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FUZZY PAIRWISE COMPACTNESS AND FUZZY PAIRWISE *a, b, g* COMPACTNESS IN FUZZY BITOPOLOGY

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ABSTRACT. This article examines the concept of fuzzy pairwise compactness in fuzzy bitopology and analyses various properties related to it. This paper establishes the notion of fuzzy pairwise α compact spaces, fuzzy pairwise b compact spaces and fuzzy pairwise g compact spaces in fuzzy bitopology and comparisons between them and discusses some interesting characteristics of these spaces. This paper also proposes the idea of FP a compact relative, FP b compact relative and FP g compact relative along with pairwise fuzzy filterbases.

1. INTRODUCTION

The idea of fuzzy bitopological spaces (fbts, in short) was introduced by Kandil and El-Shafee [10] as an extension of fuzzy topological spaces and as a generalization of bitopological spaces. Since then several bitopological notions are being generalised to the setting of fuzzy bitopological spaces using the term 'fuzzy pairwise FP'. Compactness represents a crucial part in the study of fuzzy topology. Its concept and related forms has become extremely necessary in fuzzy bitopology. In this paper we have considered fuzzy compactness in the sense of Chang [5]. The main purpose of this paper is to examine the role of pairwise compactness in fuzzy bitopological spaces and to establish various properties related to it.

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In Section 3, we discuss the characteristic features of fuzzy pairwise compact spaces using the concept of fuzzy pairwise Hausdorff spaces, fuzzy pairwise regular spaces, fuzzy pairwise normal spaces, and fuzzy pairwise homeomorphism.

In Section 4, we introduce the notion of fuzzy pairwise α compactness, fuzzy pairwise b compactness and fuzzy pairwise g compactness in fbts and analyse the relations between them. Further we also define fuzzy pairwise a, b, g compact relative concepts and their attributes involving the notion of (δ_i, δ_j) fuzzy filterbases in fbts.

2. PRELIMINARIES

Definition 2.1. *A fts (X,T) is said to be fuzzy Hausdorff or fuzzy T_2 space if and only if for any pair of distinct F-points x_r, y_s in X, there exists $A \in N^Q_{x_r}, B \in N^Q_{y_s}$ such that $A \wedge B = 0_X$*

Definition 2.2. *Let (X, δ_i, δ_j) be a bfts. A F-set μ is called $\delta_i\delta_j$-open, $\delta_i\delta_j$-closed provided $\mu \in \delta_i \vee \delta_j$, $\mu' \in \delta_i \vee \delta_j$ respectively.*

Definition 2.3. *Let $f : (X, \mathfrak{J}_1, \mathfrak{J}_2) \to (Y, \delta_1, \delta_2)$ be a function. f is called fuzzy pairwise homeomorphism iff f is a bijection, fuzzy pairwise continuous and $f^{-1} : (Y, \delta_1, \delta_2) \to (X, \mathfrak{J}_1, \mathfrak{J}_2)$ is fuzzy pairwise continuous.*

In this paper, Z is a set where we define ρ_i and ρ_j as fuzzy topologies (briefly f ts) to obtain a Fuzzy bitopological space. We denote this by $(\mathrm{Z}, \rho_i, \rho_j)$. We also use $(\mathrm{Y}, \phi_i, \phi_j)$, $(\mathrm{A}, \delta_i, \delta_j)$ and $(\mathrm{B}, \eta_i, \eta_j)$ to express fbts's. Here in this paper, $\rho_i \operatorname{int}(\mu)$ denotes the interior whereas ρ_i cl (μ), the closure with respect to the F-topology ρ_i for the F-set μ. Further {1,2} are values of i and j, when $\mathrm{i} \neq \mathrm{j}$. If $\mathrm{i} = \mathrm{j}$ then we have the obvious results in F-ts.

For certain definitions, theorems and results that are not described in this article, we refer to these papers [1-4,7,8,9,10].

3. FUZZY PAIRWISE COMPACT

Definition 3.1. *A family $\cup$ of (ρ_i, ρ_j) f-sets is a FP cover of a (ρ_i, ρ_j) F-set P iff $P \subset \{Q/Q \in U\}$.*

Definition 3.2. *A FP cover U of a fbts (Z, p_i, ρ_j) is a FP open cover (briefly FPO cover or $\rho_i\rho_j$ FPO cover) of Iff $U \subset \delta_i \vee \delta_j$ and $v_{\lambda \in U} A(z) = \tilde{1}$ for every $z \in Z$ and each* *member of U is a $\rho_i\rho_j$ F open set. A subcover of U is a subfamily of U which is also a cover.*

Definition 3.3. A *fbts* $(\mathrm{Z}, \rho_i, \rho_j)$ *said to be FP compact iff each* FPO *cover of* Z *has a finite subcover.*

Remark 3.1. *The indiscrete Fbts is FP compact.*

Remark 3.2. *A finite fuzzy set in a Fbts need not be FP compact.*

Example 1. *Let* $\mathrm{Z} = \{\mathrm{u}, \mathrm{v}, \mathrm{w}\}. (\mathrm{Z}, \rho_1, \rho_2)$ *is a F-bts where* $, \rho_1 = \{\tilde{0}, \tilde{1}, \mathrm{A}, \mathrm{B}, \mathrm{C}\}$ *and* $\rho_2 = \{\tilde{0}, \tilde{1}, \mathrm{D}, \mathrm{E}\ \ \mathrm{w}_{0.7}\}, \mathrm{D} = \{\mathrm{u}_{0.5}, \mathrm{v}_{0.7}, \mathrm{w}_{0.4}\}, \mathrm{E} = \{\mathrm{u}_{0.6}, \mathrm{v}_{0.6}, \mathrm{w}_{0.6}\}$ *and* $\mathrm{F} = \{\mathrm{u}_{0.7}, \mathrm{v}_{0.7}, \mathrm{w}_{0.7}\}$. *Let* $\mathrm{U} = \{\tilde{0}, \tilde{1}, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\ F\ \}$. *Obviously* $\mathrm{Z} \subset \{\mathrm{A}/\mathrm{A} \in \mathrm{U}\}, \mathrm{U} \subset \rho_1 \mathrm{U} \rho_2$ *and* $\mathrm{v}_{\lambda \in \mathrm{A}} \mathrm{A}(\mathrm{z}) = \tilde{1}$. *Hence* Z *is* FP *compact as* U *is a* $\rho\rho_j$ *-FO cover which contain finite subcovers. Consider the F-set* $L = \{a_1, b_1, c_{0.6}\}$. *Let* $U_1 = \{C, F\}$. *Here* $U_1 \subset \delta_1 \cup \delta_2$, $\mathrm{L} \subset \{\mathrm{A}/\mathrm{A} \in \mathrm{U}_1\}$ *and* $\mathrm{V}_{\lambda \in \mathrm{A}} \mathrm{A}(\mathrm{z}) = \tilde{1}$. *Also,* U_1 *satisfies the condition for* FP *open cover but it does not contain any finite* F *subcover of* L. *Hence* L *is not* FP *compact.*

Theorem 3.1. *Every* $\rho_i\rho_j - \mathrm{FC}$ *subset of a* FP *compact space is* FP *compact. Proof: Let* $(\mathrm{Z}, \rho_i, \rho_j)$ *be a* F *-bts which is* FP *compact. Let* A *be a* $\rho_i\rho_j - \mathrm{FC}$ *subset of* Z. *Then* $\mathrm{Z} - \mathrm{A}$ *is a* (ρ_i, ρ_j) *FO subset of* X. *Define* $\mathrm{W} = \{\mathrm{U}_\lambda, \lambda \in \Delta\}$ *to be a FPO cover of* A. *This is implies that the family* $\{\mathrm{U}_\lambda,\ \lambda \in \Delta\} \cup \{X - A\}$ *of* $\rho\rho_j - \mathrm{FO}$ *sets is a FPO cover of* Z. *since* Z *is* FP *compact, this has a finite subcover. Now* $\mathrm{A} \subset \mathrm{z}$ *and* $\mathrm{Z} - \mathrm{A}$ *covers no part of* A, *then* a *finite number of sets of* W *say* $U_1, \ldots, U_n$ *has the property that* $A \subseteq \cup_{i=1}^{n} U$, *is a finite subcover of* W. *since* W *is arbitrary,* A *is* FP *compact.*

Theorem 3.2. *If* P *and* Q *are two FP compact subsets of a fbts* (Z, ρ_i, ρ_j) *then* P *VQ is FP compact.*

Theorem 3.3. *Let* P *and* Q *be* FP *compact subsets of* (Z, ρ_i, ρ_j) , *that are* $\rho_i\rho_j - FC$ *then* $P \wedge Q$ *is* FP *compact.*

Remark 3.3. *Arbitrary union of* FP *compact sets is FP compact.*

Theorem 3.4. *Suppose* $(\mathrm{X}, \delta_i, \delta_j)$ *is a fbts. Let* K *be a* FP *compact subset of* X. *Then for every* $\rho_i\rho_j$ - F. *set* $\mathrm{F} \subset \mathrm{X}$, *the intersection* $\mathrm{F} \wedge \mathrm{K}$ *is* FP *compact.*

Proof. Follows from Theorem 3.3 □

Theorem 3.5. *Fuzzy pairwise compactness is a topological invariant.*

Proof. Let $(\mathrm{Z}, \delta_i, \delta_j)$ and $(\mathrm{Y}, \eta_i, \eta_j)$ be two fbts. Define $f : (\mathrm{Z}, \delta_i, \delta_j) \to (\mathrm{Y}, \eta_i, \eta_j)$ to be a FP homeomorphism. Assume that $(\mathrm{Z}, \delta_i, \delta_j)$ is FP compact. Let $\mathrm{F_y}$ be a FPO cover of $(\mathrm{Y}, \eta_i, \eta_j)$ Define $F_x = \{f^{-1}(V)/V \in F_Y\}$ which implies F_x is a FPO cover of Z. The fuzzy pairwise compactness nature of Z makes this to possess a finite subcover. Let it be $\mathrm{F'_x} = \{\mathrm{f^{-1}}(\mathrm{V_1}), \ldots, \mathrm{f^{-1}}(\mathrm{V_n})\}$ where $\mathrm{V_1}, \ldots, \mathrm{V_n} \in \mathrm{F_r}$. Thus the collection $\mathrm{F'_y} = \{\mathrm{V_1}, \ldots, \mathrm{V_n}\}$ is a $(\delta_i, \delta_j)\,\mathrm{F}$ finite subcover of $\mathrm{F_y}$. Hence Y is FP compact. □

Theorem 3.6. *Let $(\mathrm{Z}, \delta_i, \delta_j)$ be a fbts. If* Z *is FP Hausdorff and* $\mathrm{D} \subset \mathrm{Z}$ *is a FP compact subset, then* D *is* $\rho_i\rho_j - \mathrm{F}$ *closed in* Z *Proof: Let 'a' be a f point in D. (Z, δ_i, δ_j) is Hausdorff ensures that for every point $x \in \mathrm{Z} - \mathrm{D}$, there exist disjoint $(\delta_i, \delta_j)\,\mathrm{F}$ nbhds $\mathrm{M_{a,x}}$ and $\mathrm{N_{x,a}}$ of $'\mathrm{a}'$ and $'\mathrm{x}'$ respectively. Consider the collection $\{\mathrm{M_{a,x}} \wedge \mathrm{D}/\mathrm{a} \in \mathrm{D}\}$ which is a FPO cover of D. since D is FP compact, it has a finite subcollection $\{M_{a_1,x}, \ldots, M_{a_m,x}\}$ covering for some $a_1, \ldots, a_m \in D$. Let $N_x = \bigcap_{j=1}^{m} N_{x,a_j}$. Here N_x is a $(\delta_i, \delta_j)\,F$ nbhd of x, also N_x and D have to be disjoint. Otherwise, we can find a F-point $u \in N_x \wedge D$. Then $u \in M_{a_i,x}$ for some $i \in \{1, 2, \ldots, m\}$ Further , $\mathrm{u} \in \mathrm{N_x} \subset N_{x,a_j}$ which is not possible as $M_{a_i,x} \wedge N_{x,a_j} = \varphi$. Hence $\mathrm{Z} - \mathrm{D} = \bigcup_{x\mathrm{Z}-D} Nx$, which shows Z-D is $\delta_i\delta_j$ -FO which implies D is $\delta_i\delta_j - FC$.*

Theorem 3.7. *Let (Z, ρ_i, ρ_j) be a F -bts which is FP Hausdorff. If G is FP compact, then a fuzzy subset H $\subset$ G is FP compact iff H is $\rho_i\rho_j - F$ closed in Z.*

Proof. Assume that H is $\rho_i\rho_j - \mathrm{FC}$ in Z. $\mathrm{H} \wedge \mathrm{G}$ is FP compact and $\mathrm{H} \wedge \mathrm{G} = \mathrm{H}$ implies H is FP compact. Conversely, if H is FP compact, H is $\rho_i\rho_j - \mathrm{FC}$ in Z. □

Theorem 3.8. *Suppose the fbts (Z, ρ_i, ρ_j) is FP compact, FP Hausdorff , then a f-set A $\subset$ Z is FP compact iff it is $\rho_i\rho_j - F$ closed.*

Proof. Follows from Theorem 3.1 and Theorem 3.8 □

4. FUZZY PAIRWISE α, β, γ COMPACTNESS

Definition 4.1. *A fbts (Z, ρ_i, ρ_j) is called a fuzzy pairwise α compact (resp. β compact and γ compact) $(FP - \alpha C)$ (resp. FP- βC and $FP - \gamma C$) iff for every family μ of (ρ_i, ρ_j) fuzzy αO (resp. $\beta - O$ and $y - O$) sets satisfying $pP = 1_x$ there is a finite subfamily $\xi \subseteq \mu$ such that $p\xi^P = 1_x$ for every $x \in S(u)$.*

Remark 4.1.

1. *Every δ_i FO is (ρ_i, ρ_j) F α O , (ρ_i, ρ_j) F y O and (ρ_i, ρ_j) F β O.*
2. *Every (ρ_i, ρ_j) FαO is (ρ_i, ρ_j) FyO and every (ρ_i, ρ_j) FαO is (ρ_i, ρ_j) Fβ O.*
3. *Every (ρ_i, ρ_j) FPO or (ρ_i, ρ_j) FSO is (ρ_i, ρ_j) FyO.*
4. *The converse of the above results are not true.*

Theorem 4.1. *Let (Z, δ_i, δ_j) be a fbts. Then the subsequent statements are true.*

(a) Every FP- β compact space is FP- y compact.

(b) Every FP-y compact space is FP- α compact.

(c) Every FP- β compact space is FP- α compact.

Proof. (a) Suppose Z is FPβ compact. Let C be a (δ_i, δ_j) FγO cover of Z. Then C is a (δ_i, δ_j) FβO cover of Z.By assumption, C has a finite subcollection covering Z. This implies every (δ_i, δ_j) Fy O cover has a finite subcover of Z. (b) and (c) Similar to (a). □

Theorem 4.2. *Let (Z, δ_i, δ_j) be a fbts. Then the consecutive conditions are true.*

(a) Every FP-y compact space is either FP-S compact or FP-P compact or both.

(b) Every FP- β compact space is either FP-S compact or FP-P compact or both.

Proof.

(a) Follows from Remark 4.1(3)

(b) Follows from Remark 4.1(3) and Theorem 4.1 □

Theorem 4.3. *Let (Z, δ_i, δ_j) be a fbts. Every F pairwise- α compact space is F pairwise compact.*

Proof. Follows from Remark 4.1(1) □

Corollary 4.1.

(i) Every F pairwise - β compact space is F pairwise compact.

(ii) Every F pairwise - γ compact space is F pairwise compact.

Remark 4.2. *The counter part of the theorems 4.1, 4.2 and 4.3 does not hold always.*

Example 2. *Let (Z, δ_1, δ_2) be a F-bts where $Z = \{u, v, w\}$, $\delta_1 = \{\tilde{0}, \tilde{1}, P, Q\}$ and $\delta_2 = \{\tilde{0}, \tilde{1}, R, S\}$ are topologies with fuzzy sets $P = \{u_{0.5}, v_{0.7}, w_{0.4}\}$, $Q = \{u_{0.5}, v_{0.8}, w_{0.5}\}$, $R = \{u_{0.4}, v_{0.3}, w_{0.5}\}$ and $S = \{u_{0.6}\ \ v_{0.7}, w_{0.4}\}$. Consider $\delta_2 cl\,(\delta_1 \operatorname{int}(P)) = \delta_2 cl\,(R') = R'$ and δ_1 int $(\delta_2 cl(P)) = \delta_1 \operatorname{int}\,(R') = P$.*

Then δ_2 *cl* $(\delta_1$ *int* $(P)) \vee \delta_1$ *int* $(\delta_2 cl(P)) = R'$. *Then P is* (δ_1, δ_2) *FyO. Hence* $(\delta_1, \delta_2)\, F\gamma O$ *sets* $= \{\tilde{0}, P, Q, S\}$.

Here δ_2 *cl* $(\delta_1$ *int* $(\delta_2 y$ *cl* $(Q))) = 1$ *and* $(\delta_1, \delta_2)\, F\beta$ *o sets* $= \{\tilde{0}, \tilde{1}, P, Q, S\}$. *Also, P and Q are* $(\delta_1, \delta_2)\, F\alpha O$ *sets but* $\delta_1 \operatorname{int}(\delta_2 cl(\delta_1 int(S))) = \delta_1$ *int* $(R') = P$ *and* $S > P$.

Thus, S is not $(\delta_1, \delta_2)\, F\alpha O$. *That is , S is* $(\delta_1, \delta_2)\, F\gamma O, (\delta_1, \delta_2)\, F\beta O$ *but* not $(\delta_1, \delta_2)\, F\alpha O$ *This shows that* $FP - \alpha$ *compact need not be* $FP - \gamma$ *compact,* $FP - \beta$ *compact.*

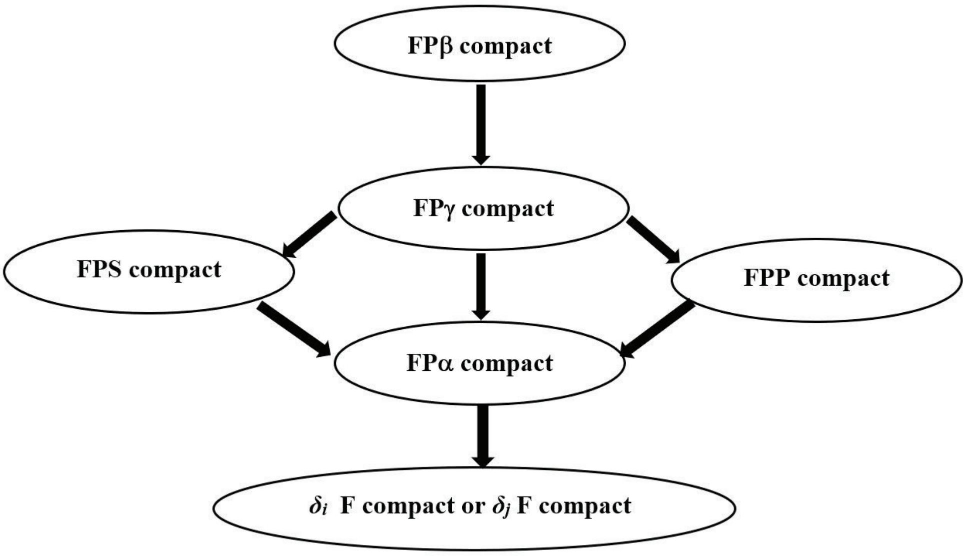

FIGURE 1. The above diagram summarizes the discussion

Example 3. *Let* $Z = \{p, q, r\}$. *Let* $\delta_1 = \{\tilde{0}, \tilde{1}, M\}$ *and* $\delta_2 = \{\tilde{0}, \tilde{1}, N\}$ *be topologies defined on X with fuzzy sets* $M = \{p_{0.2}, q_{0.5}, r_{0.4}\}$ *and* $N = \{p_{0.3}, q_{0.4}, r_{0.2}\}$. *Then* (Z, δ_1, δ_2) *is a fbts. Consider* $\delta_2 \operatorname{cl}(\delta_1 \operatorname{int}(M)) \vee \delta_1 \operatorname{int}(\delta_2 cl(M)) = MVN' = N'$. *Then M is* $(\delta_1, \delta_2)\, F\gamma O$ *Now* δ_2 *cl* $(\delta_1$ *int* $(N)) \vee \delta_1$ *int* $(\delta_2 cl(N)) = M \vee 0 = M$. *Hence* $(\delta_1, \delta_2)\, F\gamma O$ *sets* $= \{\tilde{0}, \tilde{1}, M\}$ *Here* δ_2 *cl* $(\delta_1$ *int* $(\delta_2 \gamma$ *cl* $(M))) = 1$ *and* δ_2 *cl* $(\delta_1$ *int* $(\delta_2 \gamma$ *cl* $(N))) = 1$. *Then M and N are* $(\delta_1, \delta_2)\, F\beta$ *O sets. Thus, N is* $(\delta_1, \quad \delta_2)\, F\beta O$ *but not* $(\delta_1, \delta_2)\, F\gamma O$. *This shows that* $FP - \gamma$ *compact space need not be* $FP - \beta$ *compact.*

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