi} \left| 1 + \frac{(1-\beta)b\epsilon_k}{\phi(\lambda,\delta)}\zeta^{k-1} \right|^\mu d\theta.$$

Using Littlewood subordination theorem, it is sufficient to prove that

$$1 + \sum_{k=2}^{\infty} a_k \zeta^{k-1} \prec 1 + \frac{(1-\beta)b\epsilon_k}{\phi(\lambda,\delta)}\zeta^{k-1}.$$

Let $1 + \sum_{k=2}^{\infty} a_k \zeta^{k-1} \prec 1 + \frac{(1-\beta)b\epsilon_k}{\phi(\lambda,\delta)}(w(\zeta))^{k-1}$, therefore

$$(w(\zeta))^{k-1} = \frac{\phi(\lambda,\delta)}{(1-\beta)b\epsilon_k}\sum_{k=2}^{\infty} a_k \zeta^{k-1}.$$

Hence $w(0) = 0$. Furthermore, if $f \in \mathcal{A}$ it satisfies $\phi(\lambda,\delta) \leq (1-\beta)b$,

$$|w(\zeta)|^{k-1} = \left| \frac{\phi(\lambda,\delta}{(1-\beta)b\epsilon_k} \right| \sum_{k=2}^{\infty} |a_k| \left|\zeta^{k-1}\right| \leq |\zeta| < 1.$$

Hence the theorem is completed. □

Let us define new subclass $\overline{\mathcal{P}}^{n,\lambda}_{\delta,b}(\beta) \subset \mathcal{P}^{n,\lambda}_{\delta,b}(\beta)$, which contains the function $f \in \mathcal{A}$. Now we determine the extreme points of the subclass $\overline{\mathcal{P}}^{n,\lambda}_{\delta,b}(\beta)$.

Theorem 2.5. *Let*

$$f_1(\zeta) = \zeta, f_k(\zeta) = \zeta + \sum_{k=2}^{\infty} \eta_k \frac{(1-\beta)b}{\phi(\lambda,\delta)}\zeta^k, k = 2, 3, \dots .$$

Then $f \in \overline{\mathcal{P}}^{n,\lambda}_{\delta,b}(\beta)$ *if and only if* $f(\zeta) = \sum_{k=1}^{\infty} \eta_k f_k(\zeta)$ *where* $\eta_k > 0$ *and* $\sum_{k=1}^{\infty} \eta_k = 1$.

Proof. Let

$$\begin{aligned} f(\zeta) &= \sum_{k=1}^{\infty} \eta_k f_k(\zeta) \\ &= \zeta + \sum_{k=2}^{\infty} \eta_k \frac{(1-\beta)b}{\phi(\lambda,\delta)}\zeta^k \\ &= \sum_{k=2}^{\infty} \eta_k \frac{(1-\beta)b}{\phi(\lambda,\delta)}(\phi(\lambda,\delta) \\ &= (1-\beta)b \sum_{k=1}^{\infty} \eta_k \\ &= (1-\beta)b(1-\eta_1) < (1-\beta)b, \end{aligned}$$

which shows that $f \in \overline{\mathcal{P}}^{n,\lambda}_{\delta,b}(\beta)$.

Conversely, suppose that $f \in \overline{\mathcal{P}}^{n,\lambda}_{\delta,b}(\beta)$. Since $|a_k| \le \frac{(1-\beta)b}{\phi(\lambda,\delta)}, k = 2, 3, \dots$. Let $\eta_k \le \frac{\phi(\lambda,\delta)}{(1-\beta)b}, \eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k$. Then we obtain $f(\zeta) = \sum_{k=1}^{\infty} \eta_k f_k(\zeta)$. □

Theorem 2.6. *The class* $\mathcal{P}^{n,\lambda}_{\delta,b}(\beta)$ *is convex.*

Proof. Let the function $f_j(\zeta) = \zeta + \sum_{k=2}^{\infty} a_{k,j}\zeta^k, a_{k,j} \ge 0, j = 1, 2$ lies in the class $f \in \mathcal{P}^{n,\lambda}_{\delta,b}(\beta)$. It is sufficient to prove that $h(\zeta) = (\mu+1)f_1(\zeta) - \mu f_2(\zeta), 0 \le \zeta \le 1$.

Since $h(\zeta) = \zeta + \sum_{k=2}^{\infty} [(1+\mu)a_{k,1} - \mu a_{k,2}]\,\zeta^k$, this implies that

$$\begin{aligned}&\sum_{k=2}^{\infty}(\beta b - b + 1 - k)[1+(k-1)\delta]^n \mathcal{C}(\lambda)(1+\mu)a_{k,1}\\ &\quad + (\beta b - b + 1 - k)[1+(k-1)\delta]^n \mathcal{C}(\lambda)(\mu)a_{k,2}\\ &\leq (1+\mu)(1-\beta)b + \mu(1-\beta)b\\ &\leq (1-\beta)b.\end{aligned}$$

Therefore $h \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$. Hence $\mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$ is convex. □

Theorem 2.7. *Let $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$, then f is close-to-convex of order $\sigma (0 \leq \sigma < 1)$ in the disc $|\zeta| < r_1$, where* $r_1 := \left(\frac{(1-\sigma)[(\beta b - b + 1 - k)[1+(k-1)\delta]^n C(\lambda)]}{(k)(1-\beta)b}\right)^{\frac{1}{k-1}}$.

Theorem 2.8. *Let $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$, then f is starlike of order $\sigma (0 \leq \sigma < 1)$ in the disc $|\zeta| < r_2$, where* $r_2 := inf\left(\frac{(1-\sigma)[(\beta b - b + 1 - k)[1+(k-1)\delta]^n C(\lambda)]}{(k-\sigma)(1-\beta)b}\right)^{\frac{1}{k-1}}, (k \geq 2)$.

Theorem 2.9. *Let $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$, then f is convex of order $\sigma (0 \leq \sigma < 1)$ in the disc $|\zeta| < r_3$, where* $r_3 := inf\left(\frac{(1-\sigma)[(\beta b - b + 1 - k)[1+(k-1)\delta]^n C(\lambda)]}{k(k-\sigma)(1-\beta)b}\right)^{\frac{1}{k-1}}, (k \geq 2)$.

Theorem 2.10. *Let $f(\zeta) = \zeta + \sum_{k=2}^{\infty} |a_k|\,\zeta^k$ be in the class $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$, then for $|\zeta| = r$, we have*

$$r - \frac{(1-\beta)b}{(\beta b - b - 1)(1+\delta)^n(\lambda+1)}r^2 \leq |f(\zeta)| \leq r + \frac{(1-\beta)b}{(\beta b - b - 1)(1+\delta)^n(\lambda+1)}r^2$$

and

$$1 - \frac{2(1-\beta)b}{(\beta b - b - 1)(1+\delta)^n(\lambda+1)}r \leq |f'(\zeta)| \leq 1 + \frac{2(1-\beta)b}{(\beta b - b - 1)(1+\delta)^n(\lambda+1)}r.$$

Putting $\delta = 1$ in the above theorem we obtain the result which analogue the results of M. Thirucheran, M. Vinothkumar and T. Stalin. [5]

Corollary 2.4. *Let $f(\zeta) = \zeta + \sum_{k=2}^{\infty} |a_k|\,\zeta^k$ be in the class $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$, then for $|\zeta| = r$ we have*

$$r - \frac{(1-\beta)b}{(\beta b - b - 1)(2)^n(\lambda+1)}r^2 \leq |f(\zeta)| \leq r + \frac{(1-\beta)b}{(\beta b - b - 1)(2)^n(\lambda+1)}r^2$$

and

$$1 - \frac{2(1-\beta)b}{(\beta b - b - 1)(2)^n(\lambda+1)}r \leq |f'(\zeta)| \leq 1 + \frac{2(1-\beta)b}{(\beta b - b - 1)(2)^n(\lambda+1)}r.$$

Putting $\delta = 1, b = 1$ in the above theorem we obtain result which is analogue the results of K. AlShaqsi and M. Darus. [4]

Corollary 2.5. *Let* $f \in \mathcal{P}_{\delta,b}^{n,\lambda}(\beta)$ *be in the class* $f \in \mathcal{P}_{\beta,\lambda,\delta,b}^{n}(\beta)$*, then for* $|\zeta| = r$ *we have*

$$r - \frac{(1-\beta)}{(\beta-2)(2)^n(\lambda+1)}r^2 \leq |f(\zeta)| \leq r + \frac{(1-\beta)}{(\beta-2)(2)^n(\lambda+1)}r^2$$

and

$$1 - \frac{2(1-\beta)}{(\beta-2)(2)^n(\lambda+1)}r \leq |f'(\zeta)| \leq 1 + \frac{2(1-\beta)}{(\beta-2)(2)^n(\lambda+1)}r.$$

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