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STOCHASTIC FRACTIONAL MODELS OF THE DIFFUSION OF COVID-19

MAHMOUD M. EL-BORAI[1] AND KHAIRIA EL-SAID EL-NADI

ABSTRACT. Some different stochastic and deterministic mathematical models of coronavirus disease (COVID-19) are studied. We study a nonlinear stochastic diffusion system that model of the dynamics of infections disease in particular the case of (COVID-19). Also a nonlinear deterministic fractional diffusion system related to (COVID-19) is considered. The asymptotic stability is proved for every system. In the field of infection disease, modeling, evaluating and predicting the rate of disease transmission is very important epidemic prevention and control. Our models are suitable for the considered prediction..

1. INTRODUCTION

There are different kinds of coronaviruses, most of which circulate in animals. Only seven of these viruses infect humans. But three times in the last twenty years, a corona viruses has jumped from animals to humans to cause severe disease COVID-19, a new and sometimes deadly respiratory is believed to have originated in a live animal market in China, has spread rapidly throughout that country and the world. Mathematical modeling can be used to understanding how a virus spreads within a population. The essence of mathematical modeling lies in writing down a set of mathematical equations that mimic reality. These are then solved for certain values of the parameters within the equations. The solution of the mathematical model can be refined when we use information that we already

[1]*corresponding author*

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know about the virus spread, for example, available data on reported number of infections, the reported number of hospitalizations or the confirmed number of deaths due to the infection. This process of model refinement can be done a number of times until the solutions of the mathematical equations agree with that we already know about the virus spread. Here we consider stochastic nonlinear diffusion model and a deterministic nonlinear fractional diffusion model. Our models represent generalizations of all the previous models.

This paper is prepared as follows. In section 2, we study a nonlinear stochastic diffusion system that model of the dynamics of infections disease in particular the case of (COVID-19). In section 3, we study nonlinear deterministic fractional diffusion system related to (COVID-19) and we give sufficient conditions to prove the asymptotic stability for this system.

2. Spatial stochastic model

Let $(\Omega, \Im, \Im_t, \mathcal{P})$ be a filtered probability space and let $\{W_i(t) : t \geq 0\}$ be standard independent Wiener processes adapted to the filtration $\{\Im_t : t \geq 0\}$, $i = 1, 2, \dots, m$.

In this section, we study the stochastic general epidemic model with spatial diffusion in the following form:

$$du_i(x,t) = [a_i \nabla^2 u_i(x,t) - c_i u_i(x,t) + f_i(u)]dt + \sigma_i u_i(x,t) dW_i(t), \tag{2.1}$$

where $(x,t) \in G \times (0,b], x = (x_1, x_2, \dots, x_n), b > 0$ and G is a bounded domain in the n-dimensional Euclidean space with smooth boundary ∂G, ∇^2 is the Laplace operator:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

and $a_1, \dots, a_m$, $c_1, \dots, c_m$ are positive constants, $\sigma_1, \dots, \sigma_m$ are constants. It is supposed that $f = (f_1, \dots, f_m)$ satisfies the Lipschitz condition:

$$\sum_{i=1}^{m} |f_i(u) - f_i(v)| \leq M \sum_{i=1}^{m} |u_i - v_i| \tag{2.2}$$

for all $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_m)$, where M is a positive constant. The system (2.1) can be written in the form:

$$\begin{aligned} u_i(x,t) = \varphi_i \quad &+ \quad \int_0^t [a_i \nabla^2 u_i(x,t) - c_i u_i(x,t) + f_i(u)]dt \\ &+ \int_0^t \sigma_i u_i(x,t) dW_i(t). \end{aligned} \tag{2.3}$$

Let us assume also that $u = (u_1, \ldots, u_m)$ satisfy the following Neumann homogeneous conditions:

$$\frac{\partial u_i(x,t)}{\partial \nu} \mid_{\partial G} = 0, \tag{2.4}$$

where ν is the direction of the outward normal at any point on the the surface ∂G and $\varphi_1, \ldots, \varphi_m$ are given continuous nonnegative functions defined on $G \cup \partial G$. However, our model is the generalization of the following important models:

(1) Let $m = 4$, $\sigma_1 = \ldots = \sigma_4 = 0$, $u_1, \ldots, u_4$ depend only on t, $f_1(u) = \gamma - \delta u_1 u_2$, $f_2(u) = \delta u_1 u_2$, $f_3(u) = c_2 u_2$, $f_4(u) = c_3 u_3$ where $\delta > 0$ is the infection coefficient, c_1 is the natural death rate of population, $c_2 = c_1 + \lambda_1 + \lambda_2$, $\lambda_1 \geq 0$ is the death rate due to disease and λ_2 is the recovery rate of the infection individuals, $c_3 = \lambda_2$, $c_4 = c_1$.

The parameter $\gamma \geq 0$ represents recruitment rate of population. The functions $u_1, \ldots, u_4$ satisfy equation (2.1).

In this case we get the model of Hethcote ([5], [6]) see also ([36]). It is also represents the MERS-COV (Middle East Repiratory Syndrome Corona Virus) model. The functions $u_1, \ldots, u_4$ will represent the susceptible, exposed, infections and recovered, respectively. The MERS-COV describes a chronic disease for respiration was reported in Saudi-Arabia, in 2012. Mostly it linked to the Arabian countries. This virus perhaps come from animals, like camels. Still MERS-COV have no cure or vaccination developed, so why it is fatal disease and considered pandemic.

(2) Let $m = 4$, $\sigma_1 = \cdots = \sigma_4 = 0$, $u_1, \ldots, u_4$ depend only on t and satisfy system (2.1), with $f_1(u) = \gamma - g(u)$,

$$g(u) = \frac{\delta u_1 u_2}{1 + b_1 u_1 + b_2 u_2 + b_3 u_1 u_2}$$

and $f_2(u) = g(u)$, $f_3(u) = c_2 u_2$, $f_4(u) = c_3 u_3$. In this model the function $g(u)$ represents the incidence rate. For different choices of the nonnegative parameters b_1, b_2 and b_3, we get important models due to Beddington, De Angelis and Mehdi ([1,3,8]).

(3) Let $m = 4$, $\sigma_1 = \cdots = \sigma_4 = 0$, $u_1, \ldots, u_4$ depend only on t and satisfy system (2.1), with
$f_1(u) = \gamma - g(u)$, $f_2(u) = g(u)$, where

$$g(u) = \frac{1}{N}[\delta u_1(ku_2 + u_3)], \quad \delta, k > 0.$$

where N is the total population, $N = u_1 + \cdots + u_4$ and $f_3(u) = c_2 u_2$, $f_4(u) = c_3 u_3$. This system represents a model of COVID-19 based on mobility data in Anhui, china. The model of COVID-19 studied the spread of the disease, ([2,4,7], [9]- [11], [33]- [35]) in Wuhan Hubei, China.

The mixed problem ((2.1), (2.3), (2.4)) is well-posed if there exists a vector $u = (u_1, \ldots, u_m)$ such that:

(1) All the sample paths of the processes $u_i, i = 1, 2, \ldots, m$ are continuous on $\bar{G} \times [0, b]$ where $\bar{G} = G \cup \partial G$, for almost all $w \in \Omega$, (abbreviated a.s.).
(2) All the sample paths of the functions $\frac{\partial u_i}{\partial x_r}$, $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n$ are continuous on $\bar{G} \times [0, b]$ a.s.
(3) All the sample paths of the stochastic processes $u_1, \ldots, u_m$ have continuous partial derivatives with respect to the variables $x_1, \ldots, x_n$ up to order two on $G \times [0, b]$ a.s.
(4) The sample paths of the stochastic processes $u_1, \ldots, u_m$ satisfy system (2.3) on $G \times [0, b]$ and the boundary conditions (2.4) a.s.
(5) The sample path of the vector u is unique, a.s. and all the sample paths of the stochastic processes $u_1, \ldots, u_m$ depending continuously on $\varphi_1, \ldots, \varphi_m$ a.s.

Theorem 2.1. *The mixed problem (*(2.1)*),* (2.3)*,* (2.4)*) is well-posed.*

Proof. Let

$$\begin{aligned} X_i &= e^{-\sigma_i W_i(t)}, \\ Y_i &= e^{\sigma_i W_i(t)}, \qquad i = 1, \ldots, m. \end{aligned}$$

Thus,

$$dX_i(t) = \frac{1}{2}\sigma_i^2 X_i(t) - \sigma_i X_i(t) dW_i(t) \tag{2.5}$$

Set $v_i(x,t) = X_i(t)u_i(x,t)$. Then from ((2.3),(2.4),(2.5)) and $It\hat{o}$ formula, we get:

$$\frac{\partial v_i(x,t)}{\partial t} = a_i\nabla^2 v_i(x,t) - \beta_i v_i(x,t) + X_i(t)f_i(u) \tag{2.6}$$

$$v_i(x,0) = \varphi_i(x), \tag{2.7}$$

$$v_i(x,t)\mid_{\partial G} = 0, \tag{2.8}$$

where $\beta_i = c_i + \sigma_i^2/2$ and $i = 1,2,\dots,m$.

Let $V_k(x)$, $-\lambda_k^2$ be the eigenfunctions and eigenvalues respectively of the Laplace equation:

$$\begin{aligned} \nabla^2 V_k(x) &= -\lambda_k^2 V_k(x), \quad x \in G \\ \frac{\partial V_k(x)}{\partial \nu}\mid_{\partial G} &= 0. \end{aligned}$$

The eigenfunctions $V_k, k = 1,2,\dots$ are orthonormal and the sequence of eigenvalues $\{\lambda_k\}$ tends to infinity as k tends to infinity. Thus the solution of ((2.6))-(2.8)) can be represented by

$$e^{\beta_i t} v_i(x,t) = \sum_{k=0}^{\infty} T_{ki}(t)V_k(x). \tag{2.9}$$

It is easy to prove that

$$\begin{aligned} T_{ki}(t) &= e^{a_i\lambda_k^2 t}\gamma_{ki} \\ &+ \int_0^t \int_G e^{-a_i\lambda_k^2(t-\theta)} V_k(x) e^{\beta_i\theta} X_i(\theta) f_i(u(y,\theta)) dy d\theta, \end{aligned} \tag{2.10}$$

where $\gamma_{ki} = \int_G \varphi_i(y)V_k(y)dy$.

Let $Z(\cdot)$ and $\tilde{Z}(\cdot)$ be two stochastic processes. we call $\tilde{Z}(\cdot)$ is the version of $Z(\cdot)$ if

$$\mathcal{P}(Z(t) = \tilde{Z}(t)) = 1, \quad \forall t \geq 0.$$

Any Wiener process has a version with continuous sample path a.s. Thus $X(\cdot)$ and $Y(\cdot)$ have versions with continuous sample paths a.s. on $[0,b]$. Now equation (2.9) with the help of (2.10) is of the type Volterra integral equation.

Remembering that the functions $f_1,\dots,f_m$ satisfy Lipschitz condition (2.2), thus with the properties of $X(\cdot)$ and $Y(\cdot)$, the considered Volterra integral equation (2.9) can be solved in the space $(\mathbb{C}(\bar{G}), \|\cdot\|)$, where $\mathbb{C}(\bar{G})$ is the set of all vectors

$h = (h_1, \ldots, h_m)$ such that $h_1, \ldots, h_m$ are continuous on $\bar{G}$ and $\| h(\cdot) \| = \sum_{i=1}^{m} \sup \mid h_i(x) \mid$, (the supremum is taken on $\bar{G}$).

It is easy to see that

$$u_1(x,t) = Y_1(t)v_1(x,t), u_2(x,t) = Y_2(t)v_2(x,t), \ldots, u_m(x,t) = Y_m(t)v_m(x,t),$$

satisfy the conditions $(1-4)$. To prove condition 5, (the stability of solutions), let $v = (v_1, \ldots, v_m)$, $v^* = (v_1^*, \ldots, v_m^*)$ be two solutions of (2.6) with the boundary conditions

$$\frac{\partial v_i(x,t)}{\partial \nu} \mid_{\partial G} = \frac{\partial v_i^*(x,t)}{\partial \nu} \mid_{\partial G} = 0$$

and the initial conditions

$$v_i(x,0) = \varphi_i(x), \; v_i^*(x,0) = \varphi_i^*,$$

such that

$$\sup_x \mid \varphi_i(x) - \varphi_i^* \mid \leq \varepsilon, \quad i = 1, 2, \ldots, m$$

for sufficiently small $\varepsilon > 0$.

From ((2.2),(2.9)) and (2.10), we get:

$$\| v(\cdot,t) - v^*(\cdot,t) \| \leq M_1 \varepsilon + M_2 \int_0^t \| v(\cdot,\theta) - v^*(\cdot,\theta) \| \, d\theta$$

for some constants $M_1, M_2 > 0$, (M_2 depends on $\omega \in \Omega$). Thus

$$\| v(\cdot,t) - v^*(\cdot,t) \| \leq M_1 \varepsilon e^{M_2 t}.$$

This completes the proof of the theorem. □

Theorem 2.2. *Let u be the solution of the mixed problem ((2.3),(2.4)). If $\varphi_i \geq 0$ on $\bar{G}$ and $f_i(0,0,\ldots,0) \geq 0$, $i = 1, 2, \ldots, m$, then $u_i(x,t) \geq 0$ on $\bar{G} \times [0,b]$ a.s.*

Proof. Let $Z_i(x,t)$ be the functions defined by

$$Z_i(x,t) = \begin{cases} \frac{f_i(u) - f_i(0,0,\ldots,0)}{u_i(x,t)} & , \quad \text{for } x \text{ such that } u_i(x,t) \neq 0 \\ 0 & , \quad \text{for } x \text{ such that } u_i(x,t) = 0. \end{cases}$$

According to the Lipschitz condition (2.2), one gets

$$\mid Z_i(x,t) u_i(x,t) \mid \leq M \sum_{r=1}^{m} \mid u_r(x,t) \mid$$

on $\bar{G} \times [0,b]$, for some constant $M > 0$.

Moreover,

$$f_i(u) - f_i(0,0,\ldots,0) = Z_i(x,t)u_i(x,t), \quad i = 1,2,\ldots,m.$$

Thus equation (2.6) can be written in the form

$$\begin{aligned}\frac{\partial v_i(x,t)}{\partial t} &= a_i\nabla^2 v_i(x,t) - \beta_i v_i(x,t) + Z_i(x,t)v_i(x,t)\\ &+X_i(t)f_i(0,0,\ldots,0).\end{aligned}$$

From the strong maximum principle for linear parabolic inequalities, we get

$$v_i(x,t) \geq 0, \quad \bar{G}\times[0,b], \quad i = 1,2,\ldots,m$$

So,

$$u_i(x,t) = Y_i(t)v_i(x,t) \geq 0, \quad \bar{G}\times[0,b], \quad i = 1,2,\ldots,m$$

From Theorems (2.1) and (2.2) we can say that the stochastic model (2.3) with the homogeneous Neumann boundary conditions (2.4) is suitable for predictions.

Let $\mathbb{E}(\xi)$ denote the expectation of the random variable ξ. Suppose that $f_i(0,0,\ldots,0) \geq 0$ and $\varphi_i \geq 0$, for all $i = 1,2,\ldots,m$, $x \in \bar{G}$. □

Theorem 2.3. *If the Wiener processes $W_1,\ldots,W_m$ are independent, then*

$$\begin{aligned}\sum_{i=1}^{m}\sup_x \mathbb{E}[u_i(x,t)] &\leq e^{-(c_0-M)t}\sum_{i=1}^{m}\parallel\varphi_i\parallel\\ &+\frac{1}{c_0-M}[1-e^{-(c_0-M)t}]\sum_{i=1}^{m}f_i(0,0,\ldots,0), \end{aligned} \tag{2.11}$$

where $c_0 = min(c_1,\ldots,c_m)$ and $\parallel \varphi_i \parallel = \sup_x \mid \varphi_i(x) \mid$. The supremum is taken over $\bar{G}$.

Proof. We have,

$$\mathbb{E}[\int_0^t u_i(x,\theta)dW_i(\theta)] = 0, \quad \forall x \in \bar{G}, t \geq 0 \quad i = 1,2,\ldots,m.$$

Consequently,

$$\frac{\partial}{\partial t}\mathbb{E}[u_i(x,t)] = a_i\nabla^2\mathbb{E}[u_i(x,t)] - c_i\mathbb{E}[u_i(x,t)] + \mathbb{E}[f_i(u)].$$

The expectations $\mathbb{E}[u_1(x,t)],\ldots,\mathbb{E}[u_m(x,t)]$ are given by:

$$\begin{aligned} e^{c_i t}\mathbb{E}[u_i(x,t)] &= \sum_{i=1}^{m} e^{-a_i\lambda_k^2 t}\gamma_{ki}V_k(x) \\ &\quad + \int_0^t\int_G e^{-a_i\lambda_k^2(t-\theta)}V_k(x)V_k(y)e^{c_i\theta}\mathbb{E}[f_i(u(y,\theta))dyd\theta. \end{aligned}$$

Consider the following simple problem:

$$\begin{aligned} \frac{dZ(t)}{dt} &= -(c_0-M)Z(t)+\sum_{i=1}^{m} f_i(0,0,\ldots,0) \\ Z(0) &= \sum_{i=1}^{m} \parallel \varphi_i \parallel . \end{aligned}$$

It is easy to see that

$$Z(t)=e^{-(c_0-M)t}\sum_{i=1}^{m} \parallel \varphi_i \parallel +\frac{1}{c_0-M}[1-e^{-(c_0-M)t}]\sum_{i=1}^{m} f_i(0,0,\ldots,0).$$

Remembering that the vector f satisfies the Lipschitz condition (2.2), we get the required result. □

If $c_0 > M$, we can establish the asymptotic stability of the expectations $\mathbb{E}[u_i(x,t)]$, $i=1,2,\ldots,m$. In fact, using (2.11), we get

$$\sum_{i=1}^{m}\sup_x \mathbb{E}[u_i(x,t)-u_i^*] \leq e^{-(c_0-M)t}\sum_{i=1}^{m} \parallel \varphi_i-\varphi_i^* \parallel,$$

where u_i and u_i^* are solutions of (2.3), with the initial conditions $u_i(x,0)=\varphi_i(x)$, $u_i^*(x,0)=\varphi_i^*(x)$ and the homogeneous boundary conditions (2.4).

We notice also that the zero solutions of the expectations are also asymptotically stable, (In this case $f_i(0,\ldots,0)=0$, $i=1,2,\ldots,m$).

3. Fractional epidemic model

In this section, we shall study a fractional epidemic diffusion system of the form:

$$\frac{\partial^\alpha u_i(x,t)}{\partial t^\alpha}=a_i\nabla^2 u_i(x,t)-c_iu_i(x,t)+f_i(u)$$

with the boundary conditions (2.4), where $0<\alpha\leq 1$, ([12]- [15], [32]). Compare also ([16]- [19]).

The integral form of system (12) is given in the form:

$$u_i(x,t) = \varphi_i(x) + \frac{1}{\Gamma(\alpha)}\int_0^t [a_i\nabla^2 u_i(x,t) - c_i u_i(x,t) + f_i(u)](t-\theta)^{\alpha-1}d\theta, \tag{3.1}$$

where $(x,t) \in G \times [0,b]$, $u = (u_1,\ldots,u_m)$, $\Gamma(\cdot)$ is the gamma function, $0 < \alpha \leq 1$ and

$$\frac{\partial u_i(x,t)}{\partial \nu} \mid_{\partial G} = 0, \quad i = 1,2,\ldots,m. \tag{3.2}$$

To solve the problem ((3.1),(3.2)), we set

$$u_i(x,t) = \sum_{k=0}^{\infty} \eta_{ki}(t)V_k(x), \tag{3.3}$$

where

$$\eta_{ki}(t)V_k(x) = \int_G u_i(y,t)V_k(y)dy.$$

Thus,

$$\frac{d^\alpha \eta_{ki}(t)}{dt^\alpha} = -(a_i\lambda_k^2 + c_i)\eta_{ki}(t) + \int_G f_i(u(y,t))V_k(y)dy.$$

Using ([9]), one gets

$$\eta_{ki}(t) = \int_0^\infty \zeta_\alpha(\theta)Q_{ik}(t^\alpha\theta)\gamma_{ki}d\theta + \int_0^t\int_0^\infty\int_G \mathbb{U}dyd\theta d\eta, \tag{3.4}$$

where

$$\mathbb{U} = \alpha\theta(t-\eta)^{\alpha-1}\zeta_\alpha(\theta)Q_{ik}((t-\eta)^\alpha\theta)V_k(y)f_i(u(y,\eta))$$

and

$$Q_{ik}(t^\alpha\theta) = e^{-(a_i\lambda_k^2+c_i)t^\alpha\theta}$$

and $\zeta_\alpha(\theta)$ is a probability density function, with the Laplace transform

$$\int_0^\infty e^{-pt}\zeta_\alpha(t)dt = E_\alpha(-p), \tag{3.5}$$

where

$$E_\alpha(p) = \sum_{j=0}^{\infty}\frac{p^j}{\Gamma(1+\alpha j)}$$

is the Mittag-Lefler function,

$$\gamma_{ki} = \int_G \varphi_i(y)V_k(y)dy.$$

Using (3.5), equation (3.4) can be written in the form

$$\begin{aligned} \eta_{ki}(t) &= \gamma_{ki}E_\alpha[-(a_i\lambda_k^2+c_i)t^\alpha] \\ &+ \int_0^t\int_G (t-\eta)^{\alpha-1}V_k(y)f_i(u(y,\eta))E_\alpha[-(a_i\lambda_k^2+c_i)t^\alpha]dyd\eta. \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.3) and remembering that f satisfies the Lipschitz condition (2.2), we get an equation of the type Volterra, which can be solved in the space $(\mathbb{C}(\bar{G}), \|\cdot\|)$, see ([10]).

Let us prove the stability of the solutions of the problem ((3.1),(3.2)).

Theorem 3.1. *Let u and u^* be two solutions of* (3.1)*, with the initial conditions $u_i(x,0)=\varphi_i(x)$, $u_i^*(x,0)=\varphi_i^*(x)$. Suppose that u and u^* satisfy the boundary conditions* (3.2)*. If*

$$\sum_{i=1}^m \|\ \varphi_i(\cdot)-\varphi_i^*(\cdot)\ \|\le \varepsilon,$$

then,

$$\sum_{i=1}^m \|\ u_i(\cdot,t)-u_i^*(\cdot,t)\ \|\le M\varepsilon e^{\lambda t}$$

for some positive constants M and λ, $\|\ u_i(\cdot,t)\ \|=\sup_x\ |\ u_i(x,t)\ |$, ε is sufficiently small positive number.

Proof. According to the Lipschitz condition (2.2) and the properties of the eigenfunctions, eigenvalues and the properties of the Mittag-Lifler function, we get

$$\sum_{i=1}^m \|\ u_i(\cdot,t)-u_i^*(\cdot,t)\ \|\le M\varepsilon + M\int_0^t (t-\eta)^{\alpha-1}\sum_{i=1}^m \|\ u_i(\cdot,\eta)-u_i^*(\cdot,\eta)\ \|\ d\eta$$

for some constant $M>0$. Set

$$\rho=\max_{t\in J}[e^{-\lambda t}]\sum_{i=1}^m \|\ u_i(\cdot,t)-u_i^*(\cdot,t)\ \|\ .$$

Thus,

$$\begin{aligned} \sum_{i=1}^m \|\ u_i(\cdot,t)-u_i^*(\cdot,t)\ \| &\le M\varepsilon + M\rho\int_0^{t-\frac{1}{\lambda}}(t-\eta)^{\alpha-1}e^{\lambda\eta}d\eta \\ &+ M\rho\int_{t-\frac{1}{\lambda}}^t (t-\eta)^{\alpha-1}e^{\lambda\eta}d\eta. \end{aligned}$$

So,

$$e^{-\lambda t}\sum_{i=1}^{m} \parallel u_i(\cdot,t)-u_i^*(\cdot,t)\parallel\leq M[\frac{1}{\lambda}]^{\alpha}[1+\frac{1}{\alpha}]\rho.$$

Choose λ sufficiently large such that $M[\frac{1}{\lambda}]^{\alpha}[1+\frac{1}{\alpha}]<1$, we get $\rho\leq M\varepsilon$, for some constant $M>0$. Thus

$$\sum_{i=1}^{m} \parallel u_i(\cdot,t)-u_i^*(\cdot,t)\parallel\leq M\varepsilon e^{\lambda t}.$$

Hence the required result. □

If for some i, say $i=1$, $\mid f_1(u)\mid\leq M$, for all $u=(u_1,\ldots,u_m)$. Then from (3.4), we can see that

$$\parallel u_1(\cdot,t)\parallel\leq\parallel\varphi_1(\cdot)\parallel E_{\alpha}(-c_1t^{\alpha})+\frac{\gamma}{c_1}[1-E_{\alpha}(-c_1t^{\alpha})].$$

If the recruitment $\gamma=0$ in the model of COVID-19, then it can be proved that

$$\parallel u_1(\cdot,t)\parallel\leq\parallel\varphi_1(\cdot)\parallel E_{\alpha}(-c_1t^{\alpha}),$$

which tends to zero as t tends to infinity, ([20]- [31]).

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REFERENCES

[1] D. I. DE ANGELIS, R. A. GOLDSTEN, R. NEILL: *A model for rophic interaction*, Ecology , **56** (1975), 881–892.

[2] D. S. HUI, E. I. AZHAR, T. A. MADANI, F. NTOUMI, R. KOCK, O. DAR, ET AL.: *The continuing 2019-nCoV epidemic threat of novel coronaviruses to global health-The latest 2019 novel coronavirus outbreak in Wuhan*, China, Int. J. Infect. Dis., **91** (2020), 264–266.

[3] E. M. LOTFI, M. MAZIANE, K. HATTAF, N. YOUSFI: *Partial Differential Equations of an Epidemic Model with Spatial Diffusion*, International Journal of Partial Differential Equations, **2014**, Article ID: 186437, 1–6.

[4] G. HUANG, L. LI: *A Mathematical model of infections diseases*, Ann. Oper. Res., **168** (2009), 41–80.

[5] H. W. HETHCOTE: *Asympototic behavior and stability in epidemic models*, In Lecture notes in biomathematics, **2** (1974), BerlinSpringer, 83–92.

[6] H. W. HETHCOTE, H. W. STECH, P. VAN DEN DRIESSCHE: *Stability analysis for models of disease without immunity*, Journal of Mathematical Biology, **13** (1981), 185–198.

[7] J. COHEN, D. NORMILE: *New SARS-like virus in China triggers alarm*, Science, **367** (2020), 234–235.

[8] J. R. BEDDINGTON: *Mutual interperence between parasited or predators and its effect on searching efficiency*, Journal of Animal Ecology, **44**(1) (1975), 331–341.

[9] J. TIAN, J. WU, Y. BAO, X. WENG, L. SHI, B. LIU, X. YU, L. QI, Z. LIU: *Modeling analysis of COVID-19 based on morbidity data in Anhui, China*, Mathematical Biosciences and Engineering, **17**(4), 2842–2852.

[10] K. EL-SAID EL-NADI, M. EL-SHANDIDY, Y.H. OMRALRYANY: *On some stochastic growth models with applications*, International Journal of Mathematics and Statistics Studies, **8**(2) (2020), 40–50.

[11] M. E. KILLERBY, H. M. BIGGS, C. M. MIDGLEY, S. I. GERBER, J. T. WATSON: *Middle east respiratory syndrome coronavirus transmission*, Emerg. Infect. Dis., **26** (2020), 191–198.

[12] M. M. EL-BORAI: *Some probability densities and fundamental solution of fractional evolution equations*, Chaos, Soliton and Fractals, **14** (2002), 433–440.

[13] M. M. EL-BORAI: *Evolution equations without semigroups*, J. of Appl. Math. And Comp., **149** (2004), 815–821.

[14] M. M. EL-BORAI: *Semi groups and some nonlinear fractional differential equations* , J. of Appl. Math. And Comp., **149**(3) (2004), 823–831.

[15] M. M. EL-BORAI: *The fundamental solutions for fractional evolution equations of parabolic type* , J. of Appl. Math. Stochastic Analysis, **2004**(3) 2004, 197–211.

[16] M. M. EL-BORAI, H. K. AWAD, R. H. M. ALI: *A parabolic transform and averaging methods for general partial differential equations*, J. Adv. in Math., **17** (2019), 352–361.

[17] M. M. EL-BORAI, H. K. AWAD, R. H. M. ALI: *Method of averaging for some parabolic partial differential equations*, Academic Journal of Applied Mathematical Sciences, **6** (2020), 1–4.

[18] M. M. EL-BORAI, H. K. AWAD, R. H. M. ALI: *On averaging methods for general parabolic partial differential equation*, J. Math. Computer Sci., **21** (2020), 164–175.

[19] M. M. EL-BORAI, H. K. AWAD, R. H. M. ALI: *A parabolic transform and averaging methods for integro-partial differential equations*, J. Math. Computer Sci., **22** (2021), 9–15.

[20] M. M. EL-BORAI, K. EL-SAID EL-NADI, O. LABIB, M. HAMDY: *Volterra equations with fractional stochastic integrals*, Mathematical problems in Engineering, **5** (2004), 453–468.

[21] M. M. EL-BORAI, K. EL-SAID EL-NADI, O. LABIB: *Numerical methods for some nonlinear stochastic differential equations*, Applied Math. and Comp., **168** (2005), 65–75.

[22] M. M. EL-BORAI, K. EL-SAID EL-NADI, I. G. EL-AKABAWI: *On some integro- differential equations of fractional orders*, The International J. of Contemporary Mathematics, **1**(15) (2006), 719–726.

[23] M. M. EL-BORAI, K. EL-SAID EL-NADI, I. G. EL-AKABAWY: *Approximate solutions for nonlinear fractional heat equations*, Int. J. Contemp. Math. Sciences, **2**(27) (2007), 1307–1316.

[24] M. M. EL-BORAI, K. EL-SAID EL-NADI, I. G. EL-AKABAWY: *Fractional evolution equations with non-local conditions*, Journal of Applied Mathematics and Mechanics, **4**(6) (2008), 1–12.

[25] M. M. EL-BORAI, K. EL-SAID EL-NADI, H. A. FOAD: *On some fractional stochastic delay differential equations*, Computers and Mathematics with Applications, **59** (2010), 1165–1170.

[26] M. M. EL-BORAI, K. EL-SAID EL-NADI, E. G. EL-AKABAWY: *On some fractional evolution equations*, Computers and Mathematics with Applications, **59** (2010), 1352–1355.

[27] M. M. EL-BORAI, K. EL-SAID EL-NADI, H. A. FOUAD: *Backward stochastic differential equations on a fixed finite time interval*, Special Issue SCIENCE and Mathematics with Applications, International Journal of Research and reviews in Applied Sciences, **3**(6) (2011), 247–245.

[28] M. M. EL-BORAI, K. EL-SAID EL-NADI: *A Parabolic Transform and Some Stochastic Ill-Posed Problems*, British Journal of Mathematics Computer Science, **5**(9) (2015), 418–426.

[29] M. M. EL-BORAI, K. EL-SAID EL-NADI: *The Parabolic Transform and Some singular integral evolution equations*, Journal of Mathematics and Statistics, **8**(4) (2020), 410–415.

[30] M. M. EL-BORAI, K. EL-SAID EL-NADI, H. M. AHMED, EL-SAID EL-OWAIDY, A.S. GHANEM, R. SAKTHIVEL: *Existence and Stability for fractional prabolic integro-partial differential equations with fractional Brownian motion and nonlocal conditions*, Cogent Mathematics Statistics, **5**(1) (2018), 1–9.

[31] M. M. EL-BORAI, K. EL-SAID EL-NADI: *The parabolic transformation and some singular integral evolution equations*, HRPUB: Mathematics And Statistics, (4) (2020), 410–415.

[32] M. M. EL-BORAI, O. LABIB.: *Asymptotic stability of some stochastic evolution equations*, Applied Math. And Comp., **144** (2003), 273–286.

[33] M. TAHIR, S. INAYAT, A. SHAH, G. ZAMAN, T. KHAN: *Stability Behaviour of Mathematical Model MERS Corona Virus Spread in Population*, Filomat, **33**(12) (2019), 3947–3960.

[34] P. WU, X. HAO, E. H. Y. LAU, J. Y. WONG, K.S. LEUNG, J. T. WU, ET AL.: *Teal-time tentative assessment of the epidemiological characteristics of novel coronavirus infections in Wuhan China, as at 22 January 2020*, Euro. Surveill., **25** (2020).

[35] S. ZHAO, S. S. MUSA, Q. LIN, J. RAN, G. YANG, W. WANG, ET AL.: *Estimating the unreported number of novel coronavirus (2019-nCov) cases in China in the first half of January 2020: a data-driven modelling analysis of the early outbreak*, J. Clin. Med., **9**(2) (2020), art.no. 388.

[36] W. O. KERMACK, A.G. MCKENDRICK: *A contribution to the mathematical theory of epidemics*, Proceeding of the Royal Society of London, Series A, **115** (1927), 700–721.

Department of Mathematics and Computer Science
Faculty of Science, Alexandria University
Alexandria, Egypt.
Email address: m_m_elborai@yahoo.com

Department of Mathematics and Computer Science
Faculty of Science, Alexandria University
Alexandria, Egypt.
Email address: khairia_el_said@hotmail.com