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# COMPLEMENTARY INDEPENDENT TWIN PAIRED DOMINATION NUMBER OF A GRAPH

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ABSTRACT. The set $S \subseteq V$ is said to be complementary independent twin paired dominating set, if S is a paired dominating set and $< V - S >$ is a set of independent edges. The minimum cardinality taken over all the complementary independent twin paired dominating set is called as complementary independent twin paired domination number and it is denoted by CITD(G). In this paper we initiate this parameter with application and investigate this parameter for some standard and some special type graphs.

## 1. INTRODUCTION

The basic graph notation and domination notations are refered from [1,2]. A matching in a graph G is a set of independent edges in G. A perfect matching M in G is a matching in G such that every vertex of G is incident to an edge of M. The initiation of paired domination was done by T. W. Haynes, et.al., in 1998, which is defined in [3,6] and it says that "A paired-dominating set is a set $S \subseteq V$ such that every vertex is adjacent to some vertex in S and the subgraph $< S >$ induced by S contains a perfect matching. The minimum cardinality taken over all the paired-dominating set is called as paired-domination number and is denoted by $\gamma_p(G)$." The concept of complementary perfect domination

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number was introduced by Paulraj Joseph [2–5]. A set $S \subseteq V$ is called a Complementary perfect dominating set, if $S$ is a dominating set of $G$ and the induced subgraph $< V - S >$ has a perfect matching. The minimum cardinality taken over all complementary perfect dominating set is complementary perfect domination number. In the paired-dominating set, authors haven't imposed any conditions on its complement. So as a motivation of above definition, G. Mahadevan et.al., [7] imposed the condition on the complement of paired dominating set and initiated the concept called complementary independent twin paired domination number. Also the authors obtained this number for some product related graph. The Jahangir graph $J_{m,n}(n \leq 3)$ is a graph with $mn+1$ vertices consisting of a cycle $C_{mn}$ with one additional vertex which is adjacent to $n$ vertices of $C_{mn}$ at distance $m$ to each other on $C_{mn}$. The graph $F_{(1,m)} = P_m + K_1$ is called a Fan graph. The Double Fan $F_{(2,m)}$ is defined as $P_m + 2K_1$. The Graph $l_n = P_n \times P_2$ is called the ladder graph. A Quadrilateral Snake $QS_n$ is obtained by replacing every edge of the path by $C_4$.

We have discussed CITD-number for the some standard and special type of graphs in Section 2 and in Section 3 we have discussed this number for graph operation $C_s(P_n)$.

## 2. Complementary independent twin paired domination number of a graph

In this section we have given an application for CITD-number, and discussed its bounds. Also we have found this number for standard types of graphs and some special type of graphs like fan graph, double fan and Jahangir graph.

**Definition 2.1.** *The set S $\subseteq$ V is said to be complementary independent twin paired dominating set, if S is a paired dominating set and $< V - S >$ is a set of independent edges. The minimum cardinality taken over all the complementary independent twin paired dominating set is called as complementary independent twin paired domination number and it is denoted by CITD(G).*

**Observation 2.1.**

(i) *As $S$ is a paired dominating set and $< V - S >$ is a set of independent edges, complementary independent twin paired domination number can be found only for the graphs with even number of vertices.*

(ii) *Complementary independent twin paired domination number cannot be found for graphs having more than one pendant vertices which is incident on the same vertices. Example: m-array tree.*

**Example 1.** *In the Figure 1, $S = \{v_4, v_6\}$ is the complementary independent twin paired dominating set and CITD(G)=2.*

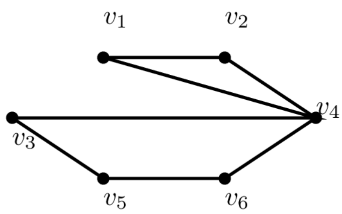


Figure 2.1: An Example

FIGURE 1

**Application**: Due to water scarcity, water is stored in tanks and sent through pipe lines. Now our aim is that all the places should get water without any obstacles. If any obstacles, there must be an immediate alternative so that the places receive the water.

Our main aim is that we want to build minimum number of water tanks. So that we take the possible obstacles in consideration. First obstacle is that if there is any damage in the pipe line. The places after the damaged pipe lines will not be able to get the water supply so this can be solved by making only two places (places without water tank) linked by a pipeline which is independently adjacent to a place with water tank. Another obstacle is that if the water tank is emptied then there must be a link with another water tank hence we choose the places such that it has a perfect matching.

Hence taking note on the obstacles and its solution we conclude that to build the minimum number of water tanks it is enough to find complementary independent twin paired domination number of its graph representation. In the following graph there are 12 places, $P_i's$ are places without water tank and $W_i's$ are places with water tank. Here the minimum number of water tank to build is 4 among 12 vertices which is the complementary independent twin paired domination number of a graph.

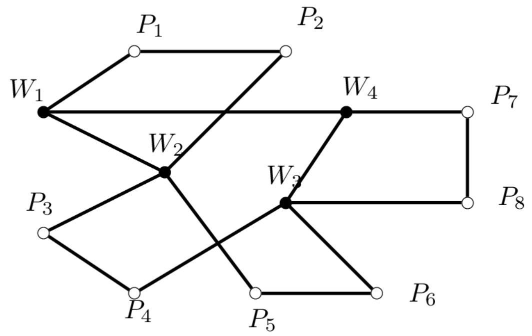

Figure 2.2: Application for CITD-number

FIGURE 2

**Theorem 2.1.** *For even path,* $CITD(P_n) = 2\lfloor \frac{n}{4} \rfloor + 2$*, where* $n \geq 6$.

*Proof.* If $n \equiv 2(mod\ 4)$, then $S = \{v_i : i \equiv 1, 2(mod\ 4)\}$ is the CITD-set. If $n \equiv 0(mod`4)$, then $S = \{v_i : i \equiv 1, 2(mod\ 4)\} \cup \{v_{n-1}, v_n\}$ is the CITD-set. In both the cases $|S| = 2\lfloor \frac{n}{4} \rfloor + 2$. In the above case, clearly $CITD(P_n) \leq |S|$. If $T \subseteq S$ is the complementary independent twin paired dominating set, then $T$ fails to satisfy the definition. Therefore S is the minimum Complementary independent twin paired dominating set, which implies $|S| \leq CITD(P_n)$ and from this it follows $|S| = CITD(P_n)$. □

**Theorem 2.2.** *For even Cycle,* $CITD(C_n) = \begin{cases} 2\lfloor \frac{n}{4} \rfloor + 2 & \text{if } n \equiv 2(mod\ 4) \\ 2\lfloor \frac{n}{4} \rfloor & \text{if } n \equiv 0(mod\ 4) \end{cases}$.

*Proof.* Let $S = \{v_i : i \equiv 1, 2(mod\ 4)\}$. If $n \equiv 2(mod\ 4)$, then S is the CITD-set and $|S| = 2\lfloor \frac{n}{4} \rfloor + 2$.If $n \equiv 0(mod\ 4)$, then S is the CITD-set and $|S| = 2\lfloor \frac{n}{4} \rfloor$. In the above cases, clearly $CITD(C_n) \leq |S|$. If $T \subseteq S$ is the complementary independent twin paired dominating set, then $T$ fails to satisfy the definition. Therefore S is the minimum complementary independent twin paired dominating set. Thus, $|S| \leq CITD(C_n)$ and $|S| = CITD(C_n)$. □

**Observation 2.2.**

(1) *For a complete graph,* $CITD(K_n) = n - 2$ *where* $n \geq 4$
(2) *For a bipartite graph,* $CITD(K_{n,s}) = n + s - 2$*, where* $n + s$ *is even.*
(3) *For a Quadrilateral snake,* $CITD(Qs_n) = n$.
(4) *For a ladder graph,*$CITD(l_n) = \begin{cases} n & \text{if } n \text{ is even} \\ n - 1 & \text{if } n \text{ is odd} \end{cases}$.

**Observation 2.3.** *For any connected graph G, $CITD(G) \geq \lceil \frac{p}{\Delta+1} \rceil$.*

**Observation 2.4.** *Every complementary independent twin paired dominating set is a restrained dominating set but converse is not true.*

**Observation 2.5.** *From above observation it can be observed that, $\gamma_{rd}(G) \leq CITD(G)$, this comparison is valid only for graphs with even number of vertices.*

**Theorem 2.3.** *For any connected graph $G$ with even number of vertices, $\lceil \frac{p}{\Delta+1} \rceil \leq CITD(G) \leq p-2$ and the bounds are sharp.*

*Proof.* Let $G$ be a connected graph with even number of vertices, if we take all the vertices as complementary independent twin paired dominating set then its complement is empty which contradicts the definition. If we take $p-1$ as the complementary independent twin paired dominating set the contradiction occurs in the set as it is not a paired dominating set. Hence $CITD(G) \leq p-2$. The lower bound is obvious from the observation. For sharpness, upper bound is attained for $K_4$ and lower bound is attained for $C_4$. □

**Proposition 2.1.** *For a fan graph, $CITD(F_{1,m}) = 2\lfloor \frac{1+m}{4} \rfloor$, where m is odd.*

*Proof.* The vertex set of $F_{1,m}$ is $\{x, v_1, v_2, \ldots ..v_m\}$ and the graph is formed by joining all the vertices of the path $P_m$ with the independent vertices $x$. As CITD-set can be found only for even number of vertices, $m$ must be odd. Clearly the vertex $x$ dominates all the vertices of the path $P_m$. As the dominating set must be perfect matching, their must be another vertex which pairs $x$, say $v_3$, and from the remaining vertices $\{v_i : i \equiv 2, 3(mod\ 4)\ and\ i \geq 4\}$ is the perfect matching. Hence $S = \{x, v_3\} \cup \{v_i : i \equiv 2, 3(mod\ 4)\ and\ i \geq 4\}$ also $< V - S >$ is a set of all independent edges implies $S$ is the required CITD-set. $|S| = 2\lfloor \frac{1+m}{4} \rfloor$. Clearly $CITD(F_{1,m}) \leq |S|$.

If $T \subseteq S$ is the complementary independent twin paired dominating set, then $T$ fails to satisfy the definition. Therefore $S$ is the minimum complementary independent twin paired dominating set. This implies $|S| \leq CITD(F_{1,m})$. Hence it follows $|S| = CITD(F_{1,m})$. □

**Proposition 2.2.** *For a double fan graph, $CITD(F_{2,m}) = 2 + 2\lfloor \frac{m}{4} \rfloor$, where $m$ is even.*

*Proof.* The vertex set of $F_{2,m}$ is $\{x, y, v_1, v_2, \ldots ..v_m\}$ and the graph is formed by joining all the vertices of the path $P_m$ with the independent vertices $x$ and $y$.

As CITD-set can be found only for even number of vertices, $m$ must be even. Clearly the vertex $x$ and $y$ dominates all the vertices of the Path $P_m$. As the dominating set must be perfect matching their must be another vertices $v_3$ and $v_4$ which pairs $x$ and $y$ respectively. And from the remaining vertices $\{v_i : i \equiv 2, 3(mod\ 4)\ and\ i \geq 4\}$ is the perfect matching. Hence $S = \{x, v_3\} \cup \{v_i : i \equiv 0, 3(mod\ 4)\ and\ i > 4\}$ also $< V - S >$ is a set of all independent edges implies $S$ is the required CITD-set . $|S| = 2 + 2\lfloor \frac{m}{4} \rfloor$. Clearly $CITD(F_{2,m}) \leq |S|$.

If $T \subseteq S$ is the complementary independent twin paired dominating set, then $T$ fails to satisfy the definition. Therefore $S$ is the minimum complementary independent twin paired dominating set. This implies $|S| \leq CITD(F_{2,m})$. Hence $|S| = CITD(F_{2,m})$ □

**Proposition 2.3.** *For a Jahangir graph, $CITD(J_{m,n}) = 2\lfloor \frac{mn}{4} \rfloor + 2$.*

*Proof.* The vertex set of $J_{m,n}$ is given by $\{v_0, v_1, v_2, \ldots .v_{mn}\}$. By the definition of Jahangir graph, and by the definition of CITD- set $m$ and $n$ must be odd. Clearly $v_0$ dominates $n$ number of vertices and as the dominating set must be paired dominating set, there exists a pair $v_n$ (say) for $v_0$. From the remaining even vertices $\{v_i : i \equiv 0, 3(mod\ 4)\} - \{vn\}$ is the dominating set. Let $S = \{v_0, v_n\} \cup \{\}\{v_i : i \equiv 0, 3(mod\ 4)\} - \{vn\}\}$. Clearly $< V - S >$ is the set of all independent edges hence $S$ is the CITD-set and $|S| = 2\lfloor \frac{mn}{4} \rfloor + 2$. Clearly $CITD(J_{m,n}) \leq |S|$.

If $T \subseteq S$ is the complementary independent twin paired dominating set, then $T$ fails to satisfy the definition. Therefore $S$ is the minimum complementary independent twin paired dominating set. this implies $|S| \leq CITD(J_{m,n})$ and therefore $|S| = CITD(J_{m,n})$. □

## 3. Complementary independent twin paired domination number for the graph of the form $C_s(P_n)$

Let $C_s$ be the cycle with $s$ number of vertices and $P_n$ be the path with $n$ number of vertices.

The graph $C_s(P_n)$ is formed by pasting the pendent vertices of $P_n$ to any one vertices of $C_s$. As said before complementary independent twin paired domination number can be found only for even number of vertices. Complementary independent twin paired domination number for $C_s(P_n)$ can be found only when

$s$ is odd and $n$ is even or $s$ is even and $n$ is odd, which will be discussed in the following two theorems.

**Theorem 3.1.** *If $s$ is odd and $n$ is even, then*

$$CITD(C_s(P_n)) = \begin{cases} 2\lfloor \frac{s+n-1}{4} \rfloor + 2 & \text{if } n \equiv 0(mod\ 4) \\ 2\lfloor \frac{s+n-1}{4} \rfloor & \text{if } n \equiv 2(mod\ 4) \end{cases}$$

*Proof.* Let $\{v_1, v_2, \ldots, v_s\}$ be the set of all vertices of $C_s$ and $\{u_1, u_2, \ldots u_n\}$ be the set of all vertices of $P_n$. Thus $\{v_1, v_2, \ldots v_s, u_1, u_2, \ldots u_{n-1}\}$ is the set of all vertices of $(C_s(P_n))$. Let $S_1 = \{u_i : i \equiv 1, 2(mod\ 4)\} - u_n$, $S_2 = \{v_i : i \equiv 2, 3(mod\ 4)\}$, $S_3 = \{v_i : i \equiv 0, 1(mod\ 4)\}$.

**Case (i):** $n \equiv 0(mod\ 4)$

$S = S_1 \cup S_2$ is the complementary independent twin paired dominating set whose cardinality is $2\lfloor \frac{s+n-1}{4} \rfloor + 2$.

**Case (ii):** $n \equiv 2(mod\ 4)$

$S = S_1 \cup S_3$ is the Complementary independent twin paired dominating set whose cardinality is $2\lfloor \frac{s+n-1}{4} \rfloor$.

In the above case, clearly $CITD(C_s(P_n)) \leq |S|$. If $T \subset S$ is the complementary independent twin paired dominating set, then $T$ fails to satisfy the definition. Therefore $S$ is the minimum complementary independent twin paired dominating set, which implies $|S| \leq CITD(C_s(P_n))$. Hence $|S| = CITD(C_s(P_n))$. $\square$

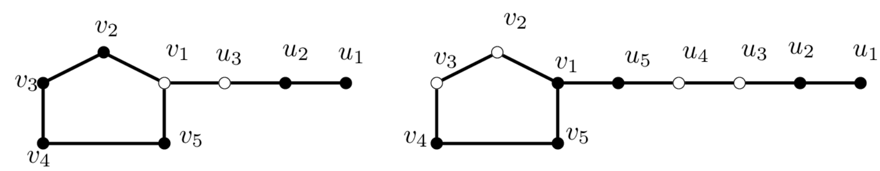

Figure 3.1: $C_5(P_4)$ Figure 3.2: $C_5(P_6)$

FIGURE 3

**Example 2.** *The darkened vertices in the Figure 3 denote the CITD-set and let the cardinality of darkened vertices be denoted as $|S|$. In the graph of the left side in Figure 3, $|S|$=6. Here $s = 5$ and $n = 4$, hence it comes under the case $s$ is odd and $n \equiv 0(mod\ 4)$. This implies, $CITD(C_s(P_n)) = 2\lfloor \frac{s+n-1}{4} \rfloor + 2 = 2\lfloor \frac{5+4-1}{4} \rfloor + 2 = 6$*

*In the graph of the right hand side, |S|=6. Here $s = 5$ and $n = 6$, hence it comes under the case $s$ is odd and $n \equiv 2(mod\ 4)$. This implies, $CITD(C_s(P_n)) = 2\lfloor\frac{s+n-1}{4}\rfloor = 2\lfloor\frac{5+6-1}{4}\rfloor = 6$.*

**Theorem 3.2.** *Let s be even and n be odd.*

(1) *If $s \equiv 0(mod\ 4)$ , then*

$$CITD(C_s(P_n)) = \begin{cases} 2\lfloor\frac{s+n-1}{4}\rfloor + 2 & \text{if } n \equiv 0(mod\ 4) \\ 2\lfloor\frac{s+n-1}{4}\rfloor & \text{if } n \equiv 2(mod\ 4) \end{cases}$$

(2) *If $s \equiv 2(mod\ 4)$ , then $CITD(C_s(P_n)) = 2\lfloor\frac{s+n-1}{4}\rfloor + 2$.*

*Proof.* Let $\{v_1, v_2, \ldots, v_s\}$ be the set of all vertices of $C_s$ and $\{u_1, u_2, \ldots u_n\}$ be the set of all vertices of $P_n$. Thus $\{v_1, v_2, \ldots v_s, u_1, u_2, \ldots u_{n-1}\}$ is the set of all vertices of $(C_s(P_n))$.

Let $S_1 = \{u_i : i \equiv 1, 2(mod\ 4)\} - \{un\}$, $S_2 = \{v_i : i \equiv 2, 3(mod\ 4)\}$, $S_3 = \{v_i : i \equiv 0, 1(mod\ 4)\} \cup \{v_{p-1}, v_p\}$. As $n$ is odd, it can be split into two cases either $n \equiv 1(mod\ 4)$ or $n \equiv 3(mod\ 4)$.

If $n \equiv 1(mod\ 4)$, then the complementary independent twin paired dominating set is $S_1 \cup S_2$.

If $n \equiv 3(mod\ 4)$, then the complementary independent twin paired dominating set is $S_1 \cup S_3$. But the cardinality differs depending upon $n$ and $s$. As $s$ is even, either $s \equiv 0(mod\ 4)$ or $s \equiv 2(mod\ 4)$.

**Case (i):** $s \equiv 0(mod\ 4)$

(1) $n \equiv 1(mod\ 4)$ $|S| = 2\lfloor\frac{s+n-1}{4}\rfloor$

(2) $n \equiv 2(mod\ 4)$ $|S| = 2\lfloor\frac{s+n-1}{4}\rfloor + 2$

**Case (ii):** $s \equiv 2(mod\ 4)$

In both the cases $|S| = 2\lfloor\frac{s+n-1}{4}\rfloor + 2$. In all the above cases, clearly $CITD(C_s(P_n)) \leq |S|$.

If $T \subseteq S$ is the complementary independent twin paired dominating set, then $T$ fails to satisfy the definition. Therefore $S$ is the minimum complementary independent twin paired dominating set. This implies $|S| \leq CITD(C_s(P_n))$ thus it follows that $|S| = CITD(C_s(P_n))$. $\square$

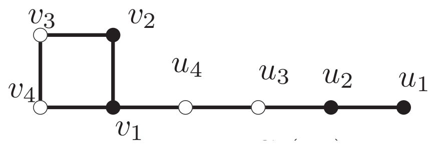


FIGURE 4. $C_4(P_5)$

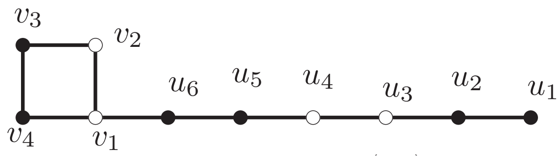


FIGURE 5. $C_4(P_7)$

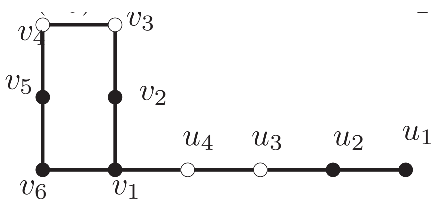


FIGURE 6. $C_6(P_5)$

**Example 3.** *The darkened vertices in the graphs of Figure* **??** *denote the CITD-set. Let it be denoted as* $|S|$.

*In the graph of the 4,* $|S|$*=4. Here* $s = 4$ *and* $n = 5$*, hence it comes under the case* $s \equiv 0(mod\ 4)$ *and* $n \equiv 1(mod\ 4)$*. This implies,* $CITD(C_s(P_n)) = 2\lfloor\frac{s+n-1}{4}\rfloor = 2\lfloor\frac{4+5-1}{4}\rfloor = 4$.

*In the graph of Figure 5,* $|S|$*=6. Here* $s = 4$ *and* $n = 7$*, hence it comes under the case* $s \equiv 0(mod\ 4)$ *and* $n \equiv 3(mod\ 4)$*. This implies,* $CITD(C_s(P_n)) = 2\lfloor\frac{s+n-1}{4}\rfloor + 2 = 2\lfloor\frac{4+7-1}{4}\rfloor + 2 = 6$.

*In Figure 6,* $|S|$*=6. Here* $s = 6$ *and* $n = 4$*, hence it comes under the* $s \equiv 2(mod\ 4)$*. This implies* $CITD(C_s(P_n)) = 2\lfloor\frac{s+n-1}{4}\rfloor + 2 = 2\lfloor\frac{6+4-1}{4}\rfloor + 2 = 6$.

## 4. CONCLUSION

In this paper, we have initiated the concept called complementary independent twin paired domination number of the graph and discussed its nature with dominating set. We have also found the bounds for this parameter and discussed its number for some standard and special type of graphs. The authors obtained this CITD - number for some results for derived graphs and product related graphs which will be reported in the subsequent papers.

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## REFERENCES

[1] F. HARARY: *Graph Theory*, Addison Wesley Reading Mass,1972.

[2] T. W. HAYNES, S. T. HEDETNIEMI, P.J.SLATER: *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[3] G. MAHADEVAN, A. NAGARAJAN, A. RAJESWARI: *Characterization of paired domination number of a graph*, International Journal of Computational Engineering Research, **2**(2012), 1070–1075.

[4] G. MAHADEVAN, B. AYISHA: *Further Results On complementary perfect domination number of a graph*, International Mathematical Forum, **8**(2013), 85–103.

[5] G. MAHADEVAN, A. I. BASIRA, M. V. SUGANTHI: *Restrained step domination number of a graph*, Advances in pure & Applied Mathematics, **54** (5) (2018), 31–40.

[6] T. W. HAYNES, P. J. SLATER: *Paired domination in graphs*, Networks, **32**(1998), 199–209.

[7] G. MAHADEVAN, M. V. SUGANTHI: *Discovering Complementary Independent Twin Paired Domination Number for Some Product Related Graphs*, Advances in Mathematics: Scientific Journal, **9**(6) (2020), 4261–4270.

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