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THE SECOND GOURAVA INDEX OF SOME GRAPH PRODUCTS

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ABSTRACT. This paper shares the topological index namely the second Gourava index. Here the precise communications for the second Gourava index of graph operations including the join, composition, cartesian and corona products of graphs will be discussed.

1. INTRODUCTION

All graphs regarded right here are simple which are connected and finite. Let $V(G), E(G)$ and $d_G(w)$ represent the vertex set, the edge set and the degree of a vertex w of a graph G respectively. p and q indicate the number of vertices and number of edges respectively in a (p,q) graph. We encourage the readers to see [3] for basic definitions and notations of a graph.

A topological index is a numerical parameter which is mathematically attained from the graph structure.

Gutman et.al., [2] introduced the first and second Zagreb indices of a graph G as follows:

$$M_1(G) = \sum_{wz\in E(G)} (d_G(w) + d_G(z)) = \sum_{w\in V(G)} d_G^2(w) \text{ and } M_2(G) = \sum_{wz\in E(G)} d_G(w)d_G(z)$$

Shirdel et.al [8] defined the hyper-Zagreb index as

$$HM(G) = \sum_{wz\in E(G)} [d_G(w) + d_G(z)]^2$$

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and presented the exact formulae for the Hyper-Zagreb index of some well-known graphs.

The F-index [1] is defined for a graph G as

$$F(G) = \sum_{w \in V(G)} d_G^3(w) = \sum_{wz \in E(G)} [d_G^2(w) + d_G^2(z)].$$

V.R. Kulli [5] computed the first and second Gourava indices and defined as

$$GO_1(G) = \sum_{wz \in E(G)} (d_G(w) + d_G(z)) + (d_G(w) d_G(z))$$

and

$$GO_2(G) = \sum_{wz \in E(G)} d_G(w) d_G(z) (d_G(w) + d_G(z))$$

M.H. Khalief et.al [4] presented the precise values for the first and second Zagreb indices of grpah operations including cartesian, join, composition, disjunction and symmetric difference of a graph. Nilanjan De et. al [6] described the accurate values for F index of some graph operations. V.R. Kulli [5] computed the first and second Gourava indices of some standard classes of graphs.

In this paper, we calculate the second Gourava index for join, composition, cartesian and corona products of graphs. We begin this paper with the following Lemma.

Lemma 1.1. *[3, 7]*

(i) $d_{G_1+G_2}(w) = \begin{cases} d_{G_1}(w) + |V(G_2)|, & w \in V(G_2) \\ d_{G_2}(w) + |V(G_1)|, & w \in V(G_2) \end{cases}$

(ii) $d_{G_1[G_2]}(w, z) = |V(G_2)| d_{G_1}(w) + d_{G_2}(z)$

(iii) $d_{G_1 \square G_2}((w_i, zj)) = d_{G_1}(w_i) + d_{G_2}(z_j)$, *where* $(w_i, z_j) \in V(G_1 \square G_2)$.

(iv)

$$d_{G_1 \odot G_2}(u) = \begin{cases} d_{G_1}(w) + p_2 & if \ w \in V(G_1) \\ d_{G_1}(w) + p_2 & if \ w \in V(G_{2,i}) \ for \ some \ 0 \leq i \leq p_1 - 1, \end{cases}$$

where $u \in V(G_1 \odot G_2)$ $G_{2,i}$ *is the* i*th copy of the graph* G_2 *in* $G_1 \odot G_2$.

2. The Second Gourava Index of Join of Graphs

Theorem 2.1. *Let $G_i, i = 1, 2$ be a (p_i, q_i) graph. Then*

$$\begin{aligned} GO_2(G_1+G_2) &= GO_2(G_1)+GO_2(G_2)+2p_2M_2(G_1)+2p_1M_2(G_2)+p_2HM(G_1) \\ &\quad + p_1HM(G_2)+(3p_2^2+2q_2+p_1p_2)M_1(G_1) \\ &\quad + (3p_1^2+2q_1+p_1p_2)M_1(G_2)+8q_1q_2(p_1+p_2)+2p_1^2p_2(2q_2+q_1) \\ &\quad + 2p_1p_2^2(2q_1+q_2)+p_2^3(2q_1+p_1^2)+p_1^3(2q_2+p_2^2) \end{aligned}$$

Proof.

$$\begin{aligned} GO_2(G_1+G_2) &= \sum_{wz\in E(G_1+G_2)} d_{G_1+G_2}(w)d_{G_1+G_2}(z)[d_{G_1+G_2}(w)+d_{G_1+G_2}(z)] \\ &= \sum_{wz\in E(G_1)} d_{G_1+G_2}(w)d_{G_1+G_2}(z)[d_{G_1+G_2}(w)+d_{G_1+G_2}(z)] \\ &\quad + \sum_{wz\in E(G_2)} d_{G_1+G_2}(w)d_{G_1+G_2}(z)[d_{G_1+G_2}(w)+d_{G_1+G_2}(z)] \\ &\quad + \sum_{w\in V(G_1)}\sum_{z\in V(G_2)} d_{G_1+G_2}(w)d_{G_1+G_2}(z)[d_{G_1+G_2}(w)+d_{G_1+G_2}(z)] \\ &= S_1+S_2+S_3 \end{aligned}$$

$$\begin{aligned} S_1 &= \sum_{wz\in E(G_1)} d_{G_1+G_2}(w)d_{G_1+G_2}(z)[d_{G_1+G_2}(w)+d_{G_1+G_2}(z)] \\ &= \sum_{wz\in E(G_1)} (d_{G_1}(w)+p_2)(d_{G_1}(z)+p_2)[d_{G_1}(w)+d_{G_1}(z)+2p_2] \\ &= GO_2(G_1)+2p_2M_2(G_1)+2p_2HM(G_1)+3p_2^2M_1(G_1)+2p_2^3q_1. \end{aligned}$$

$$\begin{aligned} S_2 &= \sum_{wz\in E(G_2)} (d_{G_2}(w)+p_1)(d_{G_2}(z)+p_1)[d_{G_2}(w)+d_{G_2}(z)+2p_1] \\ &= \sum_{wz\in E(G_2)} [d_{G_2}(w)d_{G_2}(z)+p_1[d_{G_2}(w)+d_{G_2}(z)]+p_1^2][d_{G_2}(w)+d_{G_2}(z)+2p_1] \\ &= GO_2(G_2)+2p_1M_2(G_2)+p_1HM(G_2)+3p_1^2M_1(G_2)+2p_1^3q_2. \end{aligned}$$

$$\begin{aligned}
S_3 &= \sum_{w\in V(G_1)} \sum_{z\in V(G_2)} (d_{G_1}(w)+p_2)(d_{G_2}(z)+p_1)[d_{G_1}(w)+d_{G_2}(z)+p_1+p_2] \\
&= \sum_{w\in V(G_1)} \sum_{z\in V(G_2)} [d_{G_1}(w)d_{G_2}(z)+p_1 d_{G_1}(w)+p_2 d_{G_2}(z)+p_1p_2] \\
&\quad [d_{G_1}(w)+d_{G_2}(z)+p_1+p_2] \\
&= 2q_2q_1(G_1)+2q_1q_1(G_2)+8p_1q_1q_2+8p_2q_1q_2+p_1p_2q_1(G_1)+2p_1^2p_2q_1 \\
&\quad +4p_1p_2^2q_1+p_1p_2q_1(G_2)+4p_1^2p_2q_2+2p_1p_2^2q_2+p_1^2p_2^3+p_1^3p_2^2
\end{aligned}$$

Adding S_1, S_2 and S_3 we get the required result. □

3. The Second Gourava Index of Composition of Graphs

Theorem 3.1. *Let $G_i, i=1,2$ be a (p_i,q_i) graph. Then*

$$\begin{aligned}
GO_2(G_1[G_2]) &= p_2^5 GO_2(G_1)+p_1 GO_2(G_2)+2q_1p_2 HM(G_2)+4q_2p_2^3F(G_1) \\
&\quad +8q_2p_2^3M_2(G_1)+4q_1p_2M_2(G_2)+8q_2^2p_2M_1(G_1)+4q_1q_2M_1(G_2) \\
&\quad +4p_2^2M_1(G_1)M_1(G_2)
\end{aligned}$$

Proof.

$$\begin{aligned}
GO_2(G_1[G_2]) &= \sum_{(w,k)(z,l)\in E(G_1[G_2])} d_{G_1[G_2]}(w,k)d_{G_1[G_2]}(z,l) \\
&\quad [d_{G_1[G_2]}(w,k)+d_{G_1[G_2]}(z,l)] \\
&= \sum_{k\in V(G_2)} \sum_{l\in V(G_2)} \sum_{wz\in E(G_1)} [p_2d_{G_1}(w)+d_{G_2}(k)][p_2d_{G_1}(z)+d_{G_2}(l)] \\
&\quad [p_2d_{G_1}(w)+d_{G_2}(k)+p_2d_{G_1}(z)+d_{G_2}(l)] \\
&= \sum_{w\in V(G_1)} \sum_{kl\in E(G_2)} [p_2d_{G_1}(w)+d_{G_2}(k)][p_2d_{G_1}(w)+d_{G_2}(l)] \\
&\quad [2p_2d_{G_1}(w)+d_{G_2}(k)+d_{G_2}(l)] \\
&= S_1+S_2
\end{aligned}$$

$$\begin{aligned}
S_1 &= \sum_{k\in V(G_2)} \sum_{l\in V(G_2)} \sum_{wz\in E(G_1)} [p_2 d_{G_1}(w) + d_{G_2}(k)][p_2 d_{G_1}(z) + d_{G_2}(l)] \\
&\quad [p_2 d_{G_1}(w) + d_{G_2}(k) + p_2 d_{G_1}(z) + d_{G_2}(l)] \\
&= \sum_{k\in V(G_1)} \sum_{l\in V(G_2)} \sum_{wz\in E(G_1)} [p_2^2 d_{G_1}(w) d_{G_1}(z) \\
&\quad + p_2 d_{G_1}(w) d_{G_2}(l) + p_2 d_{G_1}(z) d_{G_2}(k) + d_{G_2}(k) d_{G_2}(l)] \\
&\quad [p_2(d_{G_1}(w) + d_{G_1}(z)) + (d_{G_2}(k) + d_{G_2}(l))] \\
&= p_2^5 GO_2(G_1) + 8q_2 p_2^3 M_2(G_1) + 2q_2 p_2^3 F(G_1) + 8q_2^2 p_2 M_1(G_1) \\
&\quad + 4q_1 q_2 M_1(G_2) + p_2^2 M_1(G_1) M_1(G_2)
\end{aligned}$$

$$\begin{aligned}
S_2 &= \sum_{w\in V(G_1)} \sum_{kl\in E(G_2)} [p_2 d_{G_1}(w) + d_{G_2}(k)][p_2 d_{G_1}(w) + d_{G_2}(l)] \\
&\quad [2p_2 d_{G_1}(w) + d_{G_2}(k) + d_{G_2}(l)] \\
&= \sum_{w\in V(G_1)} \sum_{kl\in E(G_2)} [p_2^2 d_{G_1}(w) + p_2 d_{G_1}(w)[d_{G_2}(k) + d_{G_2}(l)] + d_{G_2}(k) d_{G_2}(l)] \\
&\quad [2p_2 d_{G_1}(w) + d_{G_2}(k) + d_{G_2}(l)] \\
&= 2q_2 p_2^3 F(G_1) + 3p_2^2 M_1(G_1) M_1(G_2) + 2q_1 p_2 HM(G_2) + 4q_1 p_2 M_2(G_2) \\
&\quad + p_1 GO_2(G_2).
\end{aligned}$$

Adding S_1 and S_2 we get the required result. □

4. THE SECOND GOURAVA INDEX OF CARTESIAN PRODUCT OF GRAPHS

Theorem 4.1. *Let $G_i, i = 1, 2$ be a (p_i, q_i) graph. Then*

$$\begin{aligned}
GO_2(G_1 \square G_2) &= 2q_2 F(G_1) + 2q_1 F(G_2) + 6M_1(G_1) M_1(G_2) + 2q_1 HM(G_2) \\
&\quad + 2q_2 HM(G_1) + 4q_1 M_2(G_2) + 4q_2 M_2(G_1) + p_1 GO_2(G_2) \\
&\quad + p_2 GO_2(G_1).
\end{aligned}$$

Proof.

$$\begin{aligned}
GO_2(G_1\square G_2) = & \sum_{(w,k)(z,l)\in E(G_1\square G_2)} d_{G_1\square G_2}(w,z)d_{G_1\square G_2}(z,l) \\
& [d_{G_1\square G_2}(w,k) + d_{G_1\square G_2}(z,l)] \\
= & \sum_{w\in V(G_1)} \sum_{kl\in E(G_2)} [d_{G_1}(w) + d_{G_2}(k)][d_{G_1}(w) + d_{G_2}(l)] \\
& [d_{G_1}(w) + d_{G_2}(k) + d_{G_1}(w) + d_{G_2}(l)] \\
& + \sum_{xk\in V(G_2)} \sum_{wz\in E(G_1)} [d_{G_1}(w) + d_{G_2}(k)][d_{G_1}(z) + d_{G_2}(k)] \\
& [d_{G_1}(w) + d_{G_2}(k) + d_{G_1}(z) + d_{G_2}(k)] \\
= & S_1 + S_2,
\end{aligned}$$

where S_1 and S_2 are the terms of the above sums taken in order which are calculated as follows.

$$\begin{aligned}
S_1 = & \sum_{w\in V(G_1)} \sum_{kl\in E(G_2)} [d_{G_1}(w) + d_{G_2}(k)][d_{G_1}(w) + d_{G_2}(l)] \\
& [d_{G_1}(w) + d_{G_2}(k) + d_{G_1}(w) + d_{G_2}(l)] \\
= & \sum_{w\in V(G_1)} \sum_{kl\in E(G_2)} [d^2_{G_1}(w) + d_{G_1}(w)[d_{G_2}(k) + d_{G_2}(l)] + d_{G_2}(k)d_{G_2}(l)] \\
& [2d_{G_1}(w) + d_{G_2}(k) + d_{G_2}(l)] \\
= & 2q_2F(G_1) + 3M_1(G_1)M_1(G_2) + 2q_1HM(G_2) + 4q_1M_2(G_2) + p_1GO_2(G_2)
\end{aligned}$$

$$\begin{aligned}
S_2 = & \sum_{k\in V(G_2)} \sum_{wz\in E(G_1)} [d_{G_1}(w) + d_{G_2}(k)][d_{G_1}(z) + d_{G_2}(k)] \\
& [d_{G_1}(w) + d_{G_2}(k) + d_{G_1}(z) + d_{G_2}(k)] \\
= & \sum_{k\in V(G_2)} \sum_{wz\in E(G_1)} [d_{G_1}(w)d_{G_1}(z) + d_{G_2}(k)[d_{G_1}(w) + d_{G_1}(z)] \\
& + d^2_{G_2}(k)[d_{G_1}(w) + d_{G_1}(z) + 2d_{G_2}(k)] \\
= & p_2GO_2(G_1) + 4q_2M_2(G_1) + 2q_2HM(G_1) + 3M_1(G_1)M_1(G_2) + 2q_1F(G_2)
\end{aligned}$$

Adding S_1 and S_2 we get the required result. □

5. THE SECOND GOURAVA INDEX OF CORONA PRODUCT OF GRAPHS

Theorem 5.1. *Let $G_i, i = 1, 2$ be a (p_i, q_i) graph. Then*

$$\begin{aligned} GO_2(G_1 \odot G_2) = & GO_2(G_1) + p_1 GO_2(G_2) + 2p_2 M_2(G_1) + 2p_1 M_2(G_2) \\ & + p_2 HM(G_1) + p_1 HM(G_2) + (3p_2^2 + 2q_2 + p_2)M_1(G_1) \\ & + (3p_1 + 2q_1 + p_1 p_2)M_1(G_2) + (p_2^2 + 2p_2 + 1)2q_2 p_1 \\ & + 8q_1 q_2(p_2 + 1) + 2p_2^3 q_1 + 2q_1 p_2(2p_2 + 1) + (p_1 p_2^2)(p_2 + 1). \end{aligned}$$

Proof. The copy of G_2 in $G_1 \odot G_2$ corresponding to the vertex w in G_1 is denoted by $G_{2,w}$. The edge set of $G_1 \odot G_2$ can be partitioned into three subsets as follows:

$$\begin{aligned} E_1 &= \{wz \in E(G_1 \odot G_2) : w, z \in V(G_1)\} \\ E_2 &= \{wz \in E(G_1 \odot G_2) : w \in V(G_1), z \in V(G_{2,w})\} \\ E_3 &= \{wz \in E(G_{2,w}) : w \in V(G_1)\} \end{aligned}$$

$$\begin{aligned} GO_2(G_1 \odot G_2) = & \sum_{wz \in E(G_1)} (d_{G_1}(w) + p_2)(d_{G_1}(z) + p_2)(d_{G_1}(w) + d_{G_1}(z) + 2p_2) \\ & + \sum_{w \in V(G_1)} \sum_{kl \in E(G_{2,w})} (d_{G_2}(k) + 1)(d_{G_2}(l) + 1)(d_{G_2}(k) + d_{G_2}(l) + 2) \\ & + \sum_{w \in V(G_1)} \sum_{k \in V(G_{2,w})} (d_{G_1}(w) + p_2)(d_{G_2}(k) + 1) \\ & (d_{G_1}(w) + d_{G_2}(l) + p_2 + 1)) \\ & = S_1 + S_2 + S_3, \end{aligned}$$

where S_1, S_2 and S_3 are the terms of the above sums taken in order which are calculated as follows

$$\begin{aligned} S_1 &= \sum_{wz \in E(G_1)} (d_{G_1}(w) + p_2)(d_{G_1}(z) + p_2)(d_{G_1}(w) + d_{G_1}(z) + 2p_2) \\ &= \sum_{wz \in E(G_1)} [d_{G_1}(w) d_{G_1}(z) + p_2(d_{G_1}(w) + d_{G_1}(z)) + p_2^2][d_{G_1}(w) + d_{G_1}(z) + 2p_2] \\ &= GO_2(G_1) + 2p_2 M_2(G_1) + p_2 HM(G_1) + 3p_2^2 M_1(G_1) + 2p_2^3 q_1 \end{aligned}$$

$$\begin{aligned}
S_2 &= \sum_{w\in V(G_1)} \sum_{kl\in E(G_{2,w})} (d_{G_2}(k)+1)(d_{G_2}(l)+1)(d_{G_2}(k)+d_{G_2}(l)+2)\\
&= \sum_{w\in V(G_1)} \sum_{kl\in E(G_{2,w})} [d_{G_2}(k)d_{G_2}(l)+d_{G_2}(k)+d_{G_2}(l)+1][d_{G_2}(k)+d_{G_2}(l)+2]\\
&= p_1[GO_2(G_2)+2M_2(G_2)+HM(G_2)+3M_1(G_2)+2q_2]
\end{aligned}$$

$$\begin{aligned}
S_3 &= \sum_{w\in V(G_1)} \sum_{k\in V(G_{2,w})} (d_{G_1}(w)+p_2)(d_{G_2}(k)+1)(d_{G_1}(w)+d_{G_2}(l)+p_2+1))\\
&= \sum_{w\in V(G_1)} \sum_{k\in V(G_{2,w})} [d_{G_1}(w)d_{G_2}(k)+d_{G_1}(w)+p_2 d_{G_2}(k)+p_2]\\
&\quad [d_{G_1}(w)+d_{G_2}(k)+p_2+1]\\
&= 2q_2M_1(G_1)+2q_1M_1(G_2)+8q_1q_2(p_2+1)+p_2M_1(G_1)+2q_1p_2(2p_2+1)\\
&\quad + p_1p_2M_1(G_2)+2q_2p_1(p_2^2+2p_2)+p_1p_2^2(p_2+1)
\end{aligned}$$

Adding S_1, S_2 and S_3 we get the required result. □

6. Conclusion

This paper dealt with the topological index manily the second Gourava index of a graph. we started this paper with the basic lemma which helped for our work. And the precise communications for the second Gourava index of graph operations including the join, composition, cartesian and corona products of graphs would be discussed.

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