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ESTIMATE THE NEW SUBCLASS OF NEGATIVE CO-EFFICIENT OF ANALYTIC FUNCTION DEFINED BY THE CONVOLUTION OF POLYLOGARITHM FUNCTIONS

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ABSTRACT. In this article, we contemplate and explore few properties for the class $\mathcal{M}^{n,\lambda}_{\beta,\delta,b}(\phi(\xi))$ of analytic function which is related with the operator $\mathcal{R}^{n,\lambda}_{\delta} f(\xi)$ of Polylogarithms functions and also we obtained Briot-Bouquet differential subordination, coefficient inequalities, integral means of inequalities, extreme points and distortion of the class of polylogarithms functions.

1. INTRODUCTION

Let f be the class of analytic functions defined with

$$\mathsf{f}(\xi) = \xi + \sum_{k=2}^{\infty} a_k \xi^k, \tag{1.1}$$

which belongs to the class $\mathcal{A}$, which are analytic in $\mathcal{U} = \xi : \ |\xi| < 1$, where $\mathcal{U}$ is the unit disc. Let the function $\mathsf{f}(\xi)$ is given by (1.1) and $\mathsf{g}(\xi)$ is given by

$$\mathsf{g}(\xi) = \xi + \sum_{k=2}^{\infty} b_k \xi^k,$$

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then the Hadamard product of $f(\xi)$ and $g(\xi)$ is defined by

$$(f * g)(\xi) = \xi + \sum_{k=2}^{\infty} a_k b_k \xi^k.$$

For $f \in \mathcal{A}$, Al-Oboudi [2] introduced the following differential operator:

$$\mathcal{D}_\delta^n f(\xi) = \xi + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n a_k \xi^k, (n \in N_0 = N \cup \{0\}, \delta > 0 : \xi \in \mathcal{U}).$$

For $f \in \mathcal{A}$, Ruscheweyh [9] introduced the following differential operator:

$$\mathcal{R}^\lambda f(\xi) = \frac{\xi}{(1-\xi)^{\lambda+1}} * f(\xi) = \xi + \sum_{k=2}^{\infty} \frac{(\lambda + k - 1)!}{\lambda!(k-1)!} a_k \xi^k, (\lambda > -1).$$

Now consider the Polylogarithm function $J(n, \delta)$ given by

$$J(n, \delta) = \sum_{k=1}^{\infty} \frac{\xi^k}{[1 + (k-1)\delta]^n}.$$

Note that $J(-1, 1) = \frac{\xi}{(1-\xi)^2}$ for $k = 1, 2, 3, \ldots$.is Koebe function. More for the concept about polylogarithms in the theory of univalent functions see Ponnusamy and Sabapathy [7], K. Al Shaqsi and M. Daraus [4], and Ponnusamy [8].

Now we introduce a function $J^\kappa(n, \delta)$ given by:

$$J(n, \delta) * J^\kappa(n, \delta) = \frac{\xi}{(1-\xi)^{\lambda+1}}, \lambda > -1, n \in Z$$

and define the linear operator

$$\mathcal{R}_\delta^{n,\lambda} f(\xi) = J^\kappa(n, \delta) * f(\xi). \tag{1.2}$$

Now we find the explicit form of the function

$$J^\kappa(n, \delta) = \sum_{k=1}^{\infty} [1 + (k-1)\delta]^n \frac{(\lambda + k - 1)!}{\lambda!(k-1)!} \xi^k.$$

From the equation (1.2), we define

$$\mathcal{R}_\delta^{n,\lambda} f(\xi) = \xi + \sum_{k=2}^{\infty} [1 + (k-1)\delta]^n \frac{(\lambda + k - 1)!}{\lambda!(k-1)!} a_k \xi^k.$$

Note that $\mathcal{R}_1^{n,0} = \mathcal{D}^n, \mathcal{R}_1^{0,\lambda} = \mathcal{D}^\lambda$, which give the Salagean differential operator [10] and Ruscheweyh differential operator [9] respectively. It is obvious that the operator $\mathcal{R}_\delta^{n,\lambda}$ involves two well known operators. Also note that $\mathcal{R}_1^{0,0} = f(\xi)$ and $\mathcal{R}_1^{0,1} = \mathcal{R}_1^{1,0} = \xi f'(\xi)$.

Let p denote the class of functions of the form $p(\xi) = 1+p_1\xi+p_2\xi^2+...$ analytic in $\mathcal{U}$ which satisfy the condition $Re\{p(\xi)\} > 0$.

Definition 1.1. *We define* $\mathcal{N}_{\delta,b}^{n,\lambda}(\phi(\xi))$ *be the class of functions* $\mathsf{f} \in \mathcal{A}$ *for which*

$$1 + \frac{1}{b}\left(\frac{\xi(\mathcal{R}_\delta^{n,\lambda}\mathsf{f}(\xi))'}{\mathcal{R}_\delta^{n,\lambda}\mathsf{f}(\xi)} - 1\right) \prec \phi(\xi),$$

where $n, \lambda \in N_0, \delta > 0, b > 0, \phi \in p; \xi \in \mathcal{U}$.

Definition 1.2. *For* $\phi(\xi) = \frac{1+(1-2\alpha)\xi}{(1-\xi)}$, *we define* $\mathcal{N}_{\delta,b}^{n,\lambda}(\phi(\xi)) \equiv \mathcal{N}_{\delta,b}^{n,\lambda}(\alpha)$ *be the class of functions* $\mathsf{f} \in \mathcal{A}$ *for which*

$$1 + \frac{1}{b}\left(\frac{\xi(\mathcal{R}_\delta^{n,\lambda}\mathsf{f}(\xi))'}{\mathcal{R}_\delta^{n,\lambda}\mathsf{f}(\xi)} - 1\right) > \alpha,$$

where $n, \lambda \in N_0, \delta > 0, b > 0, 0 \le \alpha \le 1;\ all\ \xi \in \mathcal{U}$.

If $\mathsf{f}(\xi)$ and $g(\xi)$ are analytic in $\mathcal{U}$, we state that $\mathsf{f}(\xi)$ is subordinate to $g(\xi)$, i.e., $\mathsf{f}(\xi) \prec g(\xi)$, if a Schwarz function $w(\xi)$ exists, with $w(0) = 0$ and $|w| < 1$ such that $\mathsf{f}(\xi) = g(w(\xi))$. Furthermore, if the function $g(\xi)$ is univalent in $\mathcal{U}$, then the above subordination equivalence holds (see [7, 8]). $\mathsf{f}(\xi) \prec g(\xi)$ if and only if $\mathsf{f}(0) = g(0)$, and $\mathsf{f}(\mathcal{U}) \subset g(\mathcal{U})$.

Definition 1.3. *We define* $\mathcal{T}$ *to represent the subclass of* $\mathcal{A}$ *that can be express in the form*

$$\mathsf{f}(\xi) = \xi - \sum_{k=2}^{\infty} a_k \xi^k.$$

Now, define the subclass $\mathcal{M}_{\delta,b}^{n,\lambda}(\alpha) = \mathcal{N}_{\delta,b}^{n,\lambda}(\alpha) \cap \mathcal{T}$. Since $\mathcal{M}_{\delta,b}^{n,\lambda}(\alpha) \subset \mathcal{N}_{\delta,b}^{n,\lambda}(\alpha)$.

Note that $\mathcal{M}_{1,1}^{n,\lambda}\phi(\xi) = \mathcal{K}_\lambda^n\phi(\xi), \mathcal{M}_{1,1}^{n,\lambda}(\alpha) = \mathcal{R}_\lambda^n(\alpha)$ was studied by K. AlShaqsi and M. Darus [4], $\mathcal{M}_{1,1}^{0,0}\phi(\xi) = \mathcal{S}^*\phi(\xi)$ was studied by Ma and Minda [6], $\mathcal{M}_{1,1}^{0,\lambda}(\alpha) = \mathcal{R}_\lambda(\alpha)$ was introduced and studied by Ahuja [1] and $\mathcal{M}_{1,1}^{n,0}(\alpha) = \mathcal{R}_n(\alpha)$ was introduced and studied by Kadioglu [3].

2. Main results

To construct Briot-Bouquet differential subordination theorem, we have to follow the next lemma:

Lemma 2.1. *Let β, v be the complex numbers. Let $\phi \in p$ be convex univalent in $\mathcal{U}$ with $\phi(0) = 1$ and $Re\left[\beta\phi(\xi) + v\right] > 0, \xi \in \mathcal{U}$.*

Theorem 2.1. *Let $n, \lambda \in N_0, \delta > 0$ and $\phi \in p$, then $\mathcal{M}_{\delta,b}^{n,\lambda+1}(\phi(\xi)) \subset \mathcal{M}_{\delta,b}^{n,\lambda}(\phi(\xi))$.*

Proof. Let $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda+1}(\phi(\xi))$ and $p(\xi) = 1 + \frac{1}{b}\left(\frac{\xi(\mathcal{R}_\delta^{n,\lambda}\mathsf{f}(\xi))'}{\mathcal{R}_\delta^{n,\lambda}\mathsf{f}(\xi)} - 1\right)$, where $p(\xi)$ is analytic in $\mathcal{U}$ with $p(0) = 1$, then

$$\xi(\mathcal{R}_\delta^{n,\lambda}\mathsf{f}(\xi))' = (\lambda+1)\mathcal{R}_\delta^{n,\lambda+1}\mathsf{f}(\xi) - (\lambda)\mathcal{R}_\delta^{n,\lambda}\mathsf{f}(\xi).$$

Hence

$$1 + \frac{1}{b}\frac{\xi(\mathcal{R}_\delta^{n,\lambda+1}\mathsf{f}(\xi))'}{\mathcal{R}_\delta^{n,\lambda+1}\mathsf{f}(\xi)} = p(\xi) + \frac{\xi p'(\xi)}{bp(\xi) + 1 - b + \lambda}.$$

Applying Lemma 2.1 we get $p \prec \phi$, that is $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\phi(\xi))$. This completes the theorem. ☐

Remark 2.1. *If we put $b = 1$ in the above theorem, it gives the result of K. AlShaqsi and M. Darus [4].*

Theorem 2.2. *Let $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda+1}(\phi(\xi))$ and c be real number such that $c > -1$, then $\mathcal{F}$ defined by $\mathcal{F} = \frac{c+1}{\xi^c}\int_0^\xi t^{c-1}\mathsf{f}(t)dt$ belongs to the class $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\phi(\xi))$.*

Proof. Let $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda+1}(\phi(\xi))$, then $q(\xi) = 1 + \frac{1}{b}\frac{\xi(\mathcal{R}_\delta^{n,\lambda+1}\mathcal{F}(\xi))'}{\mathcal{R}_\delta^{n,\lambda+1}\mathcal{F}(\xi)} \prec \phi(\xi)$, where $q(\xi)$ is analytic in $\mathcal{U}$ with $q(0) = 1$. Then we have:

$$\xi(\mathcal{R}_\delta^{n,\lambda}\mathcal{F})' = (\lambda+1)\mathcal{R}_\delta^{n,\lambda+1}\mathcal{F} - (\lambda)\mathcal{R}_\delta^{n,\lambda}\mathcal{F}.$$

Let $q(\xi) = 1 + \frac{1}{b}\left(\frac{\xi(\mathcal{R}_\delta^{n,\lambda+1}\mathcal{F}(\xi))'}{\mathcal{R}_\delta^{n,\lambda+1}\mathcal{F}(\xi)} - 1\right)$, then we get

$$1 + \frac{1}{b}\frac{\xi(\mathcal{R}_\delta^{n,\lambda+1}\mathcal{F}(\xi))'}{\mathcal{R}_\delta^{n,\lambda+1}\mathcal{F}(\xi)} = q(\xi) + \frac{\xi q'(\xi)}{bq(\xi) + 1 - b + c}.$$

Applying Lemma 2.1 we get $q \prec \phi$. Hence the theorem is complete. ☐

Theorem 2.3. *Let $\mathsf{f}(\xi)$ be defined by (1.1). Then $\mathsf{f} \in \mathcal{A}$ if and only if*

$$\sum_{k=2}^{\infty}(b - 1 + k - \alpha b)[1 + (k-1)\delta]^n \mathcal{C}(\lambda)\,|a_k| \leq (1-\alpha)b, \tag{2.1}$$

where $\mathcal{C}(\lambda) = \frac{(k+\lambda-1)!}{\lambda!(k-1)!}, 0 \leq \alpha < 1, n \in N_0 = N \cup \{0\}, \delta > 0, b > 0, \lambda \geq 0; \xi \in \mathcal{U}$.

Proof. Suppose that the inequality (2.1) is true and $|\xi| < 1$. Then it is sufficient to prove that $\left|1 + \frac{1}{b}\left(\frac{\xi(\mathcal{R}^{n,\lambda},_\delta \mathsf{f}(\xi))'}{\mathcal{R}_\delta^{n,\lambda}\mathsf{f}(\xi)} - 1\right) - \alpha + 1\right| < 1$, which gives

$$\sum_{k=2}^{\infty}(b-1+k-\alpha b)[1+(k-1)\delta]^n \mathcal{C}(\lambda)\,|a_k| \leq (1-\alpha)b.$$

It is obvious that the values of (2.1) lies in a circle centered at $w = 1$ whose radius is $(1-\alpha)b$. Hence the condition (2.1) holds.

Conversely, let us consider that the function f defined by (1.1) is in the class $\mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$. Then $Re\left(1 + \frac{1}{b}\left(\frac{\xi(\mathcal{R}_{\lambda,\delta}^n \mathsf{f}(\xi))'}{\mathcal{R}_{\lambda,\delta}^n \mathsf{f}(\xi)} - 1\right)\right) > \alpha$ by the value of ξ on the real axis. Let $\xi \to 1^-$ through real values, we obtain the result:

$$\sum_{k=2}^{\infty}(b-1+k-\alpha b)[1+(k-1)\delta]^n \mathcal{C}(\lambda)\,|a_k| \leq (1-\alpha)b.$$

Hence the result is sharp for the function $\mathsf{f}(\xi) = \xi - \frac{(1-\alpha)b}{(b-1+k-\alpha b)[1+(k-1)\delta]^n \mathcal{C}(\lambda)}$. □

Corollary 2.1. *Let* $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$*, then we have* $a_k \leq \frac{(1-\alpha)b}{(b-1+k-\alpha b)[1+(k-1)\delta]^n \mathcal{C}(\lambda)}$.

Put $b = 1$, then the class $\mathcal{M}_{\delta,b}^{n,\lambda}(\phi)$ is analogous to the class $\mathcal{K}_{\lambda,\delta,}^n(\alpha)$ introduced by M. Thirucheran, M. VinothKumar and T. Stalin [5].

Corollary 2.2. *Let* $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$*, then we have* $a_k \leq \frac{(1-\alpha)}{(k-\alpha)[1+(k-1)\delta]^n \mathcal{C}(\lambda)}$.

Put $b = 1, \delta = 1$ we obtain the corollary which is analogous to the result of K. AlShaqsi and M. Darus [4].

Corollary 2.3. *Let* $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$*, then we have* $a_k \leq \frac{(1-\alpha)}{(k-\alpha)(k)^n \mathcal{C}(\lambda)}$.

Theorem 2.4. *Let* $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$ *and suppose that* $\mathsf{f}(\xi) = \xi - \frac{(1-\alpha)b\epsilon_k}{\varphi(\lambda,\delta)}\xi^k, k = 2, 3, ..., |\epsilon_k| = 1$*, where* $\varphi(\lambda, \delta) = (b-1+k-\alpha b)[1+(k-1)\delta]^n \mathcal{C}(\lambda)$*. If there exists* $w(\xi)$ *given by* $w(\xi)^{k-1} = \frac{\varphi(\lambda,\delta)}{(1-\alpha)b\epsilon_k}\sum_{k=2}^{\infty} a_k \xi^{k-1}$*, then for* $\xi = re^{i\theta}, 0 < r < 1, \int_0^{2\pi} |\mathsf{f}(\xi)|^\mu \, d\theta \leq \int_0^{2\pi} |g(\xi)|^\mu \, d\theta, \mu > 0$.

Proof. To complete the theorem we have to prove that

$$\int_0^{2\pi}\left|1 - \sum_{k=2}^{\infty} a_k \xi^{k-1}\right|^\mu d\theta \leq \int_0^{2\pi}\left|1 - \frac{(1-\alpha)b\epsilon_k}{\varphi(\lambda,\delta)}\xi^{k-1}\right|^\mu d\theta.$$

Using Littlewood subordination theorem, it is sufficient to prove that

$$1-\sum_{k=2}^{\infty} a_k\xi^{k-1} \prec 1-\frac{(1-\alpha)b\epsilon_k}{\varphi(\lambda,\delta)}\xi^{k-1}.$$

Let

$$1-\sum_{k=2}^{\infty} a_k\xi^{k-1} \prec 1-\frac{(1-\alpha)b\epsilon_k}{\varphi(\lambda,\delta)}(w(\xi))^{k-1},$$

therefore

$$(w(\xi))^{k-1}=\frac{\varphi(\lambda,\delta)}{(1-\alpha)b\epsilon_k}\sum_{k=2}^{\infty} a_k\xi^{k-1}.$$

Hence $w(0)=0$. Furthermore, if $\mathsf{f}\in\mathcal{A}$ satisfies $\varphi(\lambda,\delta)\le(1-\alpha)b$,

$$|w(\xi)|^{k-1}=\left|\frac{\varphi(\lambda,\delta}{(1-\alpha)b\epsilon_k}\right|\sum_{k=2}^{\infty}|a_k|\left|\xi^{k-1}\right|\le|\xi|<1.$$

Hence the theorem is completed. □

Let us define new subclass $\overline{\mathcal{M}}_{\delta,b}^{n,\lambda}(\alpha)\subset\mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$, which contains the function $\mathsf{f}\in\mathcal{A}$. Now we detemine the extreme points of the subclass $\overline{\mathcal{M}}_{\delta,b}^{n,\lambda}(\alpha)$.

Theorem 2.5. *Let* $\mathsf{f}_1(\xi)=\xi, \mathsf{f}_k(\xi)=\xi-\sum_{k=2}^{\infty}\eta_k\frac{(1-\alpha)b}{\varphi(\lambda,\delta)}\xi^k, k=2,3,\ldots.$ *Then* $\mathsf{f}\in\overline{\mathcal{M}}_{\delta,b}^{n,\lambda}(\alpha)$ *if and only if* $\mathsf{f}(\xi)=\sum_{k=1}^{\infty}\eta_k\mathsf{f}_k(\xi)$ *where* $\eta_k>0$ *and* $\sum_{k=1}^{\infty}\eta_k=1$.

Proof. Let

$$\begin{aligned}\mathsf{f}(\xi)&=\sum_{k=1}^{\infty}\eta_k\mathsf{f}_k(\xi)\\&=\xi-\sum_{k=2}^{\infty}\eta_k\frac{(1-\alpha)b}{\varphi(\lambda,\delta)}\xi^k\\&=\sum_{k=2}^{\infty}\eta_k\frac{(1-\alpha)b}{\varphi(\lambda,\delta)}(\varphi(\lambda,\delta)\\&=(1-\alpha)b\sum_{k=1}^{\infty}\eta_k\\&=(1-\alpha)b(1-\eta_1)<(1-\alpha)b,\end{aligned}$$

which proves that $\mathsf{f}\in\overline{\mathcal{M}}_{\delta,b}^{n,\lambda}(\alpha)$.

Conversely, suppose that $\mathsf{f} \in \overline{\mathcal{M}}_{\delta,b}^{n,\lambda}(\alpha)$. Since $|a_k| \leq \frac{(1-\alpha)b}{\varphi(\lambda,\delta)}, k = 2, 3, \ldots$. Let $\eta_k \leq \frac{\varphi(\lambda,\delta)}{(1-\alpha)b}, \eta_1 = 1 - \sum_{k=2}^{\infty} \eta_k$. Then we obtain $\mathsf{f}(\xi) = \sum_{k=1}^{\infty} \eta_k \mathsf{f}_k(\xi)$. □

Theorem 2.6. *The class $\mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$ is convex.*

Proof. Let the function $\mathsf{f}_j(\xi) = \xi - \sum_{k=2}^{\infty} a_{k,j}\xi^k, a_{k,j} \geq 0, j = 1, 2$ lies in the class $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$, it is sufficient to prove that

$$h(\xi) = (\mu + 1)\mathsf{f}_1(\xi) - \mu \mathsf{f}_2(\xi), 0 \leq \xi \leq 1.$$

Since $h(\xi) = \xi - \sum_{k=2}^{\infty} [(1 + \mu)a_{k,1} - \mu a_{k,2}]\, \xi^k$, which implies that

$$\begin{aligned} &\sum_{k=2}^{\infty}(b - 1 + k - \alpha b)[1 + (k - 1)\delta]^n \mathcal{C}(\lambda)(1 + \mu)a_{k,1} \\ &\quad + \sum_{k=2}^{\infty}(b - 1 + k - \alpha b)[1 + (k - 1)\delta]^n \mathcal{C}(\lambda)(\mu)a_{k,2} \\ &\leq (1 + \mu)(1 - \alpha)b + \mu(1 - \alpha)b \\ &\leq (1 - \alpha)b, \end{aligned}$$

therefore $h \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$. Hence $\mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$ is convex. □

Theorem 2.7. *Let $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$, then f is close-to-convex of order $\sigma(0 \leq \sigma < 1)$ in the disc $|\xi| < r_1$, where $r_1 := \left(\frac{(1-\sigma)[(b-1+k-\alpha b)[1+(k-1)\delta]^n C(\lambda)]}{(k)(1-\alpha)b}\right)^{\frac{1}{k-1}}$.*

Theorem 2.8. *Let $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$, then f is starlike of order $\sigma(0 \leq \sigma < 1)$ in the disc $|\xi| < r_2$, where $r_2 := inf \left(\frac{(1-\sigma)[(b-1+k-\alpha b)[1+(k-1)\delta]^n C(\lambda)]}{(k-\sigma)(1-\alpha)b}\right)^{\frac{1}{k-1}}$, $(k \geq 2)$.*

Theorem 2.9. *Let $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$, then f is convex of order $\sigma(0 \leq \sigma < 1)$ in the disc $|\xi| < r_3$, where $r_3 := inf \left(\frac{(1-\sigma)[(b-1+k-\alpha b)[1+(k-1)\delta]^n C(\lambda)]}{k(k-\sigma)(1-\alpha)b}\right)^{\frac{1}{k-1}}$, $(k \geq 2)$.*

Theorem 2.10. *Let $\mathsf{f}(\xi) = \xi - \sum_{k=2}^{\infty} |a_k|\, \xi^k$ be in the class $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$, then for $|\xi| = r$, we have*

$$r - \frac{(1-\alpha)b}{(b+1-\alpha b)(1+\delta)^n(\lambda+1)}r^2 \leq |\mathsf{f}(\xi)| \leq r + \frac{(1-\alpha)b}{(b+1-\alpha b)(1+\delta)^n(\lambda+1)}r^2$$

and

$$1 - \frac{2(1-\alpha)b}{(b+1-\alpha b)(1+\delta)^n(\lambda+1)}r \leq |\mathsf{f}'(\xi)| \leq 1 + \frac{2(1-\alpha)b}{(b+1-\alpha b)(1+\delta)^n(\lambda+1)}r.$$

Putting $\delta = 1$ in the above theorem the result obtained is analogue to the results of M. Thirucheran, M. Vinothkumar and T. Stalin [5].

Corollary 2.4. *Let* $\mathsf{f}(\xi) = \xi - \sum_{k=2}^{\infty} |a_k| \xi^k$ *be in the class* $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$*, then for* $|\xi| = r$ *we have*

$$r - \frac{(1-\alpha)b}{(b+1-\alpha b)(2)^n(\lambda+1)} r^2 \leq |\mathsf{f}(\xi)| \leq r + \frac{(1-\alpha)b}{(b+1-\alpha b)(2)^n(\lambda+1)} r^2$$

and

$$1 - \frac{2(1-\alpha)b}{(b+1-\alpha b)(2)^n(\lambda+1)} r \leq |\mathsf{f}'(\xi)| \leq 1 + \frac{2(1-\alpha)b}{(b+1-\alpha b)(2)^n(\lambda+1)} r$$

Putting $\delta = 1, b = 1$ in the above theorem the result obtained is analogue the results of K. AlShaqsi and M. Darus [4].

Corollary 2.5. *Let* $\mathsf{f} \in \mathcal{M}_{\delta,b}^{n,\lambda}(\alpha)$ *be in the class* $\mathsf{f} \in \mathcal{M}_{\beta,\lambda,\delta,b}^{n}(\alpha)$*, then for* $|\xi| = r$ *we have*

$$r - \frac{(1-\alpha)}{(2-\alpha)(2)^n(\lambda+1)} r^2 \leq |\mathsf{f}(\xi)| \leq r + \frac{(1-\alpha)}{(2-\alpha)(2)^n(\lambda+1)} r^2$$

and

$$1 - \frac{2(1-\alpha)}{(2-\alpha)(2)^n(\lambda+1)} r \leq |\mathsf{f}'(\xi)| \leq 1 + \frac{2(1-\alpha)}{(2-\alpha)(2)^n(\lambda+1)} r$$

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