

FUZZY INTEGRO NABLA DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. This paper describes the study of existence and uniqueness of solutions to fuzzy integro nabla dynamic equations on time scales (FINDETs) under generalized Hukuhara nabla derivative using Banach contraction principle. Also obtain the existence result for FINDETs using Ascolis and Schauders fixed point theorem.

1. INTRODUCTION

The theory of time scales received a lot of attention and unifies the continuous and discrete analysis [4]. it has tremendous potential for applications. Unification and extension are the main features of time scale calculus. In modeling a real world phenomenon, some uncertainty happens because of inadequate data about the parameters. In order to manage this inaccuracy Zadeh, presented the theory called fuzzy sets. Hukuhara derivative is the starting point for the study of fuzzy differential equations. In [17]- [20], Vasavi et. al., introduced Hukuhara delta Δ_H -derivative, second type Hukuhara delta Δ_{sH} -derivative, genralized Hukuhara delta Δ_g -derivative and studied fuzzy dynamic equations on time scales. In [7]- [10], the authors introduced the nabla Hukuhara derivative on time scales and studied fuzzy nabla dynamic equations on time scales. For detailed study on fuzzy as well as time scale calculus, we refer [1]- [6], [11]- [22].

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2. PRELIMINARIES

Let $\mathbb{R}_{\mathcal{F}}$ denotes the family of fuzzy numbers which are convex, normal, compact and upper semi-continuous on $\mathbb{R}$ whose α -cuts $[A]_{\alpha} = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$.

For $v, w \in \mathbb{R}_{\mathcal{F}}$, exists a $u \in \mathbb{R}_{\mathcal{F}}$ satisfying $v = u + w$, where u is called the H -difference of v and w denoted by $u = v \ominus w$.

Definition 2.1. A function $\mathcal{F} : T \rightarrow \mathbb{R}_{\mathcal{F}}$ is Hukuhara differentiable at $t_0 \in T$ if there exists a $\mathcal{F}'(t_0) \in \mathbb{R}_{\mathcal{F}}$ provided the limits exist and equal to $\mathcal{F}'(t_0)$,

$$\lim_{h \rightarrow 0^+} \frac{\mathcal{F}(t_0 + h) \ominus \mathcal{F}(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\mathcal{F}(t_0) \ominus \mathcal{F}(t_0 - h)}{h} = \mathcal{F}'(t_0).$$

Definition 2.2. [4] Let $\mathbb{T}$ be a non-empty closed subset of $\mathbb{R}$. Let $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is the backward jump operator and $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$, be the graininess operator defined by $\rho(\tau) = \sup \tau_0 \in \mathbb{T} : \tau_0 < \tau$, $\mu(\tau) = \tau - \rho(\tau)$, for $\tau \in \mathbb{T}$ [4].

If $\tau > \inf(\mathbb{T})$ and $\rho(\tau) = \tau$, then τ is called left dense otherwise left scattered. $\mathbb{T}_k = \mathbb{T} - m$, if T has a left scattered minimum m . Otherwise $\mathbb{T}_k = \mathbb{T}$. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a function and $f^{\rho} : \mathbb{T} \rightarrow \mathbb{R}$ is defined by $f^{\rho}(\tau) = f(\rho(\tau))$, for each $\tau \in \mathbb{T}$.

Denote $\mathbb{T}^{[a,b]} = [a, b] \cap \mathbb{T}$, and $\mathbb{T}_k^{[a,b]} = \begin{cases} \mathbb{T}^{[a,b]}, & \text{if } a \text{ is right-dense,} \\ \mathbb{T}^{[\sigma(a),b]}, & \text{if } a \text{ is right-scattered.} \end{cases}$

Definition 2.3. [7] Let $\mathcal{F} : \mathbb{T} \rightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy function, $\xi \in \mathbb{T}_k^{[a,b]}$. Let $\nabla_{gH}\mathcal{F}(\xi) \in \mathbb{R}_{\mathcal{F}}$ (provided it exists) such that for any $\epsilon_{\xi} > 0$, there exists a neighborhood $N_{\mathbb{T}}$ of ξ for some $\nabla > 0$ such that

$$\mathcal{D}[(\mathcal{F}(\xi + h_{\xi}) \ominus_{gH} \mathcal{F}(\rho(\xi))), \nabla_{gH}\mathcal{F}(\xi)(h_{\xi} + \mu(\xi))] \leq \epsilon_{\xi}(h_{\xi} + \mu(\xi)),$$

$$\mathcal{D}[(\mathcal{F}(\rho(\xi)) \ominus_{gH} \mathcal{F}(\xi - h_{\xi})), \nabla_{gH}\mathcal{F}(\xi)(h_{\xi} - \mu(\xi))] \leq \epsilon_{\xi}(h_{\xi} - \mu(\xi)),$$

for all $\xi + h_{\xi}, \xi - h_{\xi} \in N_{\mathbb{T}^{[a,b]}}$ with $0 < h_{\xi} < \nabla$, where $\mu(\xi) = \xi - \rho(\xi)$. Then $\mathcal{F}$ is said to be ∇_{gH} -differentiable if its ∇_{gH} -derivative of $\mathcal{F}$ exists at ξ and $\mathcal{F}$ is ∇_{gH} -differentiable on $\mathbb{T}_k^{[a,b]}$ if its ∇_{gH} -derivative exists at each $\xi \in \mathbb{T}_k^{[a,b]}$.

This definition exists only for the fuzzy functions which have increasing diameter. For the fuzzy functions with decreasing diameter, second type nabla Hukuhara derivative was introduced.

Definition 2.4. [7] Let $\mathcal{F} : \mathbb{T}^{[a,b]} \rightarrow \mathbb{R}_{\mathcal{F}}$ be a fuzzy function, $\xi \in \mathbb{T}_k^{[a,b]}$. Let $\nabla_{sH}\mathcal{F}(\xi) \in \mathbb{R}_{\mathcal{F}}$ (provided it exists) such that for any $\epsilon_{\xi} > 0$, there exists a neighbourhood $N_{\mathbb{T}}$ of ξ

for some $\nabla > 0$ such that

$$\mathcal{D}[\mathcal{F}(\rho(\xi)) \ominus (\mathcal{F}(\xi + h_\xi), -\nabla_{sH}\mathcal{F}(\xi)(h_\xi + \mu(\xi)))] \leq \epsilon_\xi |-(h_\xi + \mu(\xi))|,$$

$$\mathcal{D}[\mathcal{F}(\xi - h_\xi) \ominus \mathcal{F}(\rho(\xi), -\nabla_{sH}\mathcal{F}(\xi)(h_\xi - \mu(\xi)))] \leq \epsilon_\xi |-(h_\xi - \mu(\xi))|,$$

for all $\xi + h_\xi, \xi - h_\xi \in N_{\mathbb{T}[a,b]}$ with $0 < h_\xi < \nabla$, where $\mu(\xi) = \xi - \rho(\xi)$. Then $\mathcal{F}$ is said to be ∇_{sH} -differentiable if its ∇_{sH} -derivative of $\mathcal{F}$ exists at ξ and $\mathcal{F}$ is ∇_{sH} -differentiable on $\mathbb{T}_k^{[a,b]}$ if its ∇_{sH} -derivative exists at each $\xi \in \mathbb{T}_k^{[a,b]}$.

The set of all fuzzy ld-continuous functions is denoted by $C_{ld}(\mathbb{T}^{[a,b]}) = C_{ld}(\mathbb{T}^{[a,b]}, \mathbb{R}_{\mathcal{F}})$.

3. FUZZY INTEGRO DYNAMIC EQUATIONS ON TIME SCALES

The main aim of this study is to develop the existence as well as uniqueness results of solutions for FIDETs of the form:

$$(3.1) \quad y_{gH}^\nabla(\tau) = \mathcal{F}(\tau, y(\tau)) + \int_{\tau_0}^{\tau} G(\tau, s, y(s)) \nabla s, \quad y(\tau_0) = y_0, \tau_0, s \in [\tau_0, \tau_0 + a]_{\mathbb{T}},$$

$$(3.2) \quad y_{sH}^\nabla(\tau) = \mathcal{F}(\tau, y(\tau)) + \int_{\tau_0}^{\tau} G(\tau, s, y(s)) \nabla s, \quad y(\tau_0) = y_0, \tau_0, s \in [\tau_0, \tau_0 + a]_{\mathbb{T}}.$$

where $\mathcal{F}, \mathcal{G} \in C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$, $I_{\mathbb{T}} = \mathbb{T}[\tau_0, \tau_0 + a]$, $a > 0$, $F : I_{\mathbb{T}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ and $G : I_{\mathbb{T}}^2 \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ are ld-continuous fuzzy valued functions.

Lemma 3.1.

- (i) A mapping $y : I_{\mathbb{T}} \rightarrow \mathbb{R}_{\mathcal{F}}$ is a solution of (3.1) iff it is ld-continuous and satisfies fuzzy nabla integral equation

$$y(t) = y_0 + \int_{\tau_0}^t F(s, y(s)) \nabla s + \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s, \quad t \in I_{\mathbb{T}}.$$

- (ii) y is a solution of (3.2) iff it is ld-continuous and satisfies fuzzy nabla integral equation

$$(3.3) \quad y_0 = y(t) + (-1) \int_{\tau_0}^t F(s, y(s)) \nabla s + (-1) \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s,$$

depending on the ∇_{gH}, ∇_{sH} -differentiability respectively.

Proof. We prove the case of ∇_{sH} -differentiability, the proof of the other case is similar. Let $y \in C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ be a solution to (3.2). Therefore, y is ∇_{sH} -differentiable on $I_{\mathbb{T}}$. Hence, we obtain

$$y(\tau_0) = y(\tau) + (-1) \int_{\tau_0}^{\tau} y^{\nabla_{sH}}(s) \nabla s,$$

for every $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$. Since $y(\tau_0) = y_0$ and $y^{\nabla_{sH}}(s) = \mathcal{F}(s, y(s)) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau$ for $s \in I_{\mathbb{T}}$, we obtain

$$y_0 = y(\tau) + (-1) \int_{\tau_0}^{\tau} F(s, y(s)) \nabla s + (-1) \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s.$$

To prove the converse part, let us assume that $y \in C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ is ld-continuous and satisfies (3.3), which allows that $y(\tau_0) = y_0$ and there exists the H-difference

$$y(\tau_0) \ominus (-1) \int_{\tau_0}^{\tau} F(s, y(s)) \nabla s + (-1) \int_{\tau_0}^{\tau} \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s,$$

for every $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$.

(i) If $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$ is right-dense such that $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$. Then

$$y(\tau) \ominus y(\tau + h) = (-1) \int_{\tau}^{\tau+h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s.$$

For $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$, consider

$$\begin{aligned} & y(\tau + h) + (-1) \int_{\tau}^{\tau+h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &= y(\tau_0) \ominus (-1) \int_{\tau_0}^{\tau+h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &\quad + (-1) \int_{\tau}^{\tau+h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &= y(\tau_0) \ominus (-1) \int_{\tau_0}^{\tau+h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &\quad + (-1) \int_{\tau_0}^{\tau+h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &\quad \ominus (-1) \int_{\tau_0}^{\tau} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s = y(\tau). \end{aligned}$$

for every $\tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}$. Dividing by $-h$ and passing to limit $h \rightarrow 0$,

$$\begin{aligned} & \lim_{h \rightarrow 0^+} \frac{y(t) \ominus y(t+h)}{-h} \\ &= \lim_{h \rightarrow 0^+} \frac{1}{h} \int_t^{t+h} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s. \end{aligned}$$

Since $\mathcal{F}, \mathcal{G}$ are ld-continuous, hence

$$\lim_{h \rightarrow 0^+} \frac{y(t) \ominus y(t+h)}{-h} = \mathcal{F}(t, y_t) + \int_{\tau_0}^t G(t, s, y(s)) \nabla s.$$

Hence y is ∇_{sH} -differentiable.

(ii) If t is right-scattered, then

$$y(\rho(t)) \ominus y(t) = (-1) \int_{\rho(t)}^t \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s.$$

Consider

$$\begin{aligned} & y(t) + (-1) \int_{\rho(t)}^t \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &= y(\tau_0) \ominus (-1) \int_{\rho(t)}^{\tau_0} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &\quad + (-1) \int_{\rho(t)}^{\tau_0} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &= y(\tau_0) \ominus (-1) \int_{\rho(t)}^{\tau_0} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &\quad + (-1) \int_{\rho(t)}^{\tau_0} \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \\ &\quad \ominus (-1) \int_{\rho(t)}^t \left(\mathcal{F}(s, y_s) + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s = y(t). \end{aligned}$$

Dividing by $-\mu(t)$ and passing to limit $\mu(t) \rightarrow 0$,

$$\int_{\rho(t)}^t \mathcal{F}(s, y_s) \nabla s = \mu(t) \mathcal{F}(t, y_t).$$

Hence y is ∇_{sH} -differentiable.

$$y^{\nabla_{sH}}(t) = \mathcal{F}(t, y(t)) + \int_{\rho(t)}^{\tau_0} G(t, s, y(s)) \nabla s, \text{ for every } \tau \in [\tau_0, \tau_0 + a]_{\mathbb{T}}.$$

□

Theorem 3.1. Let $F : I_{\mathbb{T}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ and $G : I_{\mathbb{T}}^2 \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ be ld-continuous fuzzy valued functions and assume that there exists a $K_1, K_2 > 0$ such that

$$\begin{aligned} D(F(t, \xi(t)), F(t, \psi(t))) &\leq K_1 D(\xi(t), \psi(t)), \\ D(G(t, \tau, \xi(\tau)), G(t, \tau, \psi(\tau))) &\leq K_2 D(\xi(t), \psi(t)), \end{aligned}$$

for each $t, \tau \in I_{\mathbb{T}}$, $\xi, \psi(i) \in \mathbb{R}_{\mathcal{F}}$, for each $i = 1, 2$. Then the FIDEts (3.1), (3.2) have two unique solutions on $I_{\mathbb{T}}$ with respect to ∇_{gH}, ∇_{sH} -derivative.

Proof. Let $C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ be the set of all ld-continuous functions from $I_{\mathbb{T}} \rightarrow \mathbb{R}_{\mathcal{F}}$, where $I_{\mathbb{T}} \in \mathbb{T}[a, b]$. Define the metric

$$d(\psi_1, \psi_2) = \sup_{\tau \in I_{\mathbb{T}}} \mathcal{D}(\psi_1(\tau), \psi_2(\tau)),$$

for all $\psi_1, \psi_2 \in C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$. Since $(\mathcal{D}, \mathbb{R}_{\mathcal{F}})$ is a complete metric space, clearly, $(C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}}), d)$ is also a complete metric. For any $\xi \in C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$, define the operator $T\xi$ as

$$(T\xi)(t) = y_0 + \int_{\tau_0}^t F(s, \xi(s)) \nabla s + \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, \xi(\tau)) \nabla \tau \right) \nabla s, \quad \forall t \in I_{\mathbb{T}}.$$

Now we will show that the operator $T\xi$ satisfies the hypothesis of the Banach fixed point theorem. For each $\tau_0, t \in I_{\mathbb{T}}$, we have

$$\begin{aligned} &\mathcal{D}(T\xi(\tau_0), T\xi(t)) \\ &= \mathcal{D} \left(y_0 + \int_{\tau_0}^t F(s, \xi(s)) \nabla s + \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, \xi(\tau)) \nabla \tau \right) \nabla s, \right. \\ &\quad \left. y_0 + \int_{\tau_0}^{\tau_0} F(s, \xi(s)) \nabla s + \int_{\tau_0}^{\tau_0} \left(\int_{\tau_0}^s G(s, \tau, \xi(\tau)) \nabla \tau \right) \nabla s \right) \\ &= \mathcal{D} \left(\int_{\tau_0}^t F(s, \xi(s)) \nabla s + \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, \xi(\tau)) \nabla \tau \right) \nabla s, \hat{0} \right) \\ &\leq \int_{\tau_0}^t \|F(s, \xi(s))\| \nabla s + \int_{\tau_0}^t \left(\int_{\tau_0}^s \|G(s, \tau, \xi(\tau))\| \nabla \tau \right) \nabla s. \end{aligned}$$

Since $\mathcal{F}, G$ satisfies Lipschitz condition, $\mathcal{F}, G$ are bounded. Therefore,

$$\leq k_1 |t - \tau_0| + k_2 |t - \tau_0| = k |t - \tau_0|, \quad \text{where } k = \max\{k_1, k_2\}.$$

Hence, $T\xi \in C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$. Now, consider

$$\begin{aligned} & d(T\xi_1, T\xi_2) \\ &= \sup_{t \in I_{\mathbb{T}}} \left\{ \mathcal{D} \left(y_0 + \int_{\tau_0}^t F(s, \xi_1(s)) \nabla s + \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, \xi_1(\tau)) \nabla \tau \right) \nabla s, \right. \right. \\ & \quad \left. \left. y_0 + \int_{\tau_0}^t F(s, \xi_2(s)) \nabla s + \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, \xi_2(\tau)) \nabla \tau \right) \nabla s \right) \right\} \\ &\leq \int_{\tau_0}^{t+a} \mathcal{D} \left[(F(s, \xi_1(s)), F(s, \xi_2(s))) \nabla s + \int_{\tau_0}^s (G(s, \tau, \xi_1(\tau)), G(s, \tau, \xi_2(\tau)) \nabla \tau) \nabla s \right] \\ &\leq akT(\xi_1, \xi_2). \end{aligned}$$

choose $I_{\mathbb{T}}$ such that $ak < 1$. Therefore, T is a contraction mapping. Hence, from Banach contraction principle there exists a unique fixed point lies in the space $C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$ which itself is the unique solution of (3.1). In a similar way, we can prove the other case. $\square$

In the following result, we establish the existence of solutions for FIDETs (3.1) and (3.2) using Ascoli's theorem and Schauder's fixed point theorem.

Theorem 3.2. *Let $F : I_{\mathbb{T}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ and $G : I_{\mathbb{T}}^2 \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{\mathcal{F}}$ be ld-continuous fuzzy valued functions and assume that there exists constants $k_1, k_2 > 0$ such that $\mathcal{D}(\mathcal{F}(t, y_t), \hat{0}) \leq k_1$, $\mathcal{D}(\mathcal{G}(t, \tau, y(\tau)), \hat{0}) \leq K_2$, $t, \tau \in I_{\mathbb{T}}, y \in \mathbb{R}_{\mathcal{F}}$. Then FIDETs (3.1), (3.2) have at least one solution on $I_{\mathbb{T}}$.*

Proof. Let $\mathcal{B}$ is a bounded set in $C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$. Then $T\mathcal{B} = [Ty] : y \in \mathcal{B}$ is totally bounded iff it is equi-continuous. For $t \in I_{\mathbb{T}}$, the set

$$T\mathcal{B}(y) = [Ty](y) : y \in \mathcal{B}$$

is totally bounded subset of $\mathbb{R}_{\mathcal{F}}$, where

$$Ty(t) = y_0 + \int_{\tau_0}^t F(s, y(s)) \nabla s + \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s, \quad \forall t \in I_{\mathbb{T}}.$$

For $t_1, t_2 \in I_{\mathbb{T}}, t_1 \leq t_2$ and $y \in \mathcal{B}$, consider

$$\begin{aligned}
& \mathcal{D}(Ty(t_1), Ty(t_2)) \\
&= \sup_{t \in I_{\mathbb{T}}} \left\{ \mathcal{D} \left(y_0 + \int_{\tau_0}^{t_1} F(s, y(s)) \nabla s + \int_{\tau_0}^{t_1} \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s, \right. \right. \\
&\quad \left. \left. y_0 + \int_{\tau_0}^{t_2} F(s, y(s)) \nabla s + \int_{\tau_0}^{t_2} \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \right) \right\} \\
&= \sup_{t \in I_{\mathbb{T}}} \left\{ \mathcal{D} \left(\int_{\tau_0}^{t_1} F(s, y(s)) \nabla s + \int_{\tau_0}^{t_1} \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s, \right. \right. \\
&\quad \left. \left. \int_{\tau_0}^{t_2} F(s, y(s)) \nabla s + \int_{\tau_0}^{t_2} \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s \right) \right\} \\
&\leq \int_{t_1}^{t_2} \|F(s, \xi(s))\| \nabla s + \int_{t_1}^{t_2} \left(\int_{\tau_0}^s \|G(s, \tau, \xi(\tau)) \nabla \tau\| \right) \nabla s. \\
&\leq k|t_2 - t_1|
\end{aligned}$$

Therefore Ty is equi-continuous on $I_{\mathbb{T}}$. For fixed $t \in I_{\mathbb{T}}$,

$$\mathcal{D}(Ty(t), Ty(t_1)) \leq k|t_2 - t_1|, t_1 \in I_{\mathbb{T}}, y \in \mathcal{B}.$$

Hence the set $[Ty](t) : y \in \mathcal{B}$ is totally bounded in $\mathbb{R}_{\mathcal{F}}$. Hence $T\mathcal{B}$ is relatively compact subset of $C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$.

Define $\mathcal{B} = \{y \in C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}}) : T(y, \hat{\cdot}) \leq ak \text{ in the metric space } C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})\}$. For $y \in C_{ld}(I_{\mathbb{T}}, \mathbb{R}_{\mathcal{F}})$,

$$\begin{aligned}
& \mathcal{D}(Ty(t), Ty(t_0)) \\
&= \sup_{t \in I_{\mathbb{T}}} \left\{ \mathcal{D} \left(y_0 + \int_{\tau_0}^t F(s, y(s)) \nabla s + \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s, y_0 \right) \right\} \\
&= \int_{\tau_0}^t \mathcal{D} \left(F(s, y(s)) \nabla s + \int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau, \hat{0} \right) \nabla s \leq \int_{\tau_0}^t k \nabla s = k|t - \tau_0| \leq ak.
\end{aligned}$$

Therefore, $T\mathcal{B} \subset \mathcal{B}$. Since T is compact, T has a fixed point which is a solution of (3.1). Therefore, FINDET (3.1) has atleast one solution on $t \in I_{\mathbb{T}}$. In a similar way, we can prove for the FINDET (3.2) by considering

$$Ty(t) = y(t) + (-1) \int_{\tau_0}^t F(s, y(s)) \nabla s + (-1) \int_{\tau_0}^t \left(\int_{\tau_0}^s G(s, \tau, y(\tau)) \nabla \tau \right) \nabla s.$$

□

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