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FIXED POINT THEOREM IN PROBABILISTICALLY CONVEX MANGER SPACE

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ABSTRACT. The main target of this paper has been to apply the concept of probabilistically convexity on manger space and deal a common fixed point theorem by using the concept of compatibility between multi- valued mappings and self mappings in the above context.

1. INTRODUCTION

In 1972, Assad and Kirk in [2] gave sufficient conditions for non-self mappings to ensure the existence of fixed point by proving a result on multi-valued contraction mappings in complete metrically convex metric space. Pai and Veeramani's works, [11] seem to be the first to establish a probabilistic analogue of Nadler's Banch contraction principle for multi-valued mappings, [10]. Hadzic and Gajic in [6], Imdad and Khan in [7], Rhoades in [12] and many others proved some fixed point theorems for non-self, multi - valued convex and sequence of set - valued mapping in metrically spaces. Our intention in this paper is to using the concept of compatibility between a multi- valued mapping and a single-valued mapping due to Kaneko and Sessa in [8] as a tool to produce some common fixed point theorems on complete probabilistically convex

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menger space. The works of Som and Mukherjee in [15], Imdad and Khan in [7] and Ahmad and Assad in [1] are very useful to decisively establish our results.

2. Preliminaries

Definition 2.1. *[13], A mapping* $\mathrm{F} : \mathrm{R} \to \mathrm{R}^+$ *is called a distribution function if it is non decreasing left continuous with*

$$\begin{aligned} \inf\{F(t); t \in R\} &= 0 \quad \textit{and} \\ \sup\{F(t); t \in R\} &= 1. \end{aligned}$$

We shall denote by L *the set of all distribution function while* H *will always* S *denote the specific distribution function defined by*

$$\mathrm{H}(t) = \begin{cases} 0; & t < 1 \\ 1; & t > 0. \end{cases}$$

Definition 2.2. *[13], A Probabilistic Menger Space(PM-space) is an ordered pair* (X, F)*, where* X *is an abstract set of elements and* $\mathrm{F} : \mathrm{X} \times \mathrm{X} \to \mathrm{L}$*, defined by* $(\mathrm{p}, \mathrm{q}) \to \mathrm{F}_{\mathrm{p},\mathrm{q}}$*,where* L *is the set of all distribution function i.e.* $\mathrm{L} = \{\mathrm{F}_{\mathrm{p},\mathrm{q}} | \mathrm{p}, \mathrm{q} \in \mathrm{X}\}$*, if the functions* $\mathrm{F}_{\mathrm{p},\mathrm{q}}$ *satisfy:*

(1) $\mathrm{F}_{\mathrm{p},\mathrm{q}}(\mathrm{x}) = 1$ *for all* $\mathrm{x} > 0$,*if and only if* $\mathrm{p} = \mathrm{q}$,
(2) $\mathrm{F}_{\mathrm{p},\mathrm{q}}(0) = 0$,
(3) $\mathrm{F}_{\mathrm{p},\mathrm{q}} = \mathrm{F}_{\mathrm{q},\mathrm{p}}$,
(4) *if* $\mathrm{F}_{\mathrm{p},\mathrm{q}}(\mathrm{x}) = 1, \text{ and } \mathrm{F}_{\mathrm{p},\mathrm{q}}(\mathrm{y}) = 1$ *then* $\mathrm{F}_{\mathrm{p},\mathrm{q}}(\mathrm{x} + \mathrm{y}) = 1$

Definition 2.3. *[13], A mapping* $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ *is called a* t*-norm if*

(1) $\Delta(\mathrm{a}, 1) = \mathrm{a}$,
(2) $\Delta(\mathrm{a}, \mathrm{b}) = \Delta(\mathrm{b}, \mathrm{a})$,
(3) $\Delta(\mathrm{c}, \mathrm{d}) \geq \Delta(\mathrm{a}, \mathrm{b}) \text{ if } \mathrm{c} \geq \mathrm{a}, \mathrm{d} \geq \mathrm{b}$,
(4) $\Delta(\Delta(\mathrm{a}, \mathrm{b}), \mathrm{c}) = \Delta(\mathrm{a}, \Delta(\mathrm{b}, \mathrm{c}))$.

It follows that $\Delta(\mathrm{a}, 0) = 0, \forall \mathrm{a} \in [0, 1]$ *in particular* $\Delta(0, 0) = 0$.

Definition 2.4. *A Menger space is a triplet* $(\mathrm{X}, \mathrm{F}, \Delta)$*, where* (X, F) *is a PM-space and* Δ *is* t*-norm such that for all* $\mathrm{p}, \mathrm{q}, \mathrm{r} \in \mathrm{X}$ *and* $\forall \mathrm{x}, \mathrm{y} \geq 0$,

$$\mathrm{F}_{\mathrm{p},\mathrm{r}}(\mathrm{x} + \mathrm{y}) \geq \Delta(\mathrm{F}_{\mathrm{p},\mathrm{q}}(\mathrm{x}), \mathrm{F}_{\mathrm{q},\mathrm{r}}(\mathrm{y})).$$

Schweizer and Sklar in [13] proved that if (X, F, Δ) is a menger space with $\sup_{0<x<1} \Delta(x, x) = 1$, then (X, F, Δ) is a Housdorff topological space in the topology τ introduced by the family of (ϵ, λ) neighborhoods.

$$\{U_p(\epsilon, \lambda) : p \in X, \in > 0, \lambda > 0\},$$

where $U_p(\epsilon, \lambda) = \{x \in X; f_{x,p}(\in) > 1 - \lambda\}$
A complete metric space can be treated as a complete menger space in the following way: Throughout this paper, we assume that (X, F, Δ) is a manger space with $(\epsilon, \lambda)-$ topology τ. Let,

$CB(X) = \{A : A$ is non empty closed and bounded subset of $X\}$
$C(X) = \{A : A$ is non empty closed and compact subset of $X\}$.

Definition 2.5. *[4], Let (X, F, Δ) be a Menger space. $A, B \in CB(X)$ and $x \in X$ we define $F_{x,A}$ and $F_{A,B}$ by*

$$\begin{aligned} F_{X,A}(t) &= \sup_{y \in A} F_{x,y}(t) \; and \\ F_{A,B}(t) &= \sup_{s<t} \Delta\{\inf_{x \in A} Sup_{y \in B} F_{x,y}(t), \inf_{y \in B} Sup_{x \in A} F_{x,y}(t)\}, \forall t \in R. \end{aligned}$$

We say that $F_{x,A}$ is the probabilistic distance from x to A and $F_{A,B}$ is the probabilistic distance from A to B induced by F.

Lemma 2.1. *[5], Let (X, F, Δ) be a Menger space Δ be a left continuous t- norm, $A \in CB(X)$ and $x, y \in X$. then we have the following*

(1) *For any $B \in CB(X)$ and $x \in A$*
$\inf_{x \in A} Sup_{y \in B} F_{x,y}(t) \leq F_{x,B}(t)$, *for all $t \in R$,*
(2) $F_{x,A}(t) = 1$ *for all $t > 0$ if and only if $x \in A$*
$F_{x,A}(t_1 + t_2) \geq \Delta(F_{x,y}(t_1), F_{y,A}(t_2))$ *for all $t_1, t_2 > 0$,*
(3) $F_{x,A}(t)$ *is left continuous function at t,*

Now, we first consider the properties of an induced manger space.

Theorem 2.1. *[14], Let (X, d) be a complete metric space and define $F : X \times X \to D^+$ (set of all distribution function)*

$$F_{x,y}(t) = H(t - d(x, y)), for \quad x, y \in E$$

then the space $(X, F, \min)$ with a left continuous t- norm $\Delta = \min$ is a τ- complete menger space and topology τ induced by the metric d coincides with the topology

τ. *And, for* $x \in X, K, C \in CB(X)$ *we can easily obtain.*

$$F_{x,K}(t) = H(t - d(x, K)) \text{ and}$$
$$F_{K,C}(t) = H(t - d_H(K, C)).$$

Proposition 2.1. *Let* (X, F, Δ) *be* τ*- complete Menger space induced by the metric* d *as follows:*

$$F_{x,y}(t) = H(t - d(x, y)), for \quad x, y \in X,$$

where Δ *is a left-continuous* $t-$ *norm such that* $\Delta(a, b) = \min\{a, b\}$.
Let $T : X \to CB(X)$ *a multi-valued mapping, then for each* $x, y \in X$ *and* $u_x \in Tx$ *there exist a* $v_y \in Ty$ *such that*

$$Fu_x, v_y(t) \geq F_{Tx,Ty}(t), t \geq 0$$

Proof. From the compactness of Ty, we can choose $v_y \in Ty$ such that

$$d(u_x, v_y) \leq d_H(Tx, Ty).$$

Hence

$$\begin{aligned} F_{u_x,v_y}(t) &= H(t - d(u_x, v_y)) \\ &\geq H(t - d_H(u_x, v_y)) \\ &= F_{Tx,Ty}(t), t \geq 0. \end{aligned}$$

□

By proposition 2.1 we can easily obtain the following.

Corollary 2.1. *Let* (X, F, Δ) *be a* τ *complete menger space induced by the matric* d *as follows:*

$$F_{x,y}(t) = H(t - d(x, y)), for \quad x, y \in X,$$

where Δ *is left- continuous* t*-norm such that* $\Delta(a, b) = \min\{a, b\}$ *and* $T : X \to CB(X)$ *is a multi-valued mapping. If for each* $x, y \in X$

$$F_{Tx,Ty}(\phi(t)) \geq F_{x,y}(t), t \geq 0.$$

Then for $u_x \in Tx$ *there exists* $v_y \in Ty$ *such that*

$$F_{u_x,v_y}(\phi(t)) \geq F_{x,y}(t), t \geq 0,$$

where $\phi : [0, +\infty) \to [0, +\infty)$ *is a function.*

Definition 2.6. *A Menger space* (X, F, Δ) *is said to be probabilistically convex if for any* $x, y \in X$ *with* $x \neq y$*, there exist* t *a point* $z \in X, x \neq z \neq y$ *such that*

$$\Delta(F_{x,z}(t_1), F_{z,y}(t_2)) = F_{x,y}(t_1 + t_2).$$

Lemma 2.2. *Let* (X, F, Δ) *is said to be complete probabilistically convex menger space. Let* K *be any non- empty closed subset of* X*. Then for any* $x \in K$ *and* $y \notin K$ *there exists a point* $z \in \partial K$ *(the boundary of* K*) such that*

$$\Delta(F_{x,z}(t_1), F_{z,y}(t_2)) = F_{x,y}(t_1 + t_2).$$

Our main theorem is prefaced with the above lemma.

Definition 2.7. *Let* K *be a non-empty subset of a menger space* (X, F, Δ) *and* $S; T : K \to X$ *the pair* $\{S, T\}$ *is said to be weakly commuting if for each* $x, y \in K$ *such that* $X = Sy$ *and* $Ty \in K$*, we have*

$$F_{Tx,STy}(t) \geq F_{Sy,Ty}(t).$$

Definition 2.8. *Let* K *be a non-empty subset of a menger space* (X, F, Δ) *and* $S, T : K \to X$ *the pair* $\{S, T\}$ *is said to be compatible if for every sequence* $\{x_n\}$ *from* K *and from relation*

$$\lim_{x\to\infty} F_{Tx_n,Sx_n}(t) = 1$$

and $Tx_n \in K, n \in N$*, it follows that*

$$\lim_{n\to\infty} F_{Ty_n,STx_n}(t) = 1,$$

for every sequence $\{y_n\}$ *from* K *such that* $y_n = Sx_n, n \in N$*. Kaneko and Sessa in [8], extended the concept of compatibility for single-valued mapping to a multi-valued mapping as follows:*

Definition 2.9. *Let* (X, F, Δ) *be a menger space. The mappings* $A : X \to CB(X)$ *and* $S : X \to X$ *are compatible if* $SA(x) \in CB(x), \forall x \in X$ *and*

$$\lim_{n\to\infty} F_{SAx_n,ASx_n}(t) = 1,$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$$Ax_n \to M \in CB(x) \quad \textit{and} \quad Sx_n \to t \in M.$$

In [3] Chang defined the family of real function ϕ as follows:
Let $\Phi = \{\phi : \mathrm{R}^+ \to \mathrm{R}^+$, ϕ is upper semi- continuous with $\phi(\mathrm{x}) < \mathrm{x}$ for each $\mathrm{x} > 0$ and $\phi(0) = 0$ },where R^+ is the set of all non negative real numbers.

Lemma 2.3. *[3], Let $\phi \in \Phi$, then there exits a strictly increasing continuous function $\psi : \mathrm{R}^+ \to \mathrm{R}^+$ such that $\phi(\mathrm{u}) \le \psi(\mathrm{u}) < \mathrm{u}$ for each $\mathrm{u} > 0, \lim_{\mathrm{n}\to\infty}\psi^{-\mathrm{n}}(\mathrm{u}) = \infty$ and $\psi(\mathrm{u}) > 0,$ for each $\mathrm{u} > 0$.*

Remark 2.1. *In the above case the function ψ is invertible if for each $\mathrm{u} > 0$, we denote $\psi^0(\mathrm{u}) = \mathrm{u}$ and $\psi^{-\mathrm{n}}(\mathrm{u}) = \psi(\psi^{-\mathrm{n}+1}(\mathrm{u}))$ for each $\mathrm{n} \in \mathrm{N}$, the $\lim_{\mathrm{n}\to\infty}\psi^{-\mathrm{n}}(\mathrm{u}) = \infty$.*

3. Main Result

Theorem 3.1. *Let $(X, \mathring{F}, \Delta)$ be a complete probabilistically convex menger space with $\Delta(a, a) \ge a$ and K be a non-empty closed convex subset of X. Let $\mathrm{A}, \mathrm{B} : \mathrm{K} \to \mathrm{CB}(\mathrm{X})$, and $\mathrm{S}, \mathrm{T} : \mathrm{K} \to \mathrm{K}$ satisfying the following conditions:*

(1) $\partial\mathrm{K} \subseteq \mathrm{SK} \cap \mathrm{K}, \mathrm{AK} \cap \mathrm{K} \subseteq \mathrm{TK}, \mathrm{BK} \cap \mathrm{K} \subseteq \mathrm{SK}$,
(2) $\mathrm{Sx} \in \partial\mathrm{K} \Rightarrow \mathrm{Ax} \subseteq \mathrm{K}, \mathrm{Tx} \in \partial\mathrm{K} \Rightarrow \mathrm{Bx} \subseteq \mathrm{K}$,
(3) (A, S) *and* (B, T) *are compatible mappings,*
(4) $\mathrm{A}, \mathrm{B}, \mathrm{S}, \mathrm{T}$ *are continuous on* K.

$$\mathrm{F}_{\mathrm{Ax},\mathrm{By}}(\phi \mathrm{t}) \ge \min(\mathrm{F}_{\mathrm{Sx},\mathrm{Ty}}(\mathrm{t}), \mathrm{F}_{\mathrm{Sx},\mathrm{Ax}}(\mathrm{t}), \mathrm{F}_{\mathrm{Ty},\mathrm{By}}(\mathrm{t}))$$

then there exists a point z in X such that $\mathrm{Sz} = \mathrm{Tz} \in \mathrm{Az} \cap \mathrm{Gz}$.

Proof. Let $\mathrm{x} \in \partial\mathrm{K}$, since $\partial\mathrm{K} \subseteq \mathrm{SK}$, there exists a point $\mathrm{x}_0 \in \mathrm{K}$ such that $\mathrm{x} = \mathrm{Sx}_0$ that is $\mathrm{Sx}_0 \in \partial\mathrm{K} \Rightarrow \mathrm{Ax}_0 \subseteq \mathrm{K}$ (from 2). Since $\mathrm{Ax}_0 \in \mathrm{AK} \Rightarrow \mathrm{Ax}_0 \subseteq \mathrm{K} \cap \mathrm{AK} \subseteq \mathrm{TK}$.. Let $\mathrm{x}_1 \in \mathrm{K}$ be such that $\mathrm{y}_1 = \mathrm{Tx}_1 \in \mathrm{Ax}_0 \subseteq \mathrm{K}$. Since $\mathrm{y}_1 \in \mathrm{Ax}_0$, there exists a point $\mathrm{y}_2 \in \mathrm{Bx}_1$ such that

$$\mathrm{F}_{\mathrm{y}_1,\mathrm{y}_2}(\mathrm{t}) \ge \mathrm{F}_{\mathrm{Ax}_0,\mathrm{Bx}_1}(\mathrm{t}).$$

Suppose $\mathrm{y}_2 \in \mathrm{K}$, then $\mathrm{y}_2 \in \mathrm{K} \cap \mathrm{BK} \subset \mathrm{SK}$ which implies that there exists a point $\mathrm{x}_2 \in \mathrm{K}$ such that $\mathrm{y}_2 = \mathrm{Sx}_2$. Otherwise if $\mathrm{y}_2 \notin \mathrm{K}$, then there exists a point $\mathrm{u} \in \partial\mathrm{K}$ such that

$$\Delta(\mathrm{F}_{\mathrm{Tx}_1,\mathrm{u}}(\mathrm{t}_1), \mathrm{F}_{\mathrm{u},\mathrm{y}_2}(\mathrm{t})) = \mathrm{F}_{\mathrm{Tx}_1,\mathrm{y}_2}(\mathrm{t}_1 + \mathrm{t}_2), \forall \mathrm{t} > 0.$$

since $\mathrm{u} \in \partial\mathrm{K} \subseteq \mathrm{SK}$, then there exist a point $\mathrm{x}_2 \in \mathrm{K}$ such that $\mathrm{u} = \mathrm{Sx}_2$ and

$$\Delta(\mathrm{F}_{\mathrm{Tx}_1,\mathrm{Sx}_2}(\mathrm{t}_1), \mathrm{F}_{\mathrm{Sx}_2,\mathrm{y}_2}(\mathrm{t}_2) = \mathrm{F}_{\mathrm{Tx}_1,\mathrm{y}_2}(\mathrm{t}_1 + \mathrm{t}_2); \forall \mathrm{t} > 0.$$

Let $y_3 \in Ax_2$ be such that

$$F_{y_2,y_3}(t) \geq F_{Bx_1,Ax_2}(t).$$

Repeating the above argument, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{2n} \in Bx_{2n-1}, y_{2n+1} \in Ax_{2n,}$

(ii) $y_{2n} \in K \Rightarrow y_{2n} = Sx_{2n}$ or $y_{2n} \notin K \Rightarrow Sx_{2n} \in \partial K$ and

$$\Delta(F_{Tx_{2n-1},Sx_{2n}}(t_1), F_{Sx_{2n},y_{2n}}(t_2) = F_{Tx_{2n-1}}, y_{2n}(t_1 + t_2).$$

(iii) $y_{2n+1} \in K, y_{2n+1} = Tx_{2n+1}$ or $y_{2n+1} \notin K$, $Sx_{2n+1} \in \partial K$

$$\Delta(F_{Sx_n,Tx_{2n+1}}(t_1), F_{Tx_{2n+1},y_{2n+1}}(t_2) = F_{Sx_{2n}}, y_{2n+1}(t_1 + t_2)$$

(iv) $F_{y_{2n-1},y_{2n}}(t) \geq F_{Bx_{2n-1},Ax_{2n-1}}(t), \quad F_{y_{2n}}, y_{2n+1}(t) \geq F_{Ax_{2n}}, Bx_{2n-1}(t)$. We denote

$$\begin{aligned}
P_0 &= \{Sx_{2i} \in \{Sx_{2n}\}; Sx_{2i} = y_{2i}\},\\
P_1 &= \{Sx_{2i} \in \{Sx_{2n}\}; Sx_{2i} \neq y_{2i}\},\\
Q_0 &= \{Tx_{2i+1} \in \{Tx_{2n+1}\}; Tx_{2i+1} = y_{2i+1}\},\\
Q_1 &= \{Tx_{2i+1} \in \{Tx_{2n+1}\}; Tx_{2i+1} \neq y_{2i+1}\}
\end{aligned}$$

First we show that $(Sx_{2n}, Tx_{2n+1}) \notin P_1 \times Q_1$ and $(Tx_{2n-1}, Sx_{2n}) \notin Q_1 \times P_1$.

If $Sx_{2n} \in P_1$ then $y_{2n} \neq Sx_{2n}$ and we have $Sx_{2n} \in \partial K$ which implies that $y_{2n+1} \in Ax_{2n} \subseteq K$. Hence $y_{2n+1} = Tx_{2n+1} \in Q_0$. Similarly we have argue that $(Tx_{2n-1}, Sx_{2n}) \notin Q_1 \times P_1$.

Case-1 If $(Sx_{2n}, Tx_{2n+1}) \in P_0 \times Q_0$ then

$$\begin{aligned}
F_{Sx_{2n},Tx_{2n+1}}(\phi t) &= Fy_{2n}, y_{2n+1}(\phi t)\\
&\geq F_{Ax_{2n},Bx_{2n+1}}(\phi t)\\
&\geq \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Ax_{2n}}(t), F_{Tx_{2n-1},Bx_{2n-1}}(t))\\
&\geq \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Tx_{2n+1}}(t), F_{Tx_{2n-1},Sx_{2n}}(t)).\\
&\geq \min(F_{Sx_{2n},Tx_{2n-1}}(t), F_{Sx_{2n},Tx_{2n+1}}(t)).
\end{aligned}$$

Case-2 If $(\mathrm{Sx_{2n}}, \mathrm{Tx_{2n+1}}) \in \mathrm{P_0} \times \mathrm{Q_1}$ then from $\mathrm{(iii)}$, we get

$$\begin{aligned}
\mathrm{F_{Sx_{2n},Tx_{2n+1}}}(\phi \mathrm{t}) &= \mathrm{F_{Sx_{2n},y_{2n+1}}}(2\phi \mathrm{t}) \\
&= \mathrm{Fy_{2n}, y_{2n+1}}(2\phi \mathrm{t}) \\
&\geq \min(\mathrm{F_{Sx_{2n-1},Ty_{2n}}}(\mathrm{t}), \mathrm{F_{Sx_{2n},Tx_{2n+1}}}(\mathrm{t})). \ [from\ case\ 1]
\end{aligned}$$

Similarly , if $(\mathrm{Tx_{2n-1}}, \mathrm{Sx_{2n}}) \in \mathrm{Q_1} \times \mathrm{P_0}$ then we show that

$$\mathrm{F_{Tx_{2n-1},Sx_{2n}}}(\phi \mathrm{t}) \geq \min(\mathrm{F_{Sx_{2n-2},Tx_{2n-1}}}(\mathrm{t}), \mathrm{F_{T_{2n-1},Sx_{2n}}}(\mathrm{t}))$$

Case-3 If $(\mathrm{Sx_{2n}}, \mathrm{Tx_{2n+1}}) \in \mathrm{P_1} \times \mathrm{Q_0}$ then $\mathrm{Tx_{2n-1}} = \mathrm{y_{2n-1}}$. Hence proceeding as in case 1, we have

$$\begin{aligned}
F_{Sx_{2n},Tx_{2n+1}}(2\phi t) = \mathrm{F_{Sx_{2n},y_{2n+1}}}(2\phi \mathrm{t}) &\geq \Delta(\mathrm{F_{Sx_{2n},y_{2n}}}(\phi \mathrm{t}), \mathrm{F_{y_{2n},y_{n+1}}}(\phi \mathrm{t})) \\
&\geq \Delta(\mathrm{F_{Sx_{2n},y_{2n}}}(\phi \mathrm{t}), \mathrm{F_{Ax_{2n},Bx_{2n-1}}}(\phi \mathrm{t})) \\
&\geq \Delta(\mathrm{F_{Sx_{2n},y_{2n}}}(\phi \mathrm{t}), \min(\mathrm{F_{Sx_{2n},Tx_{2n-1}}}(\mathrm{t}), \mathrm{F_{Sx_{2n},Ax_{2n}}}(\mathrm{t}), \\
&\geq \mathrm{F_{Tx_{2n-1},Bx_{2n-1}}}(\mathrm{t})) \\
&\geq \Delta(\mathrm{F_{Sx_{2n},y_{2n}}}(\phi \mathrm{t}), \min(\mathrm{F_{Sx_{2n},Tx_{2n-1}}}(\mathrm{t}), \mathrm{F_{Sx_{2n},Tx_{2n+1}}}(\mathrm{t})) \\
&since\ \Delta(\mathrm{F_{Tx_{2n-1},Sx_{2n}}}(\mathrm{t}), \mathrm{F_{Sx_{2n},y_{2n}}}(\mathrm{t})) = \mathrm{F_{Tx_{2n-1}y_{2n}}}(2\mathrm{t}).
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{F_{Sx_{2n},Tx_{2n+1}}}(2\phi \mathrm{t}) &\geq \Delta(\mathrm{F_{Tx_{2n-1},y_{2n}}}(2\phi \mathrm{t}), \min(\mathrm{F_{Sx_{2n},Tx_{2n-1}}}(\mathrm{t}), \mathrm{F_{Sx_{2n},Tx_{2n+1}}}(\mathrm{t})) \\
&= \Delta(\mathrm{F_{Tx_{2n-1},y_{2n}}}(2\phi \mathrm{t}), \min(\mathrm{F_{Sx_{2n},Tx_{2n-1}}}(\mathrm{t}), \mathrm{F_{Sx_{2n},Tx_{2n+1}}}(\mathrm{t})) \\
&\geq \min(\mathrm{F_{Sx_{2n},Tx_{2n-1}}}(\mathrm{t}), \mathrm{F_{Sx_{2n},Tx_{2n+1}}}(\mathrm{t})).
\end{aligned}$$

$[\text{from case } 1 \text{ and } \Delta(\mathrm{a}, \mathrm{a}) \geq \mathrm{a}]$

Thus in all cases, we put $\mathrm{z_{2n}} = \mathrm{Sx_{2n}}, \mathrm{z_{2n+1}} = \mathrm{Tx_{2n+1}}$, we have

$$\begin{aligned}
\mathrm{F_{z_{2n},z_{2n+1}}}(\phi \mathrm{t}) &\geq \min(\mathrm{F_{z_{2n},z_{2n-1}}}(\mathrm{t}), \mathrm{F_{z_{2n},z_{2n+1}}}(\mathrm{t})), \text{for all } \mathrm{t} > 0 \\
\mathrm{F_{z_{2n},z_{2n+1}}}(\mathrm{t}) &\geq \min(\mathrm{F_{z_{2n},z_{2n-1}}}(\phi^{-1}\mathrm{t}), \mathrm{F_{z_{2n},z_{2n+1}}}(\phi^{-1}\mathrm{t})), \\
&\geq \min(\mathrm{F_{z_{2n},z_{2n-1}}}(\phi^{-1}\mathrm{t}), \mathrm{F_{z_{2n},z_{2n-1}}}(\phi^{-2}\mathrm{t}), \mathrm{F_{z_{2n},z_{2n+1}}}(\phi^{-2}\mathrm{t})) \\
&= \min(\mathrm{F_{z_{2n},z_{2n-1}}}(\phi^{-1}\mathrm{t}), \mathrm{F_{z_{2n},z_{2n+1}}}(\phi^{-2}\mathrm{t}))
\end{aligned}$$

since $\mathrm{x} < \phi^{-1}(\mathrm{x}) < \phi^{-2}(\mathrm{x}) \ldots$

By repeated applications of above inequality, we obtain

$$F_{z_{2n},z_{2n+1}}(t) \geq \min(F_{z_{2n},z_{2n-1}}(\phi^{-1}t), F_{z_{2n},z_{2n+1}}(\phi^{-i}t)).$$

Since $F_{z_{2n},z_{2n+1}}(\phi^{-i}t) \to 1$ as $i \to \infty$ $\forall t > 0$ it follows that

$$F_{z_{2n},z_{2n+1}}(t) \geq F_{z_{2n},z_{2n-1}}(\phi^{-1}t), \quad \forall n \in N \text{ and } \quad \forall t > 0.$$

Therefore,

$$F_{z_{2n},z_{2n+1}}(t) \geq F_{z_{2n},z_{2n-1}}(\phi^{-1}t), \geq F_{z_{2n-1},z_{2n-2}}(\phi^{-2}t) \geq \cdots \geq F_{z_{2n_0},z_{2n_1}}(\phi^{-n}t),$$

taking limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} F_{z_{2n},z_{2n+1}}(t) = 1$$

Now $Fz_{2n}, z_{2n+p}(t) \geq \Delta(Fz_{2n}, z_{2n+p}(t/p), \ldots, F_{z_{2n+p-1},z_{2n+p}}(t/p))$. Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} F_{z_{2n},z_{2n+p}}(t) = 1.$$

This implies that z_n is a cauchy and hence converges to a point z consequently, the subsequences

$$\begin{aligned} \{z_{2n}\} &= \{Sx_{2n}\} \to z \\ \{z_{2n+1}\} &= \{Tx_{2n+1}\} \to z. \end{aligned}$$

Since (B, T) is compatible mappings

$$\begin{aligned} \lim_{n\to\infty} F_{Bx_{2n-1},Tx_{2n-1}}(t) &= 1 \\ \Rightarrow \lim_{n\to\infty} F_{TSx_{2n},BTx_{2n-1}}(t) &= 1. \end{aligned}$$

By the continuity of B and T then $F_{Tz,Bz}(t) = 1$, i.e.,

$$Tz \in Bz. \tag{3.1}$$

Similarly, the continuity and compatibility of (A, S) lead to

$$Sz \in Az. \tag{3.2}$$

Again

$$\begin{aligned} F_{Sz,Tz}(\phi t) &\geq F_{Az,Bz}(\phi t) \\ &\geq \min(F_{Sz,Tz}(t), F_{Sz,Az}(t), F_{Tz,Bz}(t)) \\ &\geq \min(F_{Sz,Tz}(t), F_{Sz,Tz}(t), F_{Tz,Sz}(t)) \\ \Rightarrow \quad F_{Sz,Tz}(\phi t) &\geq F_{Sz,Tz}(\phi t) \end{aligned}$$

which implies that

$$\text{Sz} = \text{Tz}. \tag{3.3}$$

From (3.1), (3.2) and (3.3)

$$\text{Sz} = \text{Tz} \in \text{Az} \cap \text{Bz}.$$

This complete the proof. □

Similarly, we can prove the following theorem.

Theorem 3.2. *Let* (X, F, Δ) *be a complete probabilistically convex menger space and* K *be a non-empty closed convex subset of* X. *Let* $A, B : K \to C(X)$ *and* $S, T : k \to k$ *satisfying the conditions.*

(1) $\partial K \subseteq SK \cap K, AK \cap K \subseteq TK, BK \cap K \subseteq SK$,
(2) $Sx \in \partial K \Rightarrow Ax \subseteq K, Tx \in \partial K \Rightarrow Bx \subseteq K$,
(3) (A, S) *and* (B, T) *are compatible mappings,*
(4) A, B, S, T *are continuous on* K,
(5) $F_{Ax,By}(\phi t) \geq \min\{F_{Sx,Ty}(t), F_{Sx,Ax}(t), F_{Ty,By}(t)\}$, *then there exists a point* z *in* X *such that*

$$\text{Sz} = \text{Tz} \in \text{Fz} \cap \text{Gz}.$$

Remark 3.1. *If we take* $S = T = 1$ *(identity function) and* $A = B$ *and complete menger space in theorem* 3.1 *one deduces a result due to Lee, [9].*

4. Application

Here, we study the existence of fixed point for multi-valued mappings in a metric space (X, d) using the results in the previous section.

Theorem 4.1. *Let* (X, d) *be a convex complete metric space and* $A, B : (K, d) \to (CB(X), d_H)$, $S, T : (K, d) \to (K, d)$ *satisfying the conditions.*

(1) $\partial K \subseteq SK \cap K, AK \cap K \subseteq TK, BK \cap K \subseteq SK$,
(2) $Sx \in \partial K \Rightarrow Ax \subseteq K, Tx \in \partial K \Rightarrow Bx \subseteq K$,
(3) (A, S) *and* (B, T) *are compatible mappings,*
(4) A, B, S, T *are continuous on* K,
(5) $d_H(Ax, By) \leq \phi \max\{d(Sx, Ty), d_H(Sx, Ax), d_H(Ty, By)\}$, *then there exists a point* z *in* X *such that*

$$Sz = Tz \in Az \cap Gz.$$

Proof. If we define $F : X \times X \to D^+$ such that

$$F_{A,B}(t) = H(t - d_H(A, B)), \forall A, B \in CB(X)$$

then the space $(X, F, \min)$ with $t-$ norm $\Delta = \min$ is a probabilistically convex $\tau-$ complete menger space and topology induced by the metric d coincided with the topology τ. For any $A, B \in CB(X)$, we have

$$\begin{aligned}
F_{Ax,By}(\phi t) &= H(\phi t \;\; d_H(Ax, By)) \\
&\geq H[\phi t - \max\{d(Sx, Ty), d_x(Sx, Ax), d_H(Ty, By)\}] \\
&= H[\min\{(t - d(Ax, By)), (t - d_H(Sx, Ax)), (t - d_H(Ty, By))\}] \\
&= \min[\{H(t - d(Ax, By)), H(t - d_H(Sx, Ax)), H(t - d_H(Ty, By))\}] \\
&= \min[F_{Ax,By}(t), F_{Sx,Ay}(t), F_{Tx,By}(t)].
\end{aligned}$$

Thus the Theorem 4.1 follows from theorem 3.1 immediately. Hence there exist a point $z \in X$ such that $Sz = Tz \in Az \cap Bz$. □

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