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COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH HORADAM POLYNOMIAL

D. KAVITHA[1], K. DHANALAKSHMI, AND N. ARULMOZHI

ABSTRACT. In this present article, we studied and examined the novel general subclasses of the function class Σ of bi-univalent function defined in the open unit disk, which are associated with the Horadam polynomial. This study locates estimates on the Taylor - Maclaurin coefficients $|a_2|$ *and* $|a_3|$ in functions of the class which are considered. Additionally, Fekete-Szegö inequality of functions belonging to this subclasses are also obtained.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the family of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}) \tag{1.1}$$

which are analytic in the open unit open disk $\mathbb{U} = \{z \ : \ z \in \mathbb{C}, \ |z| < 1\}$. Let S be class of all functions in $\mathcal{A}$ which are univalent and normalized by the conditions $f(0) = 0 = f'(0) - 1$ in $\mathbb{U}$. Two of the most famous subclasses of univalent functions class $\mathcal{S}$ are the class $\mathcal{S}^*(\alpha)(0 \leq \alpha < 1)$ of starlike functions of order α and the class $\mathcal{C}(\alpha)(0 \leq \alpha < 1)$ of convex functions of order α. For two functions $f(z)$ and $g(z)$,

[1]*corresponding author*

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are analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written as $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ defined on $\mathbb{U}$ with

$$w(0) = 0 \qquad and \qquad |w(z)| < 1, \qquad (z \in \mathbb{U}),$$

such that $f(z) = g(w(z))$ for all $(z \in \mathbb{U})$. Also, it is known that

$$f(z) \prec g(z) \Rightarrow f(0) = g(0) \qquad and \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

The well-known Koebe one-quarter theorem [7] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius 1/4. Hence every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z, (z \in \mathbb{U})$ and

$$f^{-1}(f(w)) = w, (|w| < r_0(f), r_0(f) \geq 1/4)\,,$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Earlier, Lewin [12] investigated the bi-univalent functions and derived that $|a_2| < 1.51$. For the brief history of functions in the class Σ, Brannan and Clunie [5], and Srivastava et al. [14] proved some results within these coefficient for different classes. Moreover, Brannan and Taha [6] introduced certain subclasses of the bi-univalent function class Σ for the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$. More Recent studies inspired by Horcum and Kocer [10], Abirami et al. [1], Alamoush [2], [3], [4] considered Horadam polynomials $h_n(x)$, which are given by the following recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x)\ ,\ (n \in \mathbb{N} \geq 2), \tag{1.3}$$

with $h_1 = a, h_2 = bx$, and $h_3 = pbx^2 + aq$ where (a, b, p, q are some real constants). By taking various values of a,b,p and q which leads to various polynomials

- when a = b = p = q = 1, we obtain the Fibonacci polynomials,

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), F_1(x) = 1, F_2(x) = x;$$

- when a = 2, b = p = q = 1, we have the Lucas polynomials,

$$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x), L_0 = 2, L_1 = x;$$

- when a = q = 1, b = p = 2, we attain the Pell polynomials,

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), p_1 = 1, p_2 = 2x;$$

- when a = b = p = 2, q = 1, we get the Pell-Lucas polynomials,

$$Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x), Q_0 = 2, Q_1 = 2x;$$

- when a = 1, b = p = 2, q = 1, we obtain the Chebyshev polynomials of second kind sequence,

$$U_{n-1}(x) = 2xU_{n-2}(x) + U_{n-3}(x), U_0 = 1, U_1 = 2x;$$

- if a = 1, b = p = 2, q = 1, we have the Chebyshev polynomials of First kind sequence,

$$T_{n-1}(x) = 2xT_{n-2}(x) + T_{n-3}(x), T_0 = 1, T_1 = x.$$

One can refer [8], [9], [11] and [13] for more details connected with these polynomials succession.

The characteristic equation of recurrence relation (1.3) is

$$t^2 - pxt - q = 0.$$

This equation has two real roots,

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2},$$

and

$$\beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Remark 1.1. *[10] Let Ω(x,z) be the generating function of the Horadam polynomials $h_n(x)$. Then*

$$\Omega(x,z) = \frac{a + (b - ap)xz}{1 - pxz - qz^2} = \sum_{n=1}^{\infty} h_n(x) z^{n-1}.$$

In this present work, we introduce $\mathcal{S}_\Sigma(\lambda, \gamma, x)$ and $\mathcal{M}_\Sigma(\lambda, \gamma, x)$ are the class of bi-univalent functions. Within this, coefficient estimates $|a_2|$ and $|a_3|$. The Fekete-Szegö problem are also derived for the function $f \in \Sigma$ belonging to the new defined subclasses.

2. Set of Main Results

We now define the new bi-univalent subclasses of analytic function.

Definition 2.1. *For* $0 \leq \lambda \leq 1$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$,*a function* $f \in \Sigma$ *given by* (1.1) *is said to be in the class* $\mathcal{S}_{\Sigma}(\lambda, \gamma, x)$*, if the following conditions are satisfied:*

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\right) \prec \Omega(x,\ z) + 1 - a$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} - 1\right) \prec \Omega(x,\ w) + 1 - a, \tag{2.1}$$

where $g = f^{-1}$ *is given by* (1.2) *and* $z, w \in \mathbb{U}$.

Definition 2.2. *For* $0 \leq \lambda \leq 1$ *and* $\gamma \in \mathbb{C} \setminus \{0\}$,*a function* $f \in \Sigma$ *given by (1.1) is said to be in the class* $\mathcal{M}_{\Sigma}(\lambda, \gamma, x)$*, if the following conditions are satisfied:*

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) \prec \Omega(x,\ z) + 1 - a$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + w^2 g''(w)}{(1-\lambda)w + \lambda w g'(w)} - 1\right) \prec \Omega(x,\ w) + 1 - a,$$

where $g = f^{-1}$ *is given by* (1.2) *and* $z, w \in \mathbb{U}$.

Theorem 2.1. *Let the function* $f \in \Sigma$ *be given by* (1.1) *be in the class* $\mathcal{S}_{\Sigma}(\lambda, \gamma, x)$. *Then*

$$|a_2| \leq \frac{|\gamma||bx|\sqrt{|bx|}}{\sqrt{|bx^2[(\lambda^2 - 3\lambda + 3)\gamma b - (2-\lambda)^2 p] - aq(2-\lambda)^2|}}, \tag{2.2}$$

$$|a_3| \leq \frac{|\gamma||bx|}{(3-\lambda)} + \frac{\gamma^2 (bx)^2}{(2-\lambda)^2}, \tag{2.3}$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{|\gamma||bx|}{(3-\lambda)} & \text{if} \quad |\eta - 1| \leq \sigma_1 \\ \dfrac{\gamma^2 |bx|^3 (\eta - 1)}{|[(\lambda^2 - 3\lambda + 3)\gamma b - p(2-\lambda)^2] bx^2 - qa(2-\lambda)^2|} & \text{if} \quad |\eta - 1| \geq \sigma_1. \end{cases}$$

Here,

$$\sigma_1 = \frac{|[(\lambda^2 - 3\lambda + 3)\gamma b - p(2-\lambda)^2] bx^2 - qa(2-\lambda)^2|}{(bx)^2 (3-\lambda)}.$$

Proof. Let $f \in \Sigma$ be given by the Taylor-Maclaurin expansion (1.1). Then, for some analytic functions Ψ and Φ such that $\Phi(0) = \psi(0) = 0$, $|\psi(z)| < 1$ and $|\phi(w)| < 1$ $z, w \in \mathbb{U}$ and using Definition2.1, we can write

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\right) = \Omega(x,\ \Phi(z)) + 1 - a \tag{2.4}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} - 1\right) = \Omega(x,\ \psi(w)) + 1 - a. \tag{2.5}$$

Equivalently,

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\right) = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \ldots$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} - 1\right) = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^2 + \ldots.$$

From (2.4) and (2.5), we obtain

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\right) = 1 + h_2(x)p_1 z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \ldots \tag{2.6}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} - 1\right) = 1 + h_2(x)q_1 w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \ldots. \tag{2.7}$$

Notice that if

$$|\Phi(z)| = |p_1 z + p_2 z^2 + p_3 z^2 + \ldots| < 1 \qquad (z \in \mathbb{U})$$

and

$$|\psi(w)| = |q_1 w + q_2 w^2 + q_3 w^3 + \ldots| < 1 \qquad (w \in \mathbb{U})\ ,$$

then

$$|p_i| \leq 1 \qquad \text{and} \qquad |q_i| \leq 1 \quad (i \in \mathbb{N}).$$

Thus, upon comparing the corresponding coefficients in (2.6) and (2.7), we have

$$\frac{(2-\lambda)}{\gamma} a_2 = h_2(x)p_1, \tag{2.8}$$

$$\frac{(3-\lambda)a_3 - \lambda(2-\lambda)a_2^2}{\gamma} = h_2(x)p_2 + h_3(x)p_1^2, \tag{2.9}$$

$$\frac{-(2-\lambda)}{\gamma}a_2 = h_2(x)q_1, \tag{2.10}$$

and

$$\frac{(3-\lambda)(2a_2^2-a_3)-\lambda(2-\lambda)a_2^2}{\gamma} = h_2(x)q_2 + h_3(x)q_1^2. \tag{2.11}$$

From (2.8) and (2.10), we find that

$$p_1 = -q_1 \tag{2.12}$$

and

$$a_2^2 = \frac{\gamma^2 h_2^2(x)(p_1^2+q_1^2)}{2(2-\lambda)^2}. \tag{2.13}$$

Adding (2.9) and (2.11), we obtain

$$\frac{2(\lambda^2-3\lambda+3)}{\gamma}a_2^2 = h_2(x)(p_2+q_2) + h_3(x)(p_1^2+q_1^2). \tag{2.14}$$

By using (2.13) in (2.14), we get

$$a_2^2 = \frac{\gamma^2 h_2^3(x)(p_2+q_2)}{2\gamma(\lambda^2-3\lambda+3)h_2^2(x) - 2h_3(x)(2-\lambda)^2}. \tag{2.15}$$

From (1.3), we have the desired inequality (2.2).

Next, by subtracting (2.11) from (2.9) and in view of (2.12), we have

$$\frac{2(3-\lambda)a_3 - 2(3-\lambda)a_2^2}{\gamma} = h_2(x)(p_2-q_2) + h_3(x)(p_1^2-q_1^2)$$

and

$$a_3 = a_2^2 + \frac{\gamma h_2(x)(p_2-q_2)}{2(3-\lambda)}.$$

Hence using (2.13) and applying (1.3), we get desired inequality (2.3).

For some $\eta \in R$, we write

$$a_3 - \eta a_2^2 = \frac{\gamma h_2(x)(p_2-q_2)}{2(3-\lambda)} + (1-\eta)a_2^2. \tag{2.16}$$

Now, by using (2.15) and (2.16), we get

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{\gamma^2[h_2(x)]^3(1-\eta)(p_2+q_2)}{2(\lambda^2-3\lambda+3)\gamma[h_2(x)]^2 - 2(2-\lambda)^2 h_3(x)} + \frac{\gamma h_2(x)(p_2-q_2)}{2(3-\lambda)} \\ &= \gamma h_2(x)\left[\left(\Theta(\eta,x) + \frac{1}{2(3-\lambda)}\right)p_2 + \left(\Theta(\eta,x) - \frac{1}{2(3-\lambda)}\right)q_2\right], \end{aligned}$$

where

$$\Theta(\eta,\ x) = \frac{\gamma[h_2(x)]^2(1-\eta)}{2(\lambda^2-3\lambda+3)\gamma[h_2(x)]^2-2(2-\lambda)^2h_3(x)}.$$

So, we conclude that

$$|a_3-\eta a_2^2| \leq \begin{cases} \dfrac{|\gamma||h_2(x)|}{3-\lambda} & if \quad 0 \leq |\Theta(\eta,x)| \leq \dfrac{1}{2(3-\lambda)} \\ 2|\gamma||h_2(x)||\Theta(\eta,x)| & if \quad |\Theta(\eta,x)| \geq \dfrac{1}{2(3-\lambda)}. \end{cases}$$

This proves Theorem 2.1. □

For $\lambda = 1$, Theorem 2.1 readily yields the following coefficient estimates:

Corollary 2.1. *Let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{S}_\Sigma(1,\gamma,x)$. Then*

$$|a_2| \leq \frac{|\gamma||bx|\sqrt{|bx|}}{\sqrt{|bx^2[\gamma b-p]-aq|}},$$

$$|a_3| \leq \frac{|\gamma||bx|}{2} + \gamma^2b^2x^2$$

and for some $\eta \in R$,

$$|a_3-\eta a_2^2| \leq \begin{cases} \dfrac{|\gamma||bx|}{2} & if \quad |\eta-1| \leq \sigma_2 \\ \dfrac{\gamma^2|bx|^3(\eta-1)}{|[\gamma b-p]bx^2-qa|} & if \quad |\eta-1| \geq \sigma_2. \end{cases}$$

Here,

$$\sigma_2 = \frac{|[\gamma b-p]bx^2-qa|}{2(bx)^2}.$$

In light of Remark 1.1, we have

Corollary 2.2. *Let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{S}_\Sigma(\lambda,\gamma,x)$. Then*

$$|a_2| \leq \frac{2|\gamma||t|\sqrt{2|t|}}{\sqrt{|[(\lambda^2-3\lambda+3)2\gamma-2(2-\lambda)^2]2t^2+(2-\lambda)^2|}},$$

$$|a_3| \leq \frac{2|\gamma||t|}{3-\lambda} + \frac{\gamma^2 4t^2}{(2-\lambda)^2}$$

and for some $\eta \in R$,

$$|a_3-\eta a_2^2| \leq \begin{cases} \dfrac{|\gamma|2|t|}{3-\lambda} & if \quad |\eta-1| \leq \sigma_3 \\ \dfrac{\gamma^2|2t|^3(\eta-1)}{|[(\lambda^2-3\lambda+3)2\gamma-2(2-\lambda)^2]2t^2+(2-\lambda)^2|} & if \quad |\eta-1| \geq \sigma_3. \end{cases}$$

Here

$$\sigma_3 = \frac{|[(\lambda^2 - 3\lambda + 3)2\gamma - 2(2-\lambda)^2]2t^2 + (2-\lambda)^2|}{4t^2(3-\lambda)}.$$

Theorem 2.2. *Let the function $f \in \Sigma$ be given by* (1.1) *be in the class $\mathcal{M}_\Sigma(\lambda, \gamma, x)$. Then*

$$|a_2| \leq \frac{|\gamma||bx|\sqrt{|bx|}}{\sqrt{|bx^2[(4\lambda^2 - 11\lambda + 9)\gamma b - (2-\lambda)^2 4p] - aq4(2-\lambda)^2|}},$$

$$|a_3| \leq \frac{|\gamma||bx|}{3(3-\lambda)} + \frac{\gamma^2(bx)^2}{4(2-\lambda)^2}$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \dfrac{|\gamma||bx|}{3(3-\lambda)} & if \quad |\eta - 1| \leq \sigma_4 \\ \dfrac{\gamma^2 |bx|^3(\eta-1)}{|[(4\lambda^2 - 11\lambda + 9)\gamma b - 4p(2-\lambda)^2]bx^2 - 4qa(2-\lambda)^2} & if \quad |\eta - 1| \geq \sigma_4. \end{cases}$$

Here,

$$\sigma_4 = \frac{|[(4\lambda^2 - 11\lambda + 9)\gamma b - 4p(2-\lambda)^2]bx^2 - qa4(2-\lambda)^2|}{(bx)^2 3(3-\lambda)}.$$

Proof. Let $f \in \mathcal{M}_\Sigma(\lambda, \gamma, x)$. be given by Taylor-Maclaurin expansion (1.1). Then for all $z, w \in \mathbb{U}$ with $\Phi(0) = \psi(0) = 0, |\Phi(z)| < 1, |\psi(w) < 1|$ such that

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = \Omega(x,\ \Phi(z)) + 1 - a$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + w^2 g''(w)}{(1-\lambda)w + \lambda w g'(w)} - 1\right) = \Omega(x,\ \psi(w)) + 1 - a.$$

Equivalently it can be written as

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \dots$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w) + w^2 g''(w)}{(1-\lambda)w + \lambda w g'(w)} - 1\right) = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \dots.$$

Making use of the inequality $|\Phi(z)| < 1$ and $|\psi(z)| < 1$, we have

$$\begin{aligned} &1 + \frac{1}{\gamma}\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right) \\ &= 1 + h_2(x)p_1 z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \dots \end{aligned} \tag{2.17}$$

and

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)+w^2g''(w)}{(1-\lambda)w+\lambda wg'(w)}-1\right) \tag{2.18}$$
$$=1+h_2(x)q_1w+[h_2(x)q_2+h_3(x)q_1^2]w^2+\dots.$$

Now comparing the like coefficients of (2.17) and (2.18), we have

$$\frac{2(2-\lambda)}{\gamma}a_2=h_2(x)p_1, \tag{2.19}$$

$$\frac{3(3-\lambda)a_3-4\lambda(2-\lambda)a_2^2}{\gamma}=h_2(x)p_2+h_3(x)p_1^2, \tag{2.20}$$

$$\frac{-2(2-\lambda)}{\gamma}a_2=h_2(x)q_1, \tag{2.21}$$

and

$$\frac{3(3-\lambda)(2a_2^2-a_3)-4\lambda(2-\lambda)a_2^2}{\gamma}=h_2(x)q_2+h_3(x)q_1^2. \tag{2.22}$$

From (2.19) and (2.21), we can observe that

$$p_1=-q_1 \tag{2.23}$$

and

$$a_2^2=\frac{\gamma^2h_2^2(x)(p_1^2+q_1^2)}{8(2-\lambda)^2}. \tag{2.24}$$

Adding (2.20) and (2.22), we get

$$\frac{2(4\lambda^2-11\lambda+9)}{\gamma}a_2^2=h_2(x)(p_2+q_2)+h_3(x)(p_1^2+q_1^2). \tag{2.25}$$

Substituting (2.24) in (2.25), we have

$$a_2^2=\frac{h_2(x)^3(u_2+v_2)}{2(1+2\lambda+6\delta)[h_2(x)]^2-2h_3(x)(1+\lambda+2\delta)^2}. \tag{2.26}$$

Using (1.3), the above equation yields

$$|a_2|\leq\frac{|\gamma||bx|\sqrt{|bx|}}{\sqrt{|bx^2[(4\lambda^2-11\lambda+9)\gamma b-(2-\lambda)^2 4p]-aq4(2-\lambda)^2|}}.$$

Similarly, upon subtracting equation(2.22) from the equation(2.20) and in view of (2.23), we obtain

$$\frac{3(3-\lambda)a_3-3(3-\lambda)(2a_2^2-a_3)}{\gamma}=h_2(x)(p_2-q_2)+h_3(x)(p_1^2-q_1^2)$$

$$a_3 = \frac{\gamma h_2(x)(p_2 - q_2)}{6(3-\lambda)} + a_2^2.$$

Applying (1.3), we deduce that

$$|a_3| \le \frac{|\gamma||bx|}{3(3-\lambda)} + \frac{\gamma^2 (bx)^2}{4(2-\lambda)^2}.$$

For any $\eta \in \mathbb{R}$,

$$a_3 - \eta a_2^2 = \frac{\gamma h_2(x)(p_2 - q_2)}{6(3-\lambda)} + (1-\eta)a_2^2. \tag{2.27}$$

Substituting (2.26) in (2.27), we have

$$a_3 - \eta a_2^2 = \gamma h_2(x)\left[(\Theta(\eta, x) + \frac{1}{6(3-\lambda)})p_2 + (\Theta(\eta, x) - \frac{1}{6(3-\lambda)}\right],$$

where

$$\Theta(\eta, x) = \frac{\gamma[h_2(x)]^2(1-\eta)}{2(4\lambda^2 - 11\lambda + 9)\gamma[h_2(x)]^2 - 8(2-\lambda)^2 h_3(x)}.$$

Hence in view of (1.3), we conclude that

$$|a_3 - \nu a_2^2| \le \begin{cases} \dfrac{|\gamma||h_2(x)|}{3(3-\lambda)}; & 0 \le |\Theta(\eta, x)| \le \dfrac{1}{6(3-\lambda)} \\ 2|\gamma||h_2(x)||\Theta(\eta, x)|; & |\Theta(\eta, x)| \ge \dfrac{1}{6(3-\lambda)} \end{cases}$$

which completes the proof of the Theorem (2.2). □

For $\lambda = 1$, Theorem 2.2 readily yields the following corollaries:

Corollary 2.3. *Let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{M}_\Sigma(1, \gamma, x)$. Then for some $\eta \in R$,*

$$|a_3 - \eta a_2^2| \le \begin{cases} \dfrac{|\gamma||bx|}{6} & \textit{if } |\eta - 1| \le \sigma_5 \\ \dfrac{\gamma^2 |bx|^3(\eta - 1)}{|[2\gamma b - 4p]bx^2 - 4qa|} & \textit{if} |\eta - 1| \ge \sigma_5. \end{cases}$$

Here

$$\sigma_5 = \frac{|[2\gamma b - 4p]bx^2 - 4qa|}{6(bx)^2}.$$

In view of Remark 1.1,

Corollary 2.4. *Let the function $f \in \Sigma$ given by (1.1) be in the class $\mathcal{M}_\Sigma(\lambda, \gamma, t)$ and for some $\eta \in R$,*

$$|a_3-\eta a_2^2| \leq \begin{cases} \dfrac{|\gamma|2|t|}{3(3-\lambda)} & if \quad |\eta-1| \leq \sigma_6 \\ \dfrac{\gamma^2|2t|^3(\eta-1)}{|[(4\lambda^2-11\lambda+9)2\gamma-8(2-\lambda)^2]2t^2+4(2-\lambda)^2|} & if \quad |\eta-1| \geq \sigma_6. \end{cases}$$

Where

$$\sigma_6 = \frac{|[(4\lambda^2-11\lambda+9)2\gamma-8(2-\lambda)^2]2t^2+4(2-\lambda)^2|}{12t^2(3-\lambda)}.$$

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DEPARTMENT OF MATHEMATICS
AUDISAKARA COLLEGE OF ENGINEERING AND TECHNOLOGY
GUDUR 524101, NELLORE DISTRICT
ANDRA PRADESH, INDIA
Email address: soundarkavitha@gmail.com

DEPARTMENT OF MATHEMATICS
THEIVANAI AMMAL COLLEGE FOR WOMEN
VILLUPURAM 605602, TAMILNADU, INDIA
Email address: ksdhanalakshmi@gmail.com

PUDUCHERRY 605501
Email address: yogiyeshvanth07@gmail.com