

## ON STUDY OF GENERALIZED FORM OF CAMASSAHOLM EQUATION AND DEGASPERIS-PROCESI EQUATIONS BY REDUCED DIFFERENTIAL TRANSFORM ALGORITHM

JAVED HUSSAIN

**ABSTRACT.** In this work, we are concerned with approximate explicit closed-form solutions of the two highly nonlinear evolution equations modeling several physical phenomena, in particular, the dynamics of shallow water. Therefore the study is twofold. Firstly, we have constructed the approximate solution to the Camassa-Holm equation subject to some suitable choice of initial data from a suitable choice of initial condition from Sobolev space of initial data. Secondly, we have obtained an approximate solution to the Degasperis-Procesi equation. To deal with both equations we have employed an efficient series solution algorithm, known as the reduced differential transform Algorithm. The graphs of obtained approximate solutions turned out to be in agreement with the known abstract results in the literature.

### 1. INTRODUCTION

This paper aims to demonstrate that how reduced differential transform algorithm, a well-known series solution method to deal with nonlinear PDEs, can be applied to famously highly nonlinear integrable equations of two last two decades i.e. Camassa-Holm (CH) equation and Degasperis-Procesi (DP) equation. In particular, the method allows to construct closed form approximate series solution to

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mentioned equations. The key problem of investigation in this paper is going to be following,

$$(1.1) \quad \begin{aligned} (1 - \partial_x^2)u_t &= F(u, u_x, u_{xx}, u_{xxx}), & (t, x) \in \mathbb{R}^+ \times \mathbb{R} \\ u(x, 0) &= u_0(x), & x \in \mathbb{R}, \end{aligned}$$

where initial data  $u_0$  lives in Sobelov space  $\mathbb{H}^3(\mathbb{R})$ , and  $F$  is a homogeneous polynomial. There are several ways we can look at Problem (1.1). In particular for following choice of  $F$  the initial value problem (1.1). referred as Generalized form of Camassa-Holm equation,

$$F(u, u_x, u_{xx}, u_{xxx}) = -3uu_x + 2u_xu_{xx} + uu_{xxx},$$

and for following choice of  $F$  the initial value problem becomes Generalized Form of the Degasperis-Procesi equation,

$$F(u, u_x, u_{xx}, u_{xxx}) = 4uu_x - 3u_xu_{xx} - uu_{xxx}.$$

This study gets its motivation from [1, 2020], where the authors studied the abstract global weak existence and uniqueness of problem (1.1) by arguing through viscosity vanishing method, also it was proved the stability of weak solutions when solutions are of higher integrability. There is a vast literature available for CH equation in concrete and abstract setting. One of prominent physical interpretation of the CH equations is that it models the shallow water equation over flat base/sheet, but CH equation equations also have applications to cellular Biology. The peakon behaviour of the CH equation makes it an interesting physical model. In [3, 1993] and [4, 1994] it was shown that CH equations possess a smooth and travelling peakon solution. Fokas in [5, 1995] shown that CH equation can also be treated as member of the Bi-Hamiltonian hierarchy of equations. There is a large literature available on construction of soliton solution of the CH equation. In [6, 2017] Rasin employed Backlund transform method, also Constantin in [10, 2009] used scattering wave method to study the soliton solution of CH equation. Recently, [Rosen 2020] applied the dressing method to construct the global solution of CH cuspon equation.

The DP equation is basically is closely related CH equation that was derived by Dega-Processi [7, 1999] using an asymptotic integrability approach. They also shown that DP equation possess bi-Hamiltonian structure and have smooth peakon

solution. Physically DP equation can be treated as an approximation to incompressible Euler equation for the dynamics of shallow water. There has been variety of studies on DP equation. A travelling wave solution was studied in [8, 2005] and [9, 2002]. In [12, 2006] Liu studied local well-posedness, global existence and blow up criterion. The long term behaviour of DP equation was studied in [11, 2018]. Long in [13, 2020] have recently studied the symmetries of solitary waves of DP equation.

DP and CH equation have several properties in common and differ by several aspect. Both equations have same asymptotic accuracy and bi-Hamiltonian structure. CH equation in reformulation of geodesic flow and diffeomorphic group cf. [14, 1999]. While there is no geometric derivation for the DP equation. Another interesting fact that makes DP equation different from CH equations that the DP equation has shock peakon shock waves, see [15, 2019], where no such result is available for CH equation.

## 2. BRIEF DESCRIPTION OF REDUCED DIFFERENTIAL TRANSFORM ALGORITHM

This section is aimed to introduce the reduced differential transform algorithm to solve nonlinear evolution equation. We will define RDTA and list its some of its basic definitions and important properties.

### 2.1. Definition and properties of differential transform.

Let us start by defining the 1-D reduced differential transform of a smooth (i.e.  $C^\infty(\mathbb{R})$ ) function.

**Definition 2.1.** [2] Suppose function  $u(t, x)$  be  $C^\infty(\mathbb{R})$ -function and is analytic, then then the differential transform of  $u(t, x)$  is following,

$$u_k(x) = \frac{1}{k!} \frac{\partial^k u}{\partial t^k}(t, x) \Big|_{t=0}, \quad k = 0, 1, 2, 3, \dots,$$

where  $u_k(x)$  can be treated as the the  $t$ -dimensional spectrum transformed function. The differential inverse transform of  $u_k(x)$  is defined as follows:

$$u(t, x) = \sum_{k=0}^{\infty} u_k(x) t^k = \sum_{k=0}^{\infty} \left( \frac{1}{k!} \frac{\partial^k u}{\partial t^k}(t, x) \Big|_{t=0} \right) t^k.$$

Based on above we have the following theorem listing the basic properties of reduce differential transform,

**Theorem 2.1.** [2] For any smooth functions  $u, v$  the reduce differential transform of  $u$  and  $v$  satisfies following properties,

- (i) *Linearity:* For any linear combination of  $u$  and  $v$ , i.e  $w(t, x) = au(t, x) + bv(t, x)$ , where  $a, b \in \mathbb{R}$ , the reduced differential transform is  $W_k(x) = aU_k(x) + V_k(x)$ ,  $k \in \mathbb{N}$ . where  $u_k, v_k$  and  $w_k$  are differential transforms of  $u, v$  and  $w$  respectively.
- (ii) If  $u(t, x) = x^m t^n$  then reduce differential transform of  $u$  is  $u_k(x) = x^m \delta(k - n)$ ,  $k \in \mathbb{N}$ .
- (iii) If  $v(x, t) = x^m t^n u(x, t)$  then reduce differential transform of  $v$  is  $v_k(x) = x^m u_{k-n}(x)$ ,  $k \in \mathbb{N}$ .
- (iv) If  $w(x, t) = u(x, t)v(x, t)$  then then reduce differential transform of  $w$  is,

$$W_k(x) = \sum_{r=0}^k V_r(x) U_{k-r}(x), \quad k \in \mathbb{N}.$$

- (v) If  $v(x, t) = \frac{\partial^r}{\partial t^r} u(x, t)$  then then reduce differential transform of  $v$  is,

$$v_k(x) = \frac{(k+r)!}{k!} u_{k+r}(x), \quad k \in \mathbb{N}.$$

- (vi) Space derivative of  $u$  is invariant under differential transform, more precisely, if  $v(t, x) = \frac{\partial u}{\partial x}(x, t)$  then reduce differential transform of  $v$  is  $v_k(x) = \frac{\partial u_k}{\partial x}(x)$ .

## 2.2. Application of differential transform to nonlinear evolution equations: A generic algorithm.

Consider the following nonlinear evolution equation,

$$\begin{aligned} u_t(t, x) &= Au(t, x) + Bu(t, x) + f(t, u(t, x)), \\ (2.1) \quad u(0, x) &= h(x), \end{aligned}$$

where  $A$  is linear operator,  $B$  is nonlinear linear operator and  $f$  is some linear smooth function of  $x$ . Suppose that variables can be separated i.e.  $u(t, x)$  can be written as product of functions of  $x$  and  $t$  i.e.,  $u(t, x) = f(x)g(t)$ , where  $f(x)$  and  $g(t)$  smooth functions of space and time variables, respectively. Based on the properties of one dimensional differential transform, the function  $u(t, x)$  can be represented as follows:

$$u(t, x) = \left( \sum_{i=0}^{\infty} F(i)x^i \right) \left( \sum_{j=0}^{\infty} G(j)t^j \right) = \sum_{k=0}^{\infty} u_k(x)t^k,$$

where  $u_k(x)$  is called  $t$ -dimensional spectrum function of  $u(t, x)$ .

Using Definition 2.1 and suitable properties from Theorem 2.1 we may take differential transform of problem (2.1) we get the following algorithm consisting of recursive set of equations,

$$\begin{aligned}(k+1)u_{k+1}(x) &= Au_k(x) + Bu_k(x) + F(u_k(x)), \quad k = 1, 2, 3, \dots \\ u_0(x) &= h(x),\end{aligned}$$

where  $Au_k(x)$ ,  $Bu_k(x)$ , and  $F(u_k(x))$  are the transformations of the functions  $Au(t, x)$ ,  $Bu(t, x)$ ,  $f(t, x)$  and respectively. Using the above relation one can compute the  $u_k$  from  $u_{k-1}$ , for all  $k = 1, 2, 3, \dots$  and get sequence of smooth function  $(u_k(x))_{k=0}^{\infty}$ . Then the series solution of the evolution equation (2.1) can be recovered from the following inverse differential transform

$$\tilde{u}_n(t, x) = \sum_{k=0}^n u_k(x) t^k.$$

Therefore, the exact solution of the problem is given by

$$(2.2) \quad u(t, x) = \lim_{n \rightarrow \infty} \tilde{u}_n(t, x) = \sum_{k=0}^{\infty} u_k(x) t^k.$$

**2.3. Convergence of RDT Algorithm.** We now an interesting recent results from [16, 2020].

**Theorem 2.2.** *The solution series  $\sum_{k=0}^{\infty} u_k(x) t^k$ , described in (2.2), converges, if there exists  $\gamma \in (0, 1)$  such that  $\|u_{k+1}(x) t^{k+1}\| \leq \gamma \|u_k(x) t^k\|$ , for all  $k \in \mathbb{N} \cup \{0\}$ . Moreover, if  $\sum_{k=0}^{\infty} u_k(x) t^k$  converges to  $u(t, x)$  then error between the truncated sum  $\sum_{k=0}^n u_k(x) t^k$  and  $u(t, x)$  can be controlled by following inequality,*

$$\|u(t, x) - \sum_{k=0}^n u_k(x) t^k\| \leq \frac{\gamma^{n+1}}{\gamma - 1} \|u_0\|.$$

### 3. GENERALIZED FORM OF CAMASSA-HOLM EQUATION

In this we aim to deal with the following Camassa-Holm equation,

$$\begin{aligned}u_t &= u_{txx} - 3uu_x + 2u_x u_{xx} + uu_{xxx}. \\ u(0, x) &= u_0(x).\end{aligned}$$

### 3.1. RDT Algorithm for Generalized Form of Camassa–Holm Equation.

Consider,

$$u_t = u_{txx} - 3uu_x + 2u_xu_{xx} + uu_{xxx}.$$

On application of properties of reduced differential transform we get the following infinite recursive sequence of second order ordinary differential equations,

$$(3.1) \quad (k+1)u_{k+1} = (k+1)\frac{\partial^2 u_{k+1}}{\partial x^2} - 3A_k + 2B_k + C_k,$$

where  $t$  is dimensional spectrum function and  $u_k(x)$  is transformed function.  $A_k$ ,  $B_k$  and  $C_k$  are transformed for nonlinear terms, can be given as,

$$A_k = \sum_{r=0}^k u_r(x) \frac{\partial u_{k-r}}{\partial x}(x), \quad B_k = \sum_{r=0}^k \frac{\partial u_r}{\partial x} \frac{\partial^2 u_{k-r}}{\partial x^2}, \quad M_k = \sum_{r=0}^k u_r(x) \frac{\partial^3 u_{k-r}}{\partial x^3}(x).$$

Solving this sequence of sequence of differential equations subject to some initial condition of Sobelov space  $\mathbb{H}^3(\mathbb{R})$  would reveal the solution of the Camassa-Holm equation.

### 3.2. Numerical Example.

To explicitly construct a solution we propose following initial condition to follow,

$$u_0 = x(4 + 2\sin(x)).$$

Indeed,  $u_0 \in C_0^\infty(\mathbb{R})$  (space of smooth function with compact support) so its weak and strong derivative agree. Hence  $u$  and all partial derivatives upto order 3, are square integrable, therefore,  $u \in H^3(\mathbb{R})$ . Using proposed initial condition above let us compute the first, second and third partial derivatives of it,

$$(3.2) \quad \begin{aligned} \frac{\partial u_0}{\partial x} &= 4 + 2\sin(x) + 2x\cos(x), \\ \frac{\partial^2 u_0}{\partial x^2} &= 4\cos(x) - 2x\sin(x), \\ \frac{\partial^3 u_0}{\partial x^3} &= -6\sin(x) - 2x\cos(x). \end{aligned}$$

Let start running algorithm (3.1).

**Step  $k = 0$ :** On substituting  $k = 0$  in relation (3.1)

$$u_1 = \frac{\partial^2 u_1}{\partial x^2} - 3A_0 + 2B_0 + C_0.$$

From (3.2) and partial derivatives (3.2) we reach at the following second order differential equations satisfied by  $u_1$  as,

$$\begin{aligned} u_1 &= \frac{\partial^2 u_1}{\partial x^2} - 3u_0 \frac{\partial u_0}{\partial x} + 2 \frac{\partial u_0}{\partial x} \frac{\partial^2 u_0}{\partial x^2} + u_0 \frac{\partial^3 u_0}{\partial x^3} \\ u_1 &= \frac{\partial^2 u_1}{\partial x^2} - 3 \left( 4x(\sin(x) \cos(x)x - \cos^2(x) + 2x \cos(x) + 4 \sin(x) + 5) \right) + \\ &\quad + 2 \left( -4 \sin(x) \cos(x)x^2 + 12 \cos^2(x)x + 8 \sin(x) \cos(x) \right. \\ &\quad \left. - 8x \sin(x) + 16 \cos(x) - 4x \right) \\ &\quad + \left( -4x(\sin(x) \cos(x)x - 3 \cos^2(x) + 2x \cos(x) + 6 \sin(x) + 3) \right). \end{aligned}$$

For sake the sake of simplicity we will always take arbitrary constants in the solution of differential equation equal to 1. On solving (3.2) we get  $u_1$  as solution as follows,

$$\begin{aligned} u_1 &= -16 \cos(x)x^2 - \frac{12}{5}x^2 \sin(2x) - 12x \sin(x) + \frac{24}{25}x \cos(2x) \\ &\quad - 12 \cos(x) - \frac{16}{125} \sin(2x) - 56x + e^x + e^{-x}. \end{aligned}$$

**Step  $k = 1$ :** On substituting  $k = 1$  in (3.1) it follows that  $u_2$  satisfies following differential equation,

$$(3.3) \quad 2u_2 = 2 \frac{\partial^2 u_2}{\partial x^2} - 3A_1 + 2B_1 + C_1.$$

To compute solution  $u_2$  from above o.d.e, and for this we need to compute  $A_1$ ,  $B_1$  and  $C_1$  using (3.2). On substituting  $A_1$ ,  $B_1$  and  $C_1$ , in differential equation (3.3), and solving o.d.e we get following solution,

$$\begin{aligned} u_2 &= \frac{816}{25}x^3 \cos(2x) - \frac{338}{5} \sin(x)x^3 - \frac{22}{625} \cos(3x) + e^x + e^{-x} \\ &\quad + \frac{18}{5}x^3 \sin(3x) - \frac{18}{5}x^2 \cos(3x) - \frac{48}{125}x \sin(3x) + x^2 e^{-x} \\ &\quad + x^2 e^x - 2x e^x \cos(x) - 2x e^{-x} \cos(x) + \frac{192}{5}x^2 \sin(2x) + \frac{5288}{625}x \cos(2x) \\ &\quad + \frac{2008}{3125} \sin(2x) - \frac{7984}{125}x \sin(x) + 736x + \frac{6125}{25} \cos(x)x^2 + 14 \cos(x). \end{aligned}$$

**Step  $k = 2$ :** On substituting  $k = 2$  in (3.1) it follows that  $u_2$  satisfies following differential equation,

$$(3.4) \quad 3u_3 = 2 \frac{\partial^2 u_3}{\partial x^2} - 3A_2 + 2B_2 + C_2.$$

On substituting  $A_2$ ,  $B_2$  and  $C_2$ , in differential equation (3.4), and solving differential equations we get following solution,

$$\begin{aligned}
 u_3 &= \frac{4}{3}xe^{-x}\sin(x) - \frac{1}{3}x^2\sin(2x)e^{-x} + \frac{14528}{75}\cos(x)x^4 + \frac{28992}{125}\sin(2x)x^4 \\
 &- \frac{4}{3}xe^x\sin(x) + \frac{28}{3}xe^x\cos(x) + \frac{7}{2}x^2\cos(2x)e^x + \frac{8}{3}x\sin(2x)e^x \\
 &+ \frac{1}{3}x^2\sin(2x)e^x - 16x^2e^x\sin(x) + \frac{8}{3}x^2e^x\cos(x) - \frac{77}{6}x^2e^x - \frac{1}{3}\cos(2x)e^{-x} \\
 &+ \frac{8}{3}e^{-x}\cos(x) - \frac{19188712112318825041}{82463372}x^2\sin(2x) - \frac{8}{3}e^x\cos(x) + \frac{5}{9}x^3e^{-x} \\
 &+ \frac{8510219998986239}{257698037760000}x^2\cos(3x) - \frac{1079016733103425529}{82433720832000}x^3\sin(3x) - \frac{4}{3}x^3e^{-x}\cos(x) \\
 &- \frac{4}{3}x^3e^x\cos(x) + \frac{29614860501655403}{82463372088832}x\sin(x) - \frac{752540942421983317}{1030792151040000}\cos(2x)x^3 \\
 &+ \frac{33833072298295297}{20615843020800}\sin(x)x^3 - \frac{354737283648905411}{164926744166400}\cos(x)x^2 \\
 &- \frac{11780666025263261977}{274877906944000000}x\cos(2x) - \frac{77}{6}x^2e^{-x} + \frac{8}{3}e^{-x}\sin(x) \\
 &+ \frac{8}{3}e^x\sin(x) - \frac{3}{2}xe^{-x} + \frac{4129920485163846803}{1080382697600012545} + x^2\sin(4x) + \frac{75456}{7225}x^3\cos(4x) \\
 &- \frac{77814038786637548273}{275469319904378880000}x\cos(4x) - \frac{528}{85}x^4\sin(4x) + \frac{4620880867666611294}{219902325552000000}\cos(3x) \\
 &- \frac{21831199193678856571}{4123168604160000000}\sin(2x) + \frac{228073544291101107}{351843720888200}\cos(x) + \frac{1}{4}e^{-x} \\
 &+ \frac{290421834767264544259}{117086196}\sin(4x) + \frac{1}{4}e^x + \frac{1}{6}x\cos(2x)e^{-x} + \frac{8}{3}x\sin(2x)e^{-x} \\
 &- 16x^2e^{-x}\sin(x) + \frac{7}{2}x^2\cos(2x)e^{-x} + \frac{28}{3}xe^{-x}\cos(x) - \frac{487481674314023017}{51539552000}x \\
 &- \frac{1728}{25}\cos(3x)x^4 + \frac{1}{3}x^4e^{-x} - 8\frac{5}{9}x^3e^x - \frac{1}{6}x\cos(2x)e^x + \frac{79164837199869}{1374385347200}x^3.
 \end{aligned}$$

On substituting the values of  $u_0$ ,  $u_1$ ,  $u_2$  and  $u_3$  in the series of inverse differential transform, we recover the exact series solution of the problem (2.1) as follows,

$$\begin{aligned}
u(t, x) &= \sum_{k=0}^{\infty} u_k(x) t^k, \\
&= x(4 + 2 \sin(x)) + \left( -16 \cos(x) x^2 - \frac{12}{5} x^2 \sin(2x) - 12x \sin(x) + \frac{24}{25} x \cos(2x) \right) t \\
&+ \left( \begin{aligned} &\frac{816}{25} x^3 \cos(2x) - \frac{338}{5} \sin(x) x^3 - \frac{22}{625} \cos(3x) + e^x + e^{-x} + \frac{18}{5} x^3 \sin(3x) \\ &- \frac{18}{5} x^2 \cos(3x) - \frac{18}{5} x^2 \cos(3x) - \frac{48}{125} x \sin(3x) + x^2 e^{-x} + \frac{5288}{625} x \cos(2x) \\ &+ \frac{2008}{3125} \sin(2x) - \frac{7984}{125} x \sin(x) + 736x + \frac{6125}{25} \cos(x) x^2 + 14 \cos(x) \end{aligned} \right) t^2 \\
&+ \left( \begin{aligned} &\frac{4}{3} x e^{-x} \sin(x) - \frac{1}{3} x^2 \sin(2x) e^{-x} + \frac{14528}{75} \cos(x) x^4 + \frac{28992}{125} \sin(2x) x^4 \\ &- \frac{4}{3} x e^x \sin(x) + \frac{28}{3} x e^x \cos(x) + \frac{7}{2} x^2 \cos(2x) e^x + \frac{8}{3} x \sin(2x) e^x \\ &+ \frac{1}{3} x^2 \sin(2x) e^x - 16x^2 e^x \sin(x) + \frac{8}{3} x^2 e^x \cos(x) - \frac{77}{6} x^2 e^x - \frac{1}{3} \cos(2x) e^{-x} \\ &+ \frac{8}{3} e^{-x} \cos(x) - \frac{19188712112318825041}{82463372} x^2 \sin(2x) - \frac{8}{3} e^x \cos(x) + \frac{5}{9} x^3 e^{-x} \\ &+ \frac{8510219998986239}{257698037760000} x^2 \cos(3x) - \frac{1079016733103425529}{82433720832000} x^3 \sin(3x) - \frac{4}{3} x^3 \\ &e^{-x} \cos(x) - \frac{4}{3} x^3 e^x \cos(x) + \frac{29614860501655403}{82463372088832} x \sin(x) - \frac{752540942421983317}{1030792151040000} \\ &\cos(2x) x^3 + \frac{33833072298295297}{20615843020800} \sin(x) x^3 - \frac{354737283648905411}{164926744166400} \cos(x) x^2 \\ &- \frac{11780666025263261977}{274877906944000000} x \cos(2x) - \frac{77}{6} x^2 e^{-x} + \frac{8}{3} e^{-x} \sin(x) \\ &+ \frac{8}{3} e^x \sin(x) - \frac{3}{2} x e^{-x} + \frac{4129920485163846803}{1080382697600012545} + x^2 \sin(4x) + \frac{75456}{7225} x^3 \cos(4x) \\ &- \frac{77814038786637548273}{275469319904378880000} x \cos(4x) - \frac{528}{85} x^4 \sin(4x) + \frac{4620880867666611294}{2199023255552000000} \cos(3x) \\ &- \frac{21831199193678856571}{4123168604160000000} \sin(2x) + \frac{228073544291101107}{351843720888200} \cos(x) + \frac{1}{4} e^{-x} \\ &+ \frac{290421834767264544259}{117086196} \sin(4x) + \frac{1}{4} e^x + \frac{1}{6} x \cos(2x) e^{-x} + \frac{8}{3} x \sin(2x) e^{-x} \\ &- 16x^2 e^{-x} \sin(x) + \frac{7}{2} x^2 \cos(2x) e^{-x} + \frac{28}{3} x e^{-x} \cos(x) - \frac{487481674314023017}{51539552000} x \\ &- \frac{1728}{25} \cos(3x) x^4 + \frac{1}{3} x^4 e^{-x} - 8 \frac{5}{9} x^3 e^x - \frac{1}{6} x \cos(2x) e^x + \frac{79164837199869}{1374385347200} x^3 \end{aligned} \right) t^3 \dots
\end{aligned}$$

#### 4. GENERLIZED FORM OF DEGASPERIS-PROCESI EQUATION

In this section we aim to deal with following version of Degasperis-Procesi Equation

$$\begin{aligned}
(1 - \partial x)^2 u_t &= 4uu_x - 3u_x u_x u_{xx} - uu_{xxx} \\
u(0, x) &= u_0(x),
\end{aligned}
\tag{4.1}$$

where  $u_0 \in \mathbb{H}^3(\mathbb{R})$ .

#### 4.1. RDT Algorithm for Generalized form of Degasperis-Procesi Equation.

Let us rewrite evolution equation (4.1), in following form,

$$\begin{aligned} u_t &= 4u_{txx} + uu_x - 3u_x u_{xx} - uu_{xxx} \\ (4.2) \quad u(0, x) &= u_0(x). \end{aligned}$$

On application of reduced differential transform properties, we get the following infinite set of recursive second order differential equations,

$$(4.3) \quad (k+1) u_{k+1} = (k+1) \frac{\partial^2 u_{k+1}}{\partial x^2} + 4A_k - 3B_k - C_k.$$

$u_k(x)$  is transformed function and dimensional spectrum function is  $t$ , and

$$A_k = \sum_{r=0}^k u_r u_{k-r}, \quad B_k = \sum_{r=0}^k \frac{\partial}{\partial x} u_r \frac{\partial^2}{\partial x^2} u_{k-r}, \quad S_k = \sum_{r=0}^k u_r \frac{\partial^3}{\partial x^3} u_{k-r}.$$

**4.2. Numerical Example.** Let us consider the following initial condition,

$$u_0 = (x+1)e^x.$$

Its not difficult to verify  $u_0 \in C_0^\infty(\mathbb{R})$  so the weak and strong derivatives of  $u_0$  agree. Hence  $u_0$  and its partial derivatives up to order 3 are square integrable, therefore,  $u_0 \in H^3(\mathbb{R})$ . Using proposed initial condition above let us compute the first, second and third partial derivatives of it,

$$(4.4) \quad \frac{\partial u_0}{\partial x} = (x+2)e^x, \quad \frac{\partial^2 u_0}{\partial x^2} = (x+3)e^x, \quad \frac{\partial^3 u_0}{\partial x^3} = (x+4)e^x.$$

**Step  $k = 0$ :** On substituting  $k = 0$  and above partial derivatives of  $u_0$  from (4.4), in (4.3) it follows that  $u_1$  satisfies following 2nd order o.d.e.

$$\begin{aligned} u_1 &= \frac{\partial^2 u_1}{\partial x^2} + 4A_0 - 3B_0 - S_0 = \frac{\partial^2 u_1}{\partial x^2} + 4u_0 \frac{\partial u_0}{\partial x} - 3 \frac{\partial u_0}{\partial x} \frac{\partial^2 u_0}{\partial x^2} - u_0 \frac{\partial^3 u_0}{\partial x^3} \\ &= \frac{\partial^2 u_1}{\partial x^2} + 4e^{2x}(x+1)(2+x) - 3e^{2x}(2+x)(3+x) - e^{2x}(x+1)(4+x) \\ (4.5) \quad &= \frac{\partial^2 u_1}{\partial x^2} - 2(4x+7)e^{2x}. \end{aligned}$$

As previously, for sake simplicity we will take all arbitrary constants as in solution of o.d.es equal to 1. On solving o.d.e (4.5) we get following

$$u_1 = e^x + e^{-x} + \frac{8}{3}e^{2x}x + \frac{10}{9}e^{2x}.$$

**Step  $k = 1$ :** On substituting  $k = 1$  in relation (4.3) we get following o.d.e satisfied by  $u_2$ ,

$$(4.6) \quad 2u_2 = 2\frac{\partial^2 u_2}{\partial x^2} + 4A_1 - 3B_1 - C_1.$$

In order to compute  $u_2$  explicitly we need to compute  $A_1$ ,  $B_1$  and  $C_1$  respectively. To do so let use the set of equations (4.4),

$$A_1 = u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} = \left( \frac{16}{3}x^2 + \frac{34}{3}x + \frac{64}{9} \right) e^{3x} + \left( \frac{11}{3}x + 1 \right) e^{2x} + e^x + e^{-x} - 1$$

$$B_1 = \frac{\partial u_0}{\partial x} \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_1}{\partial x} \frac{\partial^2 u_0}{\partial x^2} = \left( 16x^2 + \frac{172}{3}x + \frac{404}{9} \right) e^{3x} + (2x + 1) e^{2x} - 1$$

$$S_1 = u_0 \frac{\partial^3 u_1}{\partial x^3} + u_1 \frac{\partial^3 u_0}{\partial x^3} = \left( \frac{72}{3}x^2 + \frac{222}{3}x + \frac{136}{3} \right) e^{3x} + (6x + 15) e^{2x} + 3.$$

Substituting  $A_1$ ,  $B_1$  and  $C_1$  in (4.6) and solving the corresponding o.d.e gives following solution  $u_2$ ,

$$2u_2 = 2\frac{\partial^2 u_2}{\partial x^2} + 4A_1 - 3B_1 - S_1$$

$$u_2 = \left( \frac{91}{288}x^2 + \frac{59}{8}x + \frac{817}{288} \right) e^{3x} + \left( \frac{79}{27} - \frac{32}{57}x \right) e^{2x} + (1 - x)e^x + (1 + x)e^{-x} + 1.$$

**Step  $k = 2$ :** Let us switch  $k = 2$  in (4.3). Indeed, it follows that  $u_3$  satisfies following 2nd order o.d.e

$$3u_3 = 3\frac{\partial^2 u_3}{\partial x^2} + 4A_2 - 3B_2 - C_3.$$

To compute  $u_3$  explicitly, let first compute  $A_2$ ,  $B_2$  and  $C_2$  from (4.4),

$$A_2 = u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x}$$

$$= \left( \frac{91}{72}x^3 + \frac{13229}{288}x^2 + \frac{32483}{432}x + \frac{32483}{432} \right) e^{4x} + 2(4x + 3) e^{3x} + e^{2x} + e^x - e^{-2x},$$

$$B_2 = \frac{\partial u_0}{\partial x} \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_2}{\partial x} \frac{\partial^2 u_0}{\partial x^2}$$

$$= \left( \frac{91}{24}x^3 + \frac{5067}{32}x^2 + \frac{21060588}{49248}x + \frac{12920665}{49248} \right) e^{4x} + 4(4x + 5) e^{3x} + e^{2x}$$

$$\begin{aligned}
& - \left( \frac{112}{19}x + \frac{2036}{171} \right) e^x - e^{-2x} \\
S_2 = & u_0 \frac{\partial^3 u_2}{\partial x^3} + u_1 \frac{\partial^3 u_1}{\partial x^3} + u_2 \frac{\partial^3 u_0}{\partial x^3} \\
& = \left( \frac{637}{72}x^3 + \frac{83591}{288}x^2 + \frac{32635236}{49248}x + \frac{16654963}{49248} \right) e^{4x} + 6(4x+7)e^{3x} + e^{2x} \\
& - \left( \frac{32}{57}x^2 - \frac{2036}{171}x - \frac{1390}{27} \right) e^x - e^{-2x}
\end{aligned}$$

Using  $A_2$ ,  $B_2$  and  $C_2$  in 4.2 and solving the corresponding o.d.e, we get the following solution,

$$\begin{aligned}
3u_3 = & 3 \frac{\partial^2 u_3}{\partial x^2} + 4A_2 - 3B_2 - S_2 \\
u_3 = & \left( \frac{91}{270}x^3 + \frac{1715292450}{138510000}x^2 + \frac{33214478520}{138510000}x + \frac{594700171}{138510000} \right) e^{4x} + \left( \frac{5}{3}x + 2 \right) e^{3x} \\
& - \left( \frac{16}{513}x^3 + \frac{223}{513}x^2 - \frac{194}{81}x - 1 \right) e^x + e^{-x}.
\end{aligned}$$

Finally, substituting the  $u_0, u_1, u_2$  and  $u_3$  in inverse differential transform series, we recover the following solution of Degasperis-Procesi equation (4.2),

$$\begin{aligned}
u(x, t) = & \sum_{k=0}^n u_k(x) t^k = (x+1)e^x + \left( e^x + e^{-x} + \frac{8}{3}e^{2x}x + \frac{10}{9}e^{2x} \right) t \\
& + \left( \left( \frac{91}{288}x^2 + \frac{59}{8}x + \frac{817}{288} \right) e^{3x} + \left( \frac{79}{27} - \frac{32}{57}x \right) e^{2x} + (1-x)e^x + (1+x)e^{-x} + 1 \right) t^2 \\
& + \left( \left( \frac{91}{270}x^3 + \frac{1715292450}{138510000}x^2 + \frac{33214478520}{138510000}x + \frac{594700171}{138510000} \right) e^{4x} + \left( \frac{5}{3}x + 2 \right) e^{3x} \right. \\
& \left. - \left( \frac{16}{513}x^3 + \frac{223}{513}x^2 - \frac{194}{81}x - 1 \right) e^x + e^{-x} \right) t^3 \dots
\end{aligned}$$

#### 4.3. Discussion on Graphs of solution of Degasperis-Procesi and CAMASSA-HOLM Equation.

Consider the graph of  $u(t, x)$  for  $(t, x) \in [0, 1] \times [0, 1]$  (i.e. unit rectangle) up to order 3. Let us treat the Degasperis-Procesi and CAMASSA-HOLM Equations as model for dynamics of shallow water over flowing through cross section of rectangular of channel. Moreover, we treat  $u(t, x)$  as trajectory of velocity of flow of shallow, where  $x$  denotes the space coordinate/ location along the channel axis and  $t$  denotes the time. We may also assume that there is no friction and Coriolis forcing factor in flow. The graphs shows that as time passes the velocity of

the flow is increasing/ accelerating smoothly, physically water might experience smooth splashes.

If we observe  $u(t, x)$  for  $(t, x) \in [-100, 100]^2$ . Keeping in view the notation and interpretation same as above, the following second Figure (right) can be interpreted as that the solution blow up in finite time i.e., velocity flow is smoother in channel initially then it became singular in finite period of time. This result/ observation is in complete agreement with the conclusion of Proposition 3.1 of [1]. Moreover, well-known peakon and shock wave behaviour is also observed for both equations.

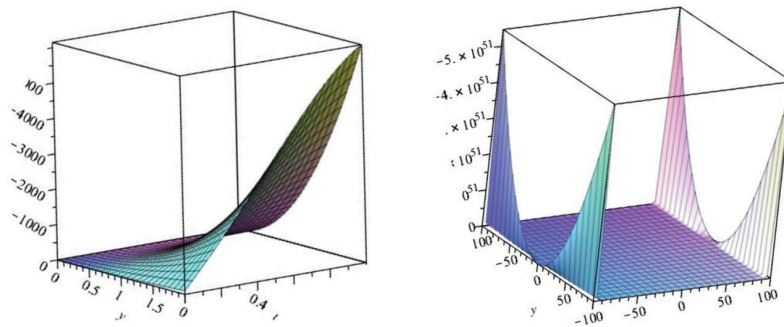


FIGURE 1. Behaviour of Solution of CAMASSAHOLM on small (left) and larger domains (right)

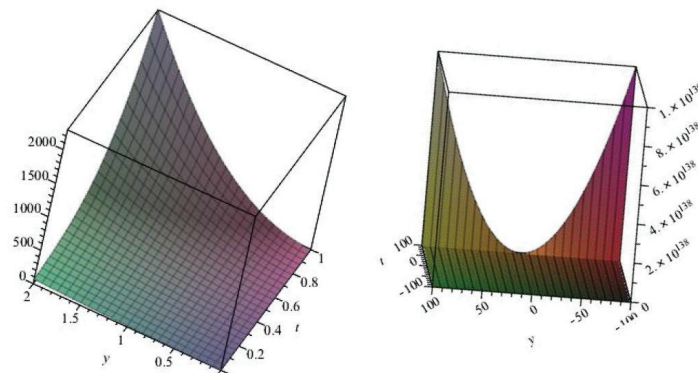


FIGURE 2. Behaviour of Solution of DEGASPERIS-PROCESI on small (left) and larger domains (right)

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DEPARTMENT OF MATHEMATICS

SUKKUR IBA UNIVERSITY

Email address: javed.brohi@iba-suk.edu.pk