

PROPERTIES OF SEQUENTIALLY COMPACT AND LOCALLY COMPACT IN STAR AND ALMOST Menger SPACES

A. KALPANA¹ AND M. SARASWATHI

ABSTRACT. Our aim of this work is to propose the concept sequentially compact and locally compact in almost and star Menger spaces. Also we have developed their properties in the same spaces.

1. INTRODUCTION

SM and strongly SSM spaces were besides examined in papers [2], [6] and [7]. To survey a few outcomes in [3], we review few ideas. For $l, e \in \Omega^\Omega$, $l \leq e$ implies that $l(n) \leq e(n)$ for each $q \in \Omega$. The minimal cardinality of a cofinal subset in $(\Omega^\Omega, \leq)$ is meant by σ , [7]. Consider $\mathcal{F}$ is a essentially different family comprising of infinite subsets of Ω .

Assume $\Psi(R) = \Omega \cup \mathcal{F}$ is the Mrowka-Isbell type space, [4], each point of Ω is isolated in $\Psi(R)$ and an neighbourhood (shortly, nhd.) at $R \in \mathcal{F}$ is of the structure $\{R\} \cup B$, where B is a cofinite subset of R . In this work, we utilize the image $Cof(Fin(k)^N)$ rather than σ_k .

¹corresponding author

2020 *Mathematics Subject Classification.* 54C10, 54D20, 54D30, 54D45, 54C18.

Key words and phrases. Almost Star Menger (ASM) spaces, 1-star compact Menger(1SCM) spaces, sequentially compact strongly star Menger (SCSSM) space, Sequentially compact 1-star Lindelöf (SC1SL) spaces, local compactness (LC).

2. PRELIMINARIES

Definition 2.1. [5], A space $\mathcal{M}$ is termed as a Star Menger (SM) (respectively, ASM, Weakly star Menger (WSM)) for every $\{\mathfrak{T}_n : q \in Q\}$ of open covers $\mathcal{M}$, there are finite subfamilies $\mathfrak{K}_n \subset \mathfrak{K}_n$ (respectively finite subsets $F_n \subset \mathcal{M}$) $\ni \{st(\mathcal{T}\mathfrak{K}_n, \mathfrak{T}_n) : q \in \Omega\}$ (respectively $\{st(F_n, \mathfrak{T}_n) : q \in \Omega\}$) is a cover of $\mathcal{M}$.

Definition 2.2. [8], A space $\mathcal{M}$ is termed as a 1-star compact (1SC) if for each open cover $\mathfrak{T}$ of $\mathcal{M}$, there is a finite subfamily $\mathfrak{K} \subset \mathfrak{T}$ (respectively a finite sub-set $F \subset \mathcal{M}$) $\ni st(\cup \mathfrak{K}, \mathfrak{T}) = \mathcal{M}$. (respectively $st(F) = \mathcal{M}$).

Definition 2.3. [8], A space $\mathcal{M}$ is termed as a 1-star Lindelöf (1SL) (respectively Strongly 1-star Lindelöf), if for each open cover $\mathfrak{K} \subset \mathfrak{T}$ (respectively a countable subset $R \subset \mathcal{M}$) $\ni st(\cup \mathfrak{K}, \mathfrak{T}) = \mathcal{M}$ (respectively $st(A, \mathfrak{T}) = \mathcal{M}$).

Definition 2.4. [5], A space $\mathcal{M}$ is termed as a Strongly star Menger (SSM) if for every seq. $\{\mathfrak{T}_n : q \in Q\}$ of open covers of $\mathcal{M}$, $\exists$ a sequence $\{F_n : q \in Q\}$ of finite sub-sets of $\mathcal{M}$, $\ni st(\{F_n, \mathfrak{T}_n\} : q \in Q)$ is an open cover of $\mathcal{M}$.

Definition 2.5. [1], A sub-set R of a space $\mathcal{M}$ is Menger relative to $\mathcal{M}$ if for any seq. $\{\mathfrak{T}_n : q \in Q\}$ of covers of R by sets open in $\mathcal{M}$, there is a sequence $\{\mathfrak{K}_n : q \in Q\}$ $\ni$ every $\mathfrak{K}_n$ is a finite sub-set of $\mathfrak{T}_n$ and $R \subset \bigcap_{q \in N} \cup \mathfrak{K}_n$.

3. SEQUENTIALLY COMPACT STAR MENDER SPACES

Definition 3.1. Let $\mathcal{M}$ be a M -space. If $t_1, t_2, \dots$ be a increasing sequence of $\mathfrak{T}_n$ and $k_1, k_2, \dots$ be a increasing sequence of $\mathfrak{K}_n$ and if each sequence of points has a convergent subsequence, then the space is termed as a sequentially compact Menger space (SCM-space).

Definition 3.2. A SM-space $\mathcal{M}$ is termed as a Sequentially Compact Star Menger space (SCSM-space) if $\{\mathfrak{T}_n\}_{n=1,2,\dots}$ be a sequence of $\mathfrak{T}_n$ and $\{\mathfrak{K}_n\}_{n=1,2,\dots}$ be a sequence of finite subfamilies of $\mathfrak{K}_n \subset \mathfrak{K}_n$ (respectively finite subsets $E_n \subset \mathcal{M}$) if each sequence of points has a convergent subsequence.

Definition 3.3. Let $\mathcal{M}$ be a 1SCM-space. If t be a subsequence of $\mathfrak{T}$ and k be a subsequence of $k \subset t$ and if both sequences converge to a point then the space $\mathcal{M}$ is termed as a sequentially compact 1-star Menger space (SC1SM-space).

Definition 3.4. A M -space $\mathcal{M}$ is termed as a sequentially compact strongly star Menger(SCSSM) space if $\{t_n\}_{n \in \mathbb{N}}$ be a finite sequence of k_n and if each sequence of points has a convergent subsequence

Definition 3.5. A M -space $\mathcal{M}$ is termed as a sequentially compact 1-star Lindelöf (SC1SL-space) (resp. SCS1SL) if $\{t\} \in \mathfrak{T}$ and $\{t\} \in \mathfrak{K}$ and if each sequence of points has a convergent subsequence.

Proposition 3.1.

- (i) Each SCSCM-space is Menger.
- (ii) Each SCS1SL-space is Menger.

In both cases the converse need not be true.

Proof. Follows from definitions(SCSSM and SC1SL-space). $\square$

Proposition 3.2. Each SCS1SL-space of cardinality less than σ is SCSSM.

Proof. First we prove the result for 1-star L-space. Let $\mathcal{M}$ be a SCS1S-Lindelöf space of cardinality less than σ . For every $q \in \Omega$ there is a countable sequence of $\mathfrak{K}_n \subset \mathfrak{T}_n \ni$ each sequence is finite and if $\{t\}$ be a sequence of t and k be a sequence of $\mathfrak{K} \subset \mathfrak{T}$ then each sequence has a convergent subsequence. For all $m \in \mathcal{M}$, we choose a function $l_x \in \Omega^\Omega \ni, St(x, m_n) \cap \mathfrak{K}_{n, l_x(n)} \neq \phi$ for all $q \in \Omega$. Therefore $\{t\}$ and $\{k\}$ both converge to m_k for $\mathcal{M}$ is SCSSM ((ie) SCSSM). $\square$

Theorem 3.1. Each regular L -space is SCSSM.

Proof. Let $\mathcal{B}$ be a basis of $\mathcal{M}$ with cardinality equal to the weight which is closed under finite union, finite intersection and complement of closure. let $t, k \in \mathcal{B}$ with $\bar{k} \subset t$ then there is open set $\{C_{tk} : t, k \in \beta \& \bar{k} \subset t\}$. Suppose $\mathcal{B}_n$ has been defined. Then for every $t, k \in \mathcal{B}_n$ with $\bar{k} \subset t$ then is an open set $C_{tk} \ni \bar{k} \subset C_{tk} \subset \overline{C_{tk}} \subset \mathfrak{T}$. Let $|\mathcal{B}_{n+1}| = \mathcal{C}(\mathcal{M})$ containing $\mathcal{B}'_n$ where $\mathcal{B}_{n+1}$ is a basis which closed under finite union, finite intersection and complement of closures. Clearly, $|\mathcal{B}| = \mathcal{C}(\mathcal{M})$. Let t be a sequence of open covers of $\mathcal{M}$. Then for each $m \in \mathcal{M} \ni \mathfrak{K}_x, \mathfrak{T}_x \in \mathcal{B} \ni, m \in \mathcal{K}_x \subset \bar{\mathfrak{K}}_x \subset \mathcal{T}_x \subset \mathfrak{T}$ for any $\mathcal{T} \in \mathfrak{T}$. By reason of $\mathcal{M}$ is Lindelöf, the $\mathcal{K}_x$'s have countable subcover say $\mathcal{K}_0, \mathcal{K}_1, \dots, \mathcal{K}_m, \dots$. Then $\bar{\mathcal{K}}_m \subset \mathcal{T}_m$ for each $m \in \Omega$. Note that $\mathcal{K}_0 \cup \mathcal{K}_1 \cup \dots \cup \mathcal{K}_k \subset \mathcal{T}_0 \cup \mathcal{T}_1 \cup \dots \cup \mathcal{T}_k$ and $\mathcal{K}_0 \cup \dots \cup \mathcal{K}_k, \mathcal{T}_0 \cup \dots \cup \mathcal{T}_k \in \mathcal{B}_n$ for some q . If $m_k \in \mathcal{K}_k \subset \mathcal{T}_k$ then each sequence of $\mathcal{K}_k$ and $\mathcal{T}_k$ converges. Hence $\mathcal{M}$ is SCSSM-space. $\square$

Theorem 3.2. *A regular almost countably SCL-space(RACSCL-space) $\mathcal{M}$ is SC.*

Proof. Consider $\mathcal{T}$ is an open cover of $\mathcal{M}$ and since $\mathcal{M}$ is regular then for every $m \in \mathcal{M}$, there is an open set $\mathcal{T}(m) \in \mathfrak{T}$ including m and a open set $\mathcal{K}(m) \ni m \in \mathcal{K}(m) \subset cl(\mathcal{K}(m)) \subset \mathcal{T}(m)$. By reason of $\mathcal{M}$ is Lindelöf, we have a countable subcover $\mathcal{C} = \{\mathcal{K}(m_n) : n \in \mathbb{N}\}$ of $\mathfrak{K}$. By reason of $\mathcal{M}$ is almost countable SC, there is a finite collection, $\{\mathcal{K}(m_{n_1}), \dots, \mathcal{K}(m_{n_k})\} \subset \mathcal{C}$ such that $\bigcup_{i=1}^k cl(\mathcal{K}(m_{n_i})) = \mathcal{M}$. By reason of $\{\mathcal{T}(m_{n_i}), i = 1, 2, \dots\}$ has a convergent sequence the space $\mathcal{M}$ is SC. $\square$

Example 1. *Each compact space is a M-space. Then the converse need not be true. Let the real line R with the topology $Z = \{0, R, (-\infty, m) : m \in \mathbb{R}\}$ is T_0 -M-space but it is not SC.*

Example 2. *The Sorgenfrey line S is a hereditarily L-space but it is not M-space.*

Lemma 3.1. *Suppose $l : \mathcal{M} \rightarrow Y$ is a continuous function where $\mathcal{M}$ is a M-space. Then $l(S_\alpha(G)) = S_\alpha(l(G))$ for each $E \subset \mathcal{M}$ and each ordinal α .*

Proof. Consider a function $l : \mathcal{M} \rightarrow Y$ is continuous function, where $\mathcal{M}$ and Y are Menger spaces. By reason of l is continuous function, follows that, $l(S_\alpha(E)) \subset S_\alpha(l(E))$. Suppose that $\alpha = 1$ then $s \in S(f(E))$ and $\exists$ a sequence (m_n) in $E \ni l(m_n)$ converges to s . By reason of E is sequentially compact, we may suppose $\text{cont-fn } l(m) = s$. Thus $s \in f(s, (E))$. Suppose that $\alpha \geq 2$. If α is a limit cardinal then, $l(S_\alpha(G)) = l(\mathcal{T}S_\gamma(G)) = \mathcal{T}(l(S_\alpha(G))) = \bigcup_{\gamma < \alpha} S_\gamma(l(G)) = S_\alpha(l(G))$ If $\alpha = \gamma + 1$ then again, $l(S_\alpha(G)) = S_\alpha(l(G))$. $\square$

4. LOCALLY COMPACT Menger SPACES

Definition 4.1. *Let $\mathcal{M}$ be a M-space. If $\{t_n\} \in \mathfrak{T}_m$, $q \in \mathbb{Q}$, $q = 1, 2, \dots$ and $\{k_n\} \in \mathfrak{K}_n$, $q \in \mathbb{Q}$, $q = 1, 2, \dots$ and for every point $m \in \mathcal{M}$ and every open nhd. contains a compact nhd. then $\mathcal{M}$ is termed as a LCM-space.*

Theorem 4.1. *A regular almost countably compact L-space(RACCL-space) $\mathcal{M}$ is compact.*

Proof. Consider $\mathfrak{T}$ is an open cover of $\mathcal{M}$. By reason of $\mathcal{M}$ is regular then for every $m \in \mathcal{M}$, $\exists$ a $\mathcal{T}(m) \in \mathfrak{T} \supset m$ including m and an open set $\mathcal{K}(m) \ni m \in$

$\mathcal{K}(m) \subset cl(\mathcal{K}(m)) \subset \mathcal{T}(m)$. If $\mathfrak{K} = \{\mathcal{K}(m) : m \in \mathcal{M}\}$ then we have a open cover of $\mathcal{M}$. By reason of $\mathcal{M}$ is Lindelöf, we get a countable subcover. $P = \{\mathcal{K}(m_n) : q \in Q\}$ of $\mathfrak{K}$ By reason of $\mathcal{M}$ is almost countably compact and locally compact, there is a finite collection, $\{\mathcal{K}(m_{n_1}), \dots, \mathcal{K}(m_{n_k})\} \subset P \ni \bigcup_{i=1}^k cl(\mathcal{K}(m_{n_i})) = \mathcal{M}$. Hence the compact nhd. $\{\mathcal{T}(m_{n_i}) : i = 1, 2, \dots, k\}$ is a finite subcover of $\mathfrak{T}$. $\square$

Example 3. Consider $R = \{(m, p) \in R \times R : 0 < m \leq 1, p = 0\}$ then R has the subspace topology τ , on $R \times R$. Then R is a M-compact space. Assume that $\mathfrak{T}_n$ is the set of all open disks in the upper half-plane of radius $\frac{1}{n}$. $\mathfrak{T}_n$ is an open cover of R . For each $m \in \mathcal{M} \ni$ nhds. $\mathcal{T}(m_{n_1}), \dots, \mathcal{T}(m_{n_k}) \in \mathfrak{T}$ and $\mathcal{K}(p_{n_1}), \mathcal{K}(p_{n_2}), \dots, \mathcal{K}(p_{n_k}) \in \mathcal{K}$. For $q \in Q$. all these open covers has countable many points in R and compact nhd.s R is a LCM-space.

Theorem 4.2. A Menger dense sub-set of a space $\mathcal{M}$ is LCM-space.

Proof. Consider R is a dense subset of $\mathcal{M}$ and $\mathcal{M}$ is a M-space and $\{\mathfrak{T}_n : q \in Q\}$ is a seq. of open covers of $\mathcal{M}$. By reason of R is a M-space in $\mathcal{M} \ni$ finite sets $\mathfrak{K}_n$, $q \in Q \ni R \subset \bigcup_{q \in N} \cup \{k : k \in \mathfrak{K}_n\}$.

By reason of $\mathcal{M}$ is compact and every point $m \in \mathcal{M}$ belongs to $R \ni$ nhd.s $\mathcal{K}(m_{n_i})$, $i = 1, 2, \dots, k$ of $\mathfrak{K}$ contains a finite subcover. Hence $\mathcal{M}$ is a LCM-space. $\square$

Theorem 4.3. Suppose $\mathcal{M}$ be an AM-space and Y be a topological space. If $l : \mathcal{M} \rightarrow Y$ is a q -irresolute function, then Y is an AM-space and also it satisfies LC.

Proof. Consider $\{\mathfrak{T}_n : q \in Q\}$ is a sequence of covers of Y by compact sets and $\mathfrak{T}'_n = \{l^{-1}(t) : \mathcal{T} \in \mathfrak{T}_n\}$ for every $q \in Q$. By the reason l is a q -irresolute function $\{\mathfrak{T}'_n : q \in Q\}$ is a sequence of open covers of $\mathcal{M}$, for each $m \in \mathcal{M} \ni$ nhd.s $\mathcal{T}(m_{n_i})$, $i = 1, 2, \dots, k \ni$ each nhd.s contains compact nhd. and $\mathcal{M}$ satisfy the condition of locally compactness. Also $\mathcal{M}$ is an AMS, $\exists$ a sequence $\{\mathfrak{K}_n : q \in Q\} \ni \bigcup_{q \in N} \mathfrak{K}_n$ is a cover of $\mathcal{M}$. If $\mathcal{K}(p_{n_i})$, $i = 1, 2, \dots, k$, $p \in Y$ contains a compact nhd. then Y is also LCM. Suppose E be a compact set in the topological space, Y and $\mathcal{K} \subset G$ then Y is a LCM-space. $\square$

Definition 4.2. A M-space $\mathcal{M}$ is termed as a M-Lindelöf (respectively Lindelöf) if each open cover $\mathfrak{T}$ of $\mathcal{M}$ has a point-countable open refinement ν .

Theorem 4.4. Each M-L space is L-space.

Proof. Suppose $\mathcal{M}$ is a M-space and also L-space. Take $\mathcal{T}$ be an open cover of $\mathcal{M}$ and γ be a point-countable open refinement of $\mathcal{T}$. By reason of $\mathcal{M}$ is a M-space, there is a sequence of open subsets $\{G_n : n \in \mathbb{N}\} \ni \bigcup_{n \in \mathbb{N}} (G_n, \gamma_n) = \mathcal{M}$. Suppose P_n be the collection of all members of γ which intersects E_n . By reason of γ is pointwise - countable and E_n is finite then P_n is countable. Then the collection $P = \bigcup_{n \in \mathbb{N}} P_n$ is countable family of γ which is a cover of $\mathcal{M}$. If $p \in P$ and $p \in \mathcal{T}_p$. Then the collection $\{\mathcal{T}_p : p \in P\}$ is a countable subcover. Then $\mathcal{M}$ is a L-space. $\square$

REFERENCES

- [1] A. SABAH, M.U.D. KHAN, L.D.R. KOINAC: *Covering properties defined by semi-open sets*, Journal of Nonlinear science and Applications, **9** (2016), 4388–4398.
- [2] M. BONANZINGA, M. METVEEV: *Some covering properties for Ψ -spaces*, Mat. Vesn., **61** (2009), 3–11.
- [3] C. CHIS, M.V. FERRER, S. HERNANDEZ, B. TSABAN: *The character of topological groups, via Pontryagin-van Kampen duality*, Journal of Algebra, **420** (2014), 86–119.
- [4] L. GILLMAN, M. JENISON: *Rings of continuous Functions*, reprint of the 1960 edition, Graduate Texts in Mathematics, springer-verlog, New york-Heidelbeg, **43** (1976).
- [5] M. SAKAI: *Star versions of the property*, Topology and its Applications, **176** (2014), 22–34.
- [6] Y.K. SONG: *Remarks on star Menger spaces II*, Houst. J. Math., **41**(1) (2015), 357–366.
- [7] E.K. VAN DOUWEN: *The integers and topology*, Handbook of set-Theoretic Topology, Elsevier Science Publishers B.V., (1984) 113–167.
- [8] Y.K. SONG: *On star-C-Menger spaces*, Quaestiones Mathematicae, **37** (2014), 337–347.

DEPARTMENT OF MATHEMATICS,
KKC-VELUR 638182,
TAMIL NADU, INDIA.
Email address: kalpanaappachi@gmail.com

DEPARTMENT OF MATHEMATICS
KKC-VELUR 638182
TAMIL NADU, INDIA.
Email address: msmathsnkl@gmail.com