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SOME COMMON FIXED POINT THEOREMS FOR EXPANSIVE MAPPINGS IN d-COMPLETE TOPOLOGICAL SPACES

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ABSTRACT. In the present paper, we entrenched common fixed point theorems for self mappings satisfying expansive condition in d-complete topological spaces. Also we prove a fixed point theorem for (ζ, α)-expansive mapping in the setting of d-complete topological spaces. Our results extend and generalize the results of Shahi et al. to d-complete topological spaces.

1. INTRODUCTION

Fixed point theory" is a fundamental theory in mathematics. It is an important tool in non-linear functional analysis. It has been widely applied to many branches in pure mathematics and applied mathematics. Fixed point theory is a beautiful mixture of analysis, topology and geometry. Wang et al. in [10] in 1984 introduced some results on "expansion mappings" in metric spaces. Further results of in [10] was generalized by Khan et al. and in [6] by using functions. After that several researchers in [9] introduced the results on expansion mappings. Shahi et al. in [8] proposed the results on "expansive mappings" in "generalized metric spaces". Also Shahi et al. in [7] proposed a new concept of (ξ, α)-expansive mappings and gave fixed point theorems (FPT) for this mapping in complete metric spaces.

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In 1975, "d-complete topological space (TS)" was proposed by Kashahara, [4, 5], as a generalization of "complete metric spaces". Hicks in [1], Hicks and Rhoades in [2] and many other authors in [3] introduced several results in d-complete topological spaces. Examples of d-complete TS are complete quasi metric spaces and complete metric spaces.

In this paper, we prove common fixed point theorems for expansion mapping and a fixed point theorem for (ζ, α)-expansive mapping in the setting of d-complete TS. Our results extend and generalize the results of Shahi et al., [7], [8] to d-complete topological spaces.

2. Preliminaries

Before going to our main results, we state some definitions.

Definition 2.1. *[1], Let, for a TS (U, τ), $d : U \times U \to [0, \infty)$ be such that $d(s, t) = 0$ iff $s = t$. If $\sum_{n=0}^{\infty} d(s_n, s_{n+1}) < \infty$ then the sequence $< s_n >$ is forenamed to be d-cauchy sequence.*

Definition 2.2. *[1], Let, for a TS (U, τ), $d : U \times U \to [0, \infty)$ be such that $d(s, t) = 0$ iff $s = t$. If $\sum_{n=0}^{\infty} d(s_n, s_{n+1}) < \infty \implies$ the sequence $< s_n >$ is convergent in (U, τ) then the triplet (U, τ, d) is forenamed to be d-complete TS.*

Definition 2.3. *[9], Two self mappings T and S on a topological space (U, τ) are forenamed to be weakly compatible if $Tu = Su$, for $u \in U \implies STu = TSu$.*

Two compatible mappings are weakly compatible but converse not true.

Definition 2.4. *[9], Let for a TS (U, τ), $d : U \times U \to [0, \infty)$ be a mapping such that $d(s, t) = 0$ if and only if $s = t$ and $T : U \to U$ be a self mapping. The mapping T is forenamed to be expansive if $\exists\ h > 1$ such that*

$$d(Ts, Tt) \geq hd(s, t),$$

$\forall\ s, t \in U$.

Here we are using the notation Γ denoting the class of all non-decreasing functions $\zeta : [0, \infty) \to [0, \infty)$ which satisfies the subsequent properties:

(i) $\sum_{n=1}^{\infty} \zeta^n(a) < \infty$ for every $a > 0$, where ζ^n represents the n^{th} iterate of ζ;

(ii) $\zeta(a) + \zeta(b) = \zeta(a+b)$.

Definition 2.5. *[7], Let for a TS (U, τ), $d : U \times U \to [0, \infty)$ be a mapping such that $d(s,t) = 0$ iff $s = t$ and $T : U \to U$ be a self mapping. Then T is an (ζ, α)-expansive mapping if there exist two functions $\zeta \in \Gamma$ and $\alpha : U \times U \to [0, \infty)$ such that*

$$\zeta(d(Ts, Tt)) \geq \alpha(s,t)d(s,t),$$

for all $s, t \in U$.

Definition 2.6. *[7], Let $T : U \to U$ and $\alpha : U \times U \to [0, \infty)$ be two mappings for a non-void set U. The mapping T is forenamed to be α-admissible if $\alpha \geq 1 \implies \alpha(Ts, Tt) \geq 1$, for $s, t \in U$.*

Example 1. *[7], Let $U = R^+ \cup \{0\}$. Outline the mapping $\alpha : U \times U \to [0, \infty)$ by*

$$\alpha(s,t) = \begin{cases} 1 & \text{if } s \geq t, \\ 0 & \text{if } s < t \end{cases}$$

and the mapping $T : U \to U$ by $Ts = s^2 \;\; \forall\, s \in U$. Then T is α-admissible.

3. Main results

This section includes our main results. Firstly, we find common FPT in d-complete TS for expansion mappings.

Theorem 3.1. *Let C, D, P and Q be self mappings on d-complete TS U satisfying the conditions:*

(i) $\varphi(d(Cs, Dt)) \geq d(Ps, Qt) \; \forall\, s, t \in U$,
(ii) *C and D are surjective,*
(iii) *the pairs $\{C, P\}$ and $\{D, Q\}$ are weakly compatible mappings,*
(iv) $d(s,t) = d(t,s) \; \forall\, s, t \in U$

where $\varphi : [0, \infty) \to [0, \infty)$ is a "non-decreasing function" with $\varphi(t) < t, \varphi(0) = 0$ and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty \; \forall\, t > 0$.
Then, C, D, P and Q have exactly one common fixed point (CFP) in U.

Proof. Due to surjective mappings of C and D, one can take a point s_1 in U for an arbitrary point s_0 in U such that

$$Cs_1 = Qs_0 = t_0.$$

For a point $s_1 \in U$, $\exists$ a point s_2 in U such that

$$Ds_2 = Ps_1 = t_1.$$

Inductively, a sequence $< t_n >$ in U can be defined such that

$$Cs_{2n+1} = Qs_{2n} = t_{2n},$$

$$Ds_{2n+2} = Ps_{2n+1} = t_{2n+1},$$

$\forall\, n \in N \cup \{0\}$, where N is the set of natural numbers .

Firstly, we will determine that the sequence $< t_n >$ constructed above is a "d-cauchy sequence".

Suppose $s_n \neq s_{n+1}$ $\forall\, n$. By use of (i), we have

$$\begin{aligned} \varphi(d(t_{2n}, t_{2n+1})) &= \varphi(d(Cs_{2n+1}, Ds_{2n+2})) \\ &\geq d(Ps_{2n+1}, Qs_{2n+2}) \\ &= (d(t_{2n+1}, t_{2n+2}) \end{aligned}$$

that is,

$$(d(t_{2n+1}, t_{2n+2}) \leq \varphi(d(t_{2n}, t_{2n+1})).$$

Similarly,

$$\begin{aligned} (d(t_{2n}, t_{2n+1}) &= (dQs_{2n}, Ps_{2n+1}) \\ &= (d(Ps_{2n+1}, Qs_{2n}) \\ &\leq \varphi(d(Cs_{2n+1}, Ds_{2n})) \\ &= \varphi(d(t_{2n}, t_{2n-1})) \\ &= \varphi(d(t_{2n-1}, t_{2n})) \end{aligned}$$

Thus, in general,

$$d(t_n, t_{n+1}) \leq \varphi^n(d(t_0, t_1)).$$

So,

$$\begin{aligned} P_n = \sum_{i=1}^{n} d(t_i, t_{i+1}) &= d(t_0, t_1) + d(t_1, t_2) + ... + d(t_n, t_{n+1}) \\ &\leq d(t_0, t_1) + \varphi(d(t_0, t_1)) + ... + \varphi^n(d(t_0, t_1)) \\ &= \sum_{i=0}^{n} \varphi^i(d(t_0, t_1)) < \sum_{i=0}^{\infty} \varphi^i(d(t_0, t_1)) < \infty. \end{aligned}$$

Thus $< t_n >$ is a d-cauchy sequence.

Since (U, τ) is a d-complete TS, it yields that $< t_n >$ and hence any subsequence of it converge to $\alpha \in U$.

So $< Cs_{2n} >, < Ds_{2n+1} >, < Ps_{2n+1} >$ and $< Qs_{2n+1} >$ converge to $\alpha \in U$. Since (U, τ) is d-complete TS, so we can choose a point $p \in U$ such that $Cp = \alpha$.

Now by use of inequality (i), we have:

$$\varphi(d(Cp, Ds_{2n+2}) \geq d(Pp, Qs_{2n+2}).$$

Letting $n \to \infty$, we have

$$\varphi(d(Cp, \alpha)) \geq d(Pp, \alpha).$$

That is,

$$0 = \varphi(0) \geq d(Sp, \alpha).$$

This determines that $Pp = \alpha$. By reason of mappings D and Q weakly compatible, $CPp = PCp$ implies $C\alpha = P\alpha$.

Now, since $D(U)$ is also a d-complete TS, so one can choose a point $p_1 \in U$ such that $Dp_1 = \alpha$.

Now consider,

$$\varphi(d(Cs_{2n+1}, Dp_1)) \geq d(Ps_{2n+1}, Qp_1).$$

Letting $n \to \infty$, we get $\varphi(d(\alpha, Dp_1)) \geq d(\alpha, Qp_1)$.

Recalling that $Dp_1 = \alpha$, we obtain $0 = \varphi(0) \geq d(\alpha, Qp_1)$.

This suggest that $Qp = \alpha$. By reason of mappings D and Q weakly compatible, $DQp_1 = QDp_1$ implies that $D\alpha = Q\alpha$.

Now, we have to prove that $C\alpha = \alpha$ and $D\alpha = \alpha$.

Let us presume that $d(C\alpha, \alpha) > 0$. So applying the inequality $\varphi(t) < t \; \forall \; t > 0$, we have

$$d(C\alpha, \alpha) > \varphi(d(C\alpha, \alpha)) \geq d(P\alpha, \alpha) = d(C\alpha, \alpha)$$

which is conflict. So $d(C\alpha, \alpha) = 0$ i.e. $C\alpha = S\alpha = \alpha$.

Similarly, let us presume that $d(D\alpha, \alpha) > 0$. Accordingly, using the fact $d(s,t)$ is symmetric and $C\alpha = P\alpha = \alpha$, we get

$$d(D\alpha, \alpha) > \varphi(d(D\alpha, \alpha)) = \varphi(d(C\alpha, D\alpha)) \geq d(P\alpha, Q\alpha) = d(C\alpha, D\alpha) = d(D\alpha, \alpha)$$

which is conflict. So,

$$d(D\alpha, \alpha) = 0.$$

This proves that

$$D\alpha = Q\alpha = \alpha.$$

Therefore,

$$C\alpha = D\alpha = P\alpha = Q\alpha = \alpha.$$

It follows that α is the exactly one CFP of C, P, D and Q. □

By taking $\varphi(t) = \frac{t}{h}$, for $h > 1$ in theorem 3.1, we get as per mentioned below Corollary:

Corollary 3.1. *Let C, D, P and Q be self mappings on d-complete TS U satisfying the conditions:*

(i) $d(Cs, Dt) \geq hd(Ps, Qt)\ \forall\ s, t \in U$ *and* $h > 1$*;*
(ii) C *and* D *are surjective;*
(iii) *the pairs* $\{D, Q\}$ *and* $\{C, P\}$ *are "weakly compatible" mappings;*
(iv) $d(s, t) = d(t, s)\ \forall\ s, t \in U$.

Then, C, D, P*and* Q *have exactly one CFP in* U.

If we take $P = Q = I$ in theorem 3.1, we have as per mentioned below Corollary:

Corollary 3.2. *Let C, D be surjective self mappings on d-complete TS U satisfying the inequality*

$$\varphi(d(Cs, Dt)) \geq d(s, t),$$

$\forall\ s, t \in U$*, where* $\varphi : [0, \infty) \to [0, \infty)$*, a "non-decreasing function" satisfying* $\varphi(t) < t, \varphi(0) = 0$ *and* $\sum_1^\infty \varphi^n(t) < \infty\ \forall\ t > 0$.
Then C and D have exactly one CFP.

If we put $\varphi(t) = \frac{t}{h}, h > 1$ and $P = Q = I$ in Theorem 3.1, we obtain the subsequent result:

Corollary 3.3. *Let C and D be surjective self mappings on a d-complete TS U. Assuming that $\exists\ h > 1$ to the extent that*

$$d(Cs, Dt)) \geq hd(s, t)$$

for all $s, t \in U$. Then, C and D have exactly one CFP.

Theorem 3.2. *Let C and D be surjective self mappings on d-complete TS U satisfying the inequalities*

$$\varphi(d(CDs, Ds)) \geq d(Ds, s), \tag{3.1}$$

$$\varphi(d(DCs, Cs)) \geq d(Cs, s) \tag{3.2}$$

$\forall\ s \in U$, where $\varphi : [0,\infty] \to [0,\infty]$, a "non-decreasing continuous function" with

$$\sum_{n=1}^{\infty} \varphi^n(t) < \infty,$$

then C and D have a CFP.

Proof. Because of the surjective mappings C and D, for an arbitrary point $s_0 \in U$, one can choose points $s_1 \in f^{-1}(s_0)$ and $s_2 \in g^{-1}(s_1)$. Pursuing like this, we attain the sequence $< s_n >$ with $s_{2n+1} \in C^{-1}(s_{2n})$ and $s_{2n+2} \in D^{-1}(s_{2n+1})$.

If $s_n = s_{n+1}$ for some n, then s_n is a FP of C and D. And if $s_{2n} = s_{2n+1}$ for some $n \geq 0$, then s_{2n} is a FP of C. And from inequality (i), we get

$$\begin{aligned}
\varphi(d(s_{2n}, s_{2n+1})) &= \varphi(d(Cs_{2n+1}, Ds_{2n+2})) && (3.3)\\
&= \varphi(d(CDs_{2n+2}, Ds_{2n+2})) && (3.4)\\
&\geq d(Ds_{2n+2}, s_{2n+2}) && (3.5)\\
&= d(s_{2n+1}, s_{2n+2}) && (3.6)
\end{aligned}$$

that is,

$$d(s_{2n+1}, s_{2n+2}) \leq \varphi(d(s_{2n}, s_{2n+1})).$$

Similarly,

$$\begin{aligned}
\varphi(d(s_{2n+1}, s_{2n+2})) &= \varphi(d(Ds_{2n+2}, Cs_{2n+3}))\\
&= \varphi(d(DCs_{2n+3}, Cs_{2n+3}))\\
&\geq d(Cs_{2n+3},\ s_{2n+3})\\
&= d(s_{2n+2}, s_{2n+3})
\end{aligned}$$

that is,

$$d(s_{2n+2},\ s_{2n+3}) \leq \varphi(d(s_{2n+1},\ s_{2n+2})).$$

Thus, we obtain

$$d(s_n, s_{n+1}) \leq \varphi(d(s_{n-1}, s_n)),$$

for $n = 1, 2, 3, \ldots$. Thus,

$$d(s_n, s_{n+1}) \leq \varphi^n(d(s_0, s_1)),$$

for $n = 1, 2, 3, \ldots$. So,

$$\begin{aligned} P_n &= \sum_{i=1}^{n} d(s_i, s_{i+1}) \\ &= d(s_0, s_1) + d(s_1, s_2) + ...d(s_n, s_{n+1}) \\ &\leq d(s_0, s_1) + \varphi(d(s_0, s_1)) + ... + \varphi^n(d(s_0, s_1)) \\ &= \sum_{i=0}^{n} \varphi^n(d(s_i, s_{i+1})). \end{aligned}$$

Since

$$\sum_{n=0}^{\infty} \varphi^n(t) < \infty$$

$\forall\ t > 0$, therefore

$$P_n \leq \sum_{i=0}^{\infty} \varphi^k(d(s_i, s_{i+1})) < \infty.$$

In this way, $< s_n >$ is a "d-cauchy sequence".

After all, (U, τ, d) is "d-complete", it gives that $< s_n >$ and hence any subsequence of it converge to $\alpha \in U$.

Now, we suppose that C is continuous. As $s_{2n} = Cs_{2n+1}$, it pursues that

$$\alpha = \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} Cs_{2n+1} = C\alpha.$$

Thus α is a FP of C. Because of the surjective mapping D, one can choose a point $t \in U$ such that $Dt = \alpha$. Thus, from inequality (3.1) ,we have

$$0 = \varphi(0) = \varphi(d(\alpha, \alpha)) = \varphi(d(C\alpha, D\alpha)) = \varphi(d(CDt, Dt)) \geq d(Dt, t) = d(\alpha, t),$$

which implies $d(\alpha, t) = 0$ and so $t = \alpha$. Thus, $D\alpha = \alpha$. Therefore, we have proven that α is a CFP of C and D. Similarly, it can be seen by considering the continuity of D that C and D have a CFP.

This finalizes the proof. □

Taking $\varphi(t) = \frac{t}{h}$, where $h = \max\{a, b\} > 1$ in Theorem 3.2, we get as per mentioned below Corollary:

Corollary 3.4. *Let C and D be surjective self mappings on d-complete TS U satisfying the inequalities*

$$d(CDs, Ds) \geq ad(Ds, s)$$

$$d(DCs, Cs) \geq bd(Cs, s)$$

for all $s \in U$, where $a, b > 1$. If either C or D is continuous, then C and D have a CFP.

Putting $C = D$ in Theorem 3.2, we get the subsequent Corollary:

Corollary 3.5. *Let C be a continuous surjective self mapping on d-complete TS U satisfying the inequality*

$$\varphi(d(C^2 s, Cs)) \geq d(Cs, s),$$

$\forall\ s \in U$, where $\varphi : [0, \infty] \to [0, \infty]$, a non-decreasing function with the position

$$\sum_{n=1}^{\infty} \varphi^n(t) < \infty$$

$\forall\ t > 0$. Then C has a "fixed point".

Set $k = \min\{a, b\}$ and $C = D$ in Corollary 3.4, we get the subsequent Corollary:

Corollary 3.6. *Let C be a surjective self mapping on d-complete TS U satisfying the inequality*

$$d(C^2 s, Cs) \geq k d(Cs, s),$$

$\forall\ s \in U$, where $k > 1$. Then continuity of $C \implies C$ has a FP.

Now, we prove fixed point theorem by using (ζ, α)-expansive mapping in the setting of d-complete TS.

Theorem 3.3. *Let C be a continuous bijective, (ζ, α)-expansive mapping on a d-complete TS U gratifying the situations:*

(i) *C^{-1} is an "α-admissible" mapping;*
(ii) *$\exists\ s_0 \in U$ s.t. $\alpha(s_0, C^{-1}s_0) \geq 1$.*

Then C has a FP.

Proof. Outline a sequence $< s_n >$ in U as

$$s_n = Cs_{n+1}$$

$\forall\ n \in N$. Here s_0 is such that $\alpha(s_0, Q^{-1}s_0) \geq 1$. If $s_n = s_{n+1}$ for any $n \in N$, then clearly s_n is a FP of C. Suppose that $s_n \neq s_{n+1}$ for $n \in N$. From given condition (ii),

$$\alpha(s_0, s_1) = \alpha(s_0, C^{-1}s_0) \geq 1,$$

therefore

$$\alpha(C^{-1}s_0, C^{-1}s_1) = \alpha(s_1, s_2) \geq 1,$$

because C^{-1} is α-admissible mapping. Thus by induction,

$$\alpha(s_n, s_{n+1}) \geq 1, \forall n \in N.$$

Using inequality (3.3) and the condition of (ζ, α)-expansive mapping, we have

$$d(s_n, s_{n+1}) \leq \alpha(s_n, s_{n+1})d(s_n, s_{n+1}) \leq \zeta(d(Cs_n, Cs_{n+1})) = \zeta(d(s_{n-1}, s_n)).$$

Thus, by repeating the inequality (3.4), we have

$$d(s_n, s_{n+1}) \leq \zeta^n(d(s_0, s_1)),$$

for all $n \in N$.

$$\begin{aligned} P_n &= d(s_1, s_2) + d(s_2, s_3) + ... + d(s_n, s_{n+1}) \\ &= \zeta(d(s_0, s_1)) + \zeta^2(d(s_0, s_1)) + ... + \zeta^n(d(s_0, s_1)) \\ &= \sum_{i=1}^{n} \zeta^i(d(s_0, s_1)). \end{aligned}$$

When $n \to \infty$,

$$\sum_{i=1}^{\infty} \zeta^i(d(s_0, s_1)) < \infty$$

since

$$\sum_{n=1}^{\infty} \zeta^n(a) < \infty$$

for each $a > 0$.

Hence, $< s_n >$ is a d-cauchy sequence. So, there exists $\alpha \in U$ such that $s_n \to \alpha$ as $n \to \infty$. Using the continuity of C, $s_n = Cs_{n+1} \to C\alpha$ as $n \to \infty$. Using the uniqueness of limit, we get $\alpha = C\alpha$. Thus α is a FP of C.

This finalizes the proof. $\square$

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