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KNOWING NARAYANA COWS SEQUENCE

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ABSTRACT. Among several types of interesting sequences that exist in mathematics, Fibonacci sequence is the most famous and well known sequence. There was an equally important but not well known sequence named after Indian Mathematician Narayana Panditha of 14^{th} Century CE. This paper discusses the mathematical aspects of the sequence provided by him. This sequence will eventually produce a limiting constant similar to that of Golden Ratio corresponding to Fibonacci sequence.

1. INTRODUCTION

During the period from 1325 – 1400, a notable mathematician named Narayana Panditha from India made substantial contributions to mathematics. His two main works were Ganita Kaumudi (The Moonlight of Mathematics) and Bijaganitha Vatamsa (History of Algebra). These ideas were considered to be forerunner to modern Combinatorics. In this paper, we consider an important sequence named after him. Narayana's cows sequence is a sequence of positive integers which has startling resemblance to Fibonacci sequence. In the making of Fibonacci Sequence, Leonardo considered rabbits, whereas, for this sequence, Narayana Panditha considered cows, hence the name Narayana's cows sequence. Narayana Panditha described the sequence as the number of cows present each year, starting from one cow in the first year, where every cow has one baby cow every year starting

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in its fourth year of life. We will formally define Narayana's cows sequence and study its mathematical consequences.

2. Definition

Narayana's cows sequence is a sequence of positive integers which are defined by:

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, 872, 1278, \ldots. \tag{2.1}$$

The sequence mentioned in (2.1), is such that, the number of cows in each year is equal to the number of cows in the previous year, plus the number of cows three years ago. Using this description, we can form a recursive relation describing its terms by the equation:

$$C_0 = 0, C_1 = 1, C_2 = 1, C_3 = 1, C_{n+3} = C_{n+2} + C_n, \quad n \geq 1, \tag{2.2}$$

where C_n is the number of cows in the n^{th} month.

3. Characteristic Equation and Root

Using the shift operator E we define $E^r(C_n) = C_{n+r}$ and using equation (2.2), the characteristic equation corresponding to (2.2) is given by:

$$(E^3 - E^2 - 1)C_n = 0. \tag{3.1}$$

The auxiliary equation corresponding to (3.1) is given by:

$$m^3 - m^2 - 1 = 0. \tag{3.2}$$

Using Descarte's rule of signs, we see that equation (3.2) has only one positive real root, the other two being complex conjugates of each other. We now try to locate that positive root.

3.1. **Determining the root.**

$$f(m) = m^3 - m^2 - 1. \tag{3.3}$$

Let (3.3) be the given function representing the characteristic equation obtained in (3.1). We need to find a positive value m such that $f(m) = 0$.

First, we see that $f(1) = -1 < 0,\ f(2) = 3 > 0$. Since there is a sign change between $f(1)$ and $f(2)$, the root of (3.3) lies between 1 and 2. Also $f'(m) =$

$3m^2 - 2m$. By Newton's method, the root of (3.3) can be determined from the following recursive equation:

$$m_{n+1} = m_n - \frac{f(m_n)}{f'(m_n)}, n = 0, 1, 2, 3, 4, \ldots.$$

Taking $m_0 = 1$ we get

$$m_1 = m_0 - \frac{f(m_0)}{f'(m_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{(-1)}{1} = 2.$$

Taking $m_1 = 2$ we get

$$m_2 = m_1 - \frac{f(m_1)}{f'(m_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{8} = 1.625.$$

Taking $m_2 = 1.625$ we get

$$m_3 = m_2 - \frac{f(m_2)}{f'(m_2)} = 1.625 - \frac{f(1.625)}{f'(1.625)} = 1.625 - \frac{0.650390625}{4.671875} = 1.4857.$$

Taking $m_3 = 1.4857$ we get

$$m_4 = m_3 - \frac{f(m_3)}{f'(m_3)} = 1.4857 - \frac{f(1.4857)}{f'(1.4857)} = 1.4857 - \frac{0.072087790973}{3.65051347} = 1.46595.$$

Taking $m_4 = 1.46595$ we get

$$\begin{aligned} m_5 = m_4 - \frac{f(m_4)}{f'(m_4)} &= 1.46595 - \frac{f(1.46595)}{f'(1.46595)} \\ &= 1.46595 - \frac{0.001330931094875}{3.5151282075} = 1.46557137. \end{aligned}$$

Taking $m_5 = 1.46557137$ we get

$$\begin{aligned} m_6 = m_5 - \frac{f(m_5)}{f'(m_5)} &= 1.46595 - \frac{f(1.46557137)}{f'(1.46557137)} \\ &= 1.46557137 - \frac{0.00000061034}{3.51255558169} = 1.46557119. \end{aligned}$$

From these calculations, we see that the positive root of equation (3.2) is 1.46557 approximately. The number 1.46557 is called the characteristic root of equation (3.1).

3.2. **Geometric Verification.** We saw that the positive root of the polynomial equation $x^3 - x^2 - 1 = 0$ is 1.46557 approximately. We can also verify this fact through Figure 1 (shown below) where the graph of the function $f(x) = x^3 - x^2 - 1$ meets the X – axis at 1.46557 approximately thereby verifying our calculation in section 3.1

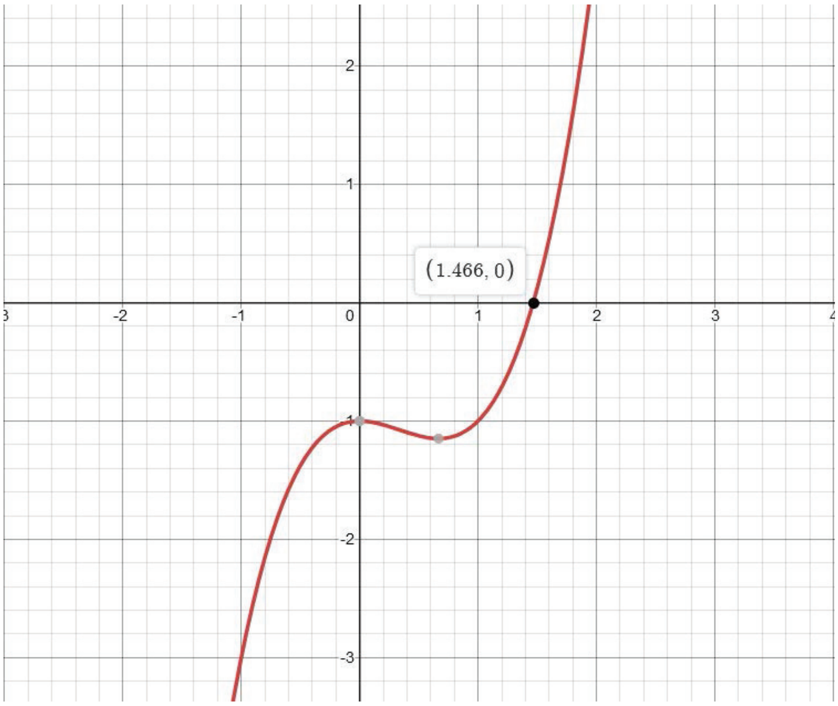

FIGURE 1. Graph of $f(x) = x^3 - x^2 - 1$

4. LIMITING RATIO

Assuming that the ratio of the consecutive terms of the Narayana's cows sequence described in (2.1) as well as through recursive relation (2.2) is constant, we shall try to determine that constant.

First we assume that $\lim \frac{C_{n+1}}{C_n} = \lambda$ then as $n \to \infty$

$$\lim \frac{C_{n+k}}{C_n} = \lim \left[\frac{C_{n+k}}{C_{n+k-1}} \times \frac{C_{n+k-1}}{C_{n+k-2}} \times \cdots \times \frac{C_{n+1}}{C_n} \right] = \lambda \times \lambda \cdots \times \lambda = \lambda^k \tag{4.1}$$

From Recursive Relation (2.2), we have as $n \to \infty$

$$\lim(C_{n+3}) = \lim(C_{n+2} + C_n).$$

That is,

$$\lim \frac{C_{n+3}}{C_n} = \lim \frac{C_{n+2}}{C_n} + 1.$$

Now using (4.1), we have $\lambda^3 = \lambda^2 + 1$ leading to $\lambda^3 - \lambda^2 - 1 = 0$. But this is precisely equation (3.2), whose positive root we found in section 3 to be 1.46557 approximately. Thus $\lambda = 1.46557....$ and this is the ratio of the consecutive terms of the Naryana's cows sequence. We call the limiting ratio 1.46557... as the Supergolden Ratio in view of extension of Golden Ratio obtained as limiting ratio of two consecutive terms of Fibonacci sequence.

CONCLUSION

Considering Narayana Panditha's definition as in recursive relation (2.2), we determined the positive root of the characteristic equation pertaining to (2.2). Interestingly, in section 4, we proved that the limiting ratio of two consecutive terms of Naryana's cows sequence is the number 1.46557... called Supergolden Ratio. This sequence provides us with an scope of extending definition (2.2) to arrive at other special constants to explore.

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