

STRONG PRIME IDEALS IN TERNARY SEMIRING

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ABSTRACT. This article introduces the notion of strong prime ideals in ternary semiring; an m -system corresponding to the above strong primeness and expose some results in completely prime ideals in ternary semiring.

1. INTRODUCTION

The writing of the theory of ternary operations is huge and disperse over various disciplines of mathematics. Ternary generalization of mathematical structures are the exceptionally characteristic ways for additional turn of events and inside and out cognizance of their essential attributes. Cayley just because spearheaded and propelled first ternary mathematical operations in the manner, thinking back to the nineteenth century. Cayley's thoughts elucidated and created n -ary generalization of matrices and their determinants [9, 13] and general theory of n -ary algebras [3, 10] and ternary rings [11]. Ternary structures and their generalizations creat a few expectations in view of their chance of utilizations in material science. A couple of significant physical applications are recorded in [1, 2, 6, 7]. In compatibility of Lister's generalization of ternary rings presented in 1971, T. K. Dutta and S. Kar concocted the thought of ternary semirings. T. K. Dutta and S. Kar started prime ideals and prime radical of

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ternary semirings in [4]. Similar analysts propelled semiprime ideals and irreducible ideals of ternary semirings in [5]. Moreover S. Kar in [8] concocted the thought of quasi-ideals and bi-ideals in ternary semirings. Thus, M. Shabir and M. Bano coasted prime bi-ideals in ternary semigroups in [12].

2. PRELIMINARIES

For the basic terminology about ternary semirings see [4–8]. Throughout the paper $\mathcal{T}_S$ means a ternary semiring with zero.

3. STRONG PRIME IDEAL IN TERNARY SEMIRING

In this section we introduce the notion of SPI in $\mathcal{T}_S$. Further ltI , llI and rtI means left ideal, lateral ideal and right ideal respectively.

Definition 3.1. An ideal P of $\mathcal{T}_S$ is said to be SPI if for any ltI L , llI M and rtI R of $\mathcal{T}_S$ $RML \subseteq P \Rightarrow R \subseteq P$ or $M \subseteq P$ or $L \subseteq P$.

Any element $a \in RML$ is of the form $a = \sum_i r_i m_i l_i$ for $r_i \in R$, $m_i \in M$ and $l_i \in L$. Clearly every SPI is prime but not the converse, and $SxS + SSxSS$ is a llI of $\mathcal{T}_S$.

Theorem 3.1. Let P be a proper ideal of $\mathcal{T}_S$. Then the conditions bellow are equivalent:

- (1) P is SPI.
- (2) For every $l, m, n \in \mathcal{T}_S$ such that $lSmSn + lSSmSSn \subseteq P$ then $l \in P$ or $m \in P$ or $n \in P$.
- (3) If $\langle l \rangle$, $\langle m \rangle$ and $\langle n \rangle$ are principle ideals in $\mathcal{T}_S \ni \langle l \rangle \langle m \rangle \langle n \rangle \subseteq P$, then $l \in P$ or $m \in P$ or $n \in P$.

Proof.

(1) $\Rightarrow$ (2): Let $l, m, n \in \mathcal{T}_S \ni$ such that $lSmSn + lSSmSSn \subseteq P$. We know that lSS is rtI , $SmS + SSmSS$ is a llI and SSc is a ltI . Now:
 $lSS(SmS + SSmSS)SSn = lSSSmSSSn + lSSSSmSSSn =$
 $lSmSn + lSSmSSn \subseteq P \Rightarrow lSS \subseteq P$ or $(SmS + SSmSS) \subseteq P$ or $SSl \subseteq P$. If $lSS \subseteq P$, then $l^3 + lSS \subseteq P$.

Consider rtI , llI and ltI generated by l ; $\langle l \rangle_R$, $\langle l \rangle_M$, $\langle l \rangle_L$. Now, $\langle l \rangle_R \langle l \rangle_M \langle l \rangle_L = \{(n_0l + lSS)(n_1l + SlS + SS lSS)(n_2l + SS l)\} \subseteq l^3 + lSS \subseteq P$. (Here $n_i, i = 0, 1, 2$ represent elements in $\mathbb{N}$.) That implies $\langle l \rangle_R \subseteq P$ or $\langle l \rangle_M \subseteq P$ or $\langle l \rangle_L \subseteq P \Rightarrow l \in P$.

Likewise we can prove for $SmS + SS mSS \subseteq P$ and $SSn \subseteq P$. That is $\langle m \rangle_R \langle m \rangle_M \langle m \rangle_L = (n_0m + mSS)(n_1m + SmS + SS mSS)(n_2m + SS m) \subseteq SmS + SS mSS \subseteq P \Rightarrow \langle m \rangle_R \subseteq P$ or $\langle m \rangle_M \subseteq P$ or $\langle m \rangle_L \subseteq P \Rightarrow m \in P$ and similarly the other case arrive, that is $n \in P$.

(2) $\Rightarrow$ (3) Let $\langle l \rangle \langle m \rangle \langle n \rangle \subseteq P$. Now $lSmSn + lSSmSSn \subseteq \langle l \rangle \langle m \rangle \langle n \rangle \subseteq P$. Therefore by hypothesis $l \in P$ or $m \in P$ or $n \in P$.

(2) $\Rightarrow$ (1) Let for every $l, m, n \in \mathcal{T}_S$ such that $lSmSn + lSSmSSn \subseteq P \Rightarrow l \in P$ or $m \in P$ or $n \in P$. Then we prove that P is a SPI. Let $RML \subseteq P$ for any $R - rtI$, $M - llI$ and $L - ltI$ of $\mathcal{T}_S$. Suppose not; $R \not\subseteq P$, $M \not\subseteq P$ and $L \not\subseteq P$. Then there exists $l \in R \setminus P$, $m \in M \setminus P$ and $n \in L \setminus P$. Now $lSmSn + lSSmSSn \subseteq RML + RML = RML \subseteq P$ that implies $l \in P$ or $m \in P$ or $n \in P$, which is absurd. This implies $R \subseteq P$ or $M \subseteq P$ or $L \subseteq P$. Therefore P is a SPI. $\square$

Definition 3.2. $m_1 - system$: Let $M \subseteq \mathcal{T}_S$. M is an $m_1 - system$ if given $l, m, n \in M$ there exists $l_1 \in \langle l \rangle_R, m_1 \in \langle m \rangle_M$ and $n_1 \in \langle n \rangle_L \ni l_1 m_1 n_1 \in M$.

Theorem 3.2. Let P be an ideal of a $\mathcal{T}_S$. Then P is SPI iff $S \setminus P$ is an $m_1 - system$.

Proof. Let P be a SPI of $\mathcal{T}_S$. Let $l, m, n \in S \setminus P$. Suppose $l_1 m_1 n_1 \notin S \setminus P$ for all $l_1 \in \langle l \rangle_R, m_1 \in \langle m \rangle_M$ and $n_1 \in \langle n \rangle_L$. Then $\langle l \rangle_R \langle m \rangle_M \langle n \rangle_L \subseteq P$. Since P is SPI $\langle l \rangle_R \subseteq P$ or $\langle m \rangle_M \subseteq P$ or $\langle n \rangle_L \subseteq P$. This implies $l \in P$ or $m \in P$ or $n \in P$. which is a contradiction. Therefore $l_1 m_1 n_1 \in S \setminus P$ for some $l_1 \in \langle l \rangle_R, m_1 \in \langle m \rangle_M$ and $n_1 \in \langle n \rangle_L$.

Conversely, let L, M, N be rtI, llI and ltI of $\mathcal{T}_S$ respectively such that $LMN \subseteq P$. Suppose $L \not\subseteq P, M \not\subseteq P$ and $N \not\subseteq P$. Set $l \in L \setminus P, m \in M \setminus P$ and $n \in N \setminus P$. Then $l, m, n \in S \setminus P$ and since $S \setminus P$ is an $m_1 - system$ $l_1 m_1 n_1 \in S \setminus P$ for some $l_1 \in \langle l \rangle_L, m_1 \in \langle m \rangle_M$ and $n_1 \in \langle n \rangle_N$.

But $l_1 m_1 n_1 \in \langle l \rangle_L \langle m \rangle_M \langle n \rangle_N \subseteq LMN \subseteq P$. This in turn implies $l_1 m_1 n_1 \in P$. which is a contradiction to the assumption that $l_1 m_1 n_1 \in S \setminus P$. Hence $L \subseteq P$ or $M \subseteq P$ or $N \subseteq P$. $\square$

Theorem 3.3. Let $\mathcal{T}_S$ has an ideal I , and A be an m -system such that $M \cap I = \phi$, then $\mathcal{T}_S$ has a PI P such that $I \subseteq P$ with $P \cap A = \phi$.

Proof. Consider $J = \{K \subseteq S/I \subseteq K, M \cap K = \phi\}$. K is an ideal in $\mathcal{T}_S$. Clearly $J \neq \phi$, since I is one such ideal. Suppose $J_1, J_2, \dots \in J$ such that $J_1 \subseteq J_2 \subseteq \dots \subseteq J_n \dots$. Let $J = \bigcup J_i$ then we prove that J is an ideal. Let $a \in J$, then $a \in J_i$ for some i . As J_i is an ideal in S , for $s_1, s_2 \in S$, $s_1 s_2 x \in J_i$, $s_1 x s_2 \in J_i$ and $x s_1 s_2 \in J_i$. And so $s_1 s_2 x \in J$, $s_1 x s_2 \in J$ and $x s_1 s_2 \in J$. Therefore given set is a non-empty. By Zorn's lemma there exists a maximal ideal $P \ni M \cap P = \phi$.

We shall prove P is PI. Suppose $\langle a \rangle \not\subseteq P$, $\langle b \rangle \not\subseteq P$ and $\langle c \rangle \not\subseteq P$ for $\langle a \rangle$ -an ideal generated by a , $\langle b \rangle$ - an ideal generated by b and $\langle c \rangle$ - an ideal generated by c . By the maximality theory there exists $a_1 \in M$ such that $a_1 \in P + \langle a \rangle$, $b_1 \in M$ such that $b_1 \in P + \langle b \rangle$ and $c_1 \in M$ such that $c_1 \in P + \langle c \rangle$.

Since M is an m -system there exists $a'_1 \in \langle a_1 \rangle$, $b'_1 \in \langle b_1 \rangle$, and $c'_1 \in \langle c_1 \rangle$ such that $a'_1 b'_1 c'_1 \in M$. Moreover $a'_1 b'_1 c'_1 \in (P + \langle a_1 \rangle)(P + \langle b_1 \rangle)(P + \langle c_1 \rangle) \subseteq P$ only when $\langle a_1 \rangle \langle b_1 \rangle \langle c_1 \rangle \subseteq P$ that would simply $a'b'c' \in P$. This is impossible, since $M \cap P = \phi$. Therefore $\langle a_1 \rangle \langle b_1 \rangle \langle c_1 \rangle \not\subseteq P$ and hence P is prime. $\square$

Theorem 3.4. Let $\mathcal{T}_S$ has an ideal U and M be an m_1 - system $\ni M \cap U = \phi$. Then there exists a SPI $P \ni U \subseteq P$ with $M \cap P = \phi$.

Proof. Construct set of ideals K as in Theorem 3.3 $\ni U \subseteq K$ and $M \cap K = \phi$. The set of all such ideal is non-empty, since U is in the set. By the Zone's lemma we have a maximal ideal $P \ni M \cap P = \phi$.

Suppose $\langle l \rangle_R \not\subseteq P$, $\langle m \rangle_M \not\subseteq P$ and $\langle n \rangle_L \not\subseteq P$ for $\langle l \rangle_R$ - a rtI generated by l , $\langle m \rangle_M$ - a llI generated by m and $\langle n \rangle_L$ - a ltI generated by n . By the maximality theory, there exists $l_1 \in M \ni l_1 \in (P + \langle l \rangle_R)$, $m_1 \in M \ni m_1 \in (P + \langle m \rangle_M)$ and $n_1 \in M \ni n_1 \in (P + \langle n \rangle_L)$. Since M is a m_1 - system for $l_1, m_1, n_1 \in M$ there exists $l'_1 \in \langle l_1 \rangle_R$, $m'_1 \in \langle m_1 \rangle_M$ and $n'_1 \in \langle n_1 \rangle_L$ such that $l'_1 m'_1 n'_1 \in M$.

Moreover $[l'_1 m'_1 n'_1 \in (P + \langle l \rangle)(P + \langle m \rangle)(P + \langle n \rangle) \subseteq P]$ iff $\langle l \rangle_R \langle m \rangle_M \langle n \rangle_L \subseteq P$. That indeed imply $l'_1 m'_1 n'_1 \in P$, which contradicts the fact that $M \cap P = \phi$. Therefore $\langle l \rangle_R \langle m \rangle_M \langle n \rangle_L \not\subseteq P$. So P is a SPI. $\square$

Theorem 3.5. If M is an m - system then M is an m_1 - system.

Proof. Let $l, m, n \in M$. Consider an element $l_1 \in \langle l \rangle_R$ a rtI generated by l , $m_1 \in \langle m \rangle_M$ a llI generated by m and $n_1 \in \langle n \rangle_L$ a ltI generated by n . Since $\langle l \rangle_R \subseteq \langle l \rangle$, $\langle m \rangle_M \subseteq \langle m \rangle$ and $\langle n \rangle_L \subseteq \langle n \rangle \Rightarrow l_1 \in \langle l \rangle$, $m_1 \in \langle m \rangle$ and $n_1 \in \langle n \rangle$ and as M is an m -system $\Rightarrow l_1 m_1 n_1 \in M$. Therefore M is an m_1 - system. $\square$

Definition 3.3. An ideal P of a $\mathcal{T}_S$ is said to be CSPI if $p^3 \in P \Rightarrow p \in P$.

Theorem 3.6. If U is an ideal and V is CSPI in $\mathcal{T}_S$, then $(U : V)$ is an ideal.

Proof. For $u \in (U : V)$, $s_1, s_2 \in \mathcal{T}_S$, $us_1s_2VV = u(s_1s_2V)V \subseteq uVV \subseteq U$ (since U is an ideal). Therefore $us_1s_2 \in (U : V)$. This implies $(U : V)$ is a rtI . For $s_1, s_2, s_3, s_4 \in \mathcal{T}_S$ and $u \in (U : V)$, consider $(s_1us_2b_1b_2)^3 = s_1us_2(b_1b_2s_1us_2b_1b_2s_1u)s_2b_1b_2$. Now,

$$(b_1b_2s_1us_2b_1b_2s_1u)^3 = b_1b_2s_1us_2b_1b_2s_1(ub_1b_2)s_1us_2b_1b_2s_1ub_1b_2s_1us_2b_1b_2s_1u \in U$$

Thus $(b_1b_2s_1us_2b_1b_2s_1u) \in U$. Therefore $(U : V)$ is an ideal. $\square$

Theorem 3.7. If U is a CSPI then for any $V \subseteq \mathcal{T}_S$, $(U : V)$ is an ideal.

Proof. From Theorem 3.6 $(U : V)$ is a ltI . Let $b_1, b_2 \in V$ and $s_1, s_2 \in \mathcal{T}_S$. For $x \in (U : V)$, $(xs_1s_2b_1b_2)^3 = xs_1s_2b_1b_2xs_1s_2(b_1b_2x)s_1s_2b_1b_2$. Now, $(b_1b_2x)^3 = b_1b_2(xb_1b_2)xb_1b_2x \in U$. Therefore $(b_1b_2x)^3 \in U \Rightarrow b_1b_2x \in U$ and $(xs_1s_2b_1b_2)^3 \in U \Rightarrow xs_1s_2b_1b_2 \in U \Rightarrow xs_1s_2 \in (U : V)$. Therefore $(U : V)$ is a rtI . Let $s_1, s_2 \in S$, $x \in (U : V)$ and $b_1, b_2 \in V$ such that $(s_1xs_2b_1b_2)^3 = s_1xs_2(b_1b_2s_1xs_2b_1b_2s_1x)s_2b_1b_2$. Consider $(b_1b_2s_1xs_2b_1b_2s_1x)^3 = b_1b_2s_1xs_2b_1b_2s_1(xb_1b_2)s_1xs_2b_1b_2s_1xb_1b_2s_1xs_2b_1b_2s_1x \in U$, that implies $b_1b_2s_1xs_2b_1b_2s_1x \in U$, so $s_1xs_2b_1b_2 \in U$. Thus $s_1xs_2 \in (U : V)$. Therefore $(U : V)$ is II . Hence $(U : V)$ is an ideal. $\square$

Theorem 3.8. An ideal P is CPI iff P is a SPI and a CSPI.

Proof. Let P be CPI and $R, M, L \ni RML \subseteq P$. Suppose $R \not\subseteq P, M \not\subseteq P$. Let $a \in R \setminus P, b \in R \setminus P$ and $c \in L$ with $abc \in P$. As P is CPI $a \in P$ or $b \in P$ or $c \in P$ but $a \notin P, b \notin P$ implies $c \in P \Rightarrow L \subseteq P$. Let $A^3 \subseteq P$ for an ideal, then $AAA \subseteq P$. Clearly $a^3 \in P \Rightarrow a \in P$.

Conversely, let P be a SPI and CSPI. Let $abc \in P$. Consider $asbsc + assbssc$ such that $(asbsc + assbssc) \in P$. Now $(asbsc + assbssc)^3 = (asbsc + assbssc)(asbsc + assbssc)(asbsc + assbssc)$. Take any element of the product $asbs(casbscasb)sc$, $(casbscasb)^3 = casbscasbcas(bca)sbscasbscasb$. Now $(bca)^3 = bc(abc)abca \in P \Rightarrow bca \in P \Rightarrow casbscasb \in P \Rightarrow asbscasbscasbsc \in P$.

Similarly, if we take $asbs(cassb)sscasbsc$ in the product, $(cassb)^3 = cass(bca)ssbcassb$. Here $bca \in P \Rightarrow cassb \in P \Rightarrow asbscassbsscasbsc \in P$ and similarly if we take $assbsscassbassbsscscsc$ in the product $cassb \in P \Rightarrow asbscassbsscasbsc \in P$. So $(asbsc + assbssc)^3 \in P \Rightarrow (asbsc + assbssc) \in P$. This implies $(aSbSc + aSSbSSc) \subseteq P$. Therefore from Theorem 3.1 $a \in P, b \in P$ and $c \in P$. Thus P is CPI. $\square$

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