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STUDY OF THE DYNAMIC RESPONSE OF THE SOIL-PILE BEHAVIORAL MODEL UNDER AXIAL AND LATERAL LOAD COUPLING

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ABSTRACT. This work aims to extend and improve our previous study on mathematical and numerical analysis of stationary Winkler model under axial and lateral load coupling. In this paper a dynamic response of Winkler model is considered. On the one hand we establish the existence and the uniqueness of the solution of the problem by using the results of spectral theory and the Lax-Milgram theorem and on the other hand the finite element method is used to determinate the numerical results. Furthermore, the influence of soil parameter K_p and the length l of the pile on the displacement is studied numerically at any time t_n.

1. INTRODUCTION

Deep foundations on piles, widely used in the construction of structures, are experiencing increasing development. The progress made in dimensioning methods, technological innovations in the construction of piles, the increasingly mediocre quality of the land left to builders and the large dimensions of the structures are at the origin of this development. In practice, these structures are dimensioned in order to take both axial and lateral forces and moments. Nowadays, although complex, the study of the mechanical behavior of piles has already been the subject of several research works [1–8]. These have resulted in modeling and calculation methods used for the design of such structures. Numerical methods by

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finite elements or by finite difference make it possible to solve soil-pile interaction problems with more rigor while including the effects of loadings on the interface, of the inclination of the piles and of the stiffness of the soil. In our previous studies [6, 7], we worked on stationary behavioral models of soil-pile interaction. So, this present contribution on a dynamic model of soil-pile interaction aims primarily to establish first by appropriate mathematical tools (the results of spectral theory and the Lax-Milgram theorem) the existence and uniqueness of the solution of the problem posed and then to present a rigorous numerical method based on the finite element method and Newmark's method in order to determine the displacements of the structure at each instant by taking into account a large number of parameters relative to piles and soils.

2. Presentation of the model

The dynamic response of Winkler model is defined as follows.
Find: $u : \Omega =]0, l[\times\mathbb{R}_+^* \to \mathbb{R}$ such that:

$$(2.1) \quad \begin{cases} \mathbf{m}\dfrac{\partial^2 u(z,t)}{\partial t^2} + E_pI_p\dfrac{\partial^4 u(z,t)}{\partial z^4} + K_p u(z,t) = P(z,t) \ \forall t > 0, \forall z \in]0,l[, \\ u(z,0) = u_0(z) \ \forall \ z \in]0,l[, \\ \dfrac{\partial u(z,0)}{\partial t} = u_1(z) \ \forall \ z \in]0,l[, \\ u(0,t) = \dfrac{\partial u(0,t)}{\partial z} = 0 \ \forall t > 0, \\ \dfrac{\partial^2 u(l,t)}{\partial z^2} = \frac{M}{E_pI_p} \ \forall t > 0, \\ \dfrac{\partial^3 u(l,t)}{\partial z^3} = \frac{H}{E_pI_p} \ \forall t > 0. \end{cases}$$

Here, $u(z,t)$ is the longitudinal deflection of the beam in terms of m, z is the space coordinate measured along the length of the beam in m, t is the time in s, E_pI_p is the flexural rigidity of the beam in $(N.m^2)$, $\mathbf{m}$ is the mass per unit length of the beam in (kg/m), $P(z,t)$ is the applied external load per unit length in (N/m), K_p is the spring constant (the first parameter) of the soil per unit beam length in terms of (N/m^2), H the head trenchant effort of the free pious in (N) and M the bending moment in $(N.m)$. We see the description of the soil-pile interaction in the following figure 1.

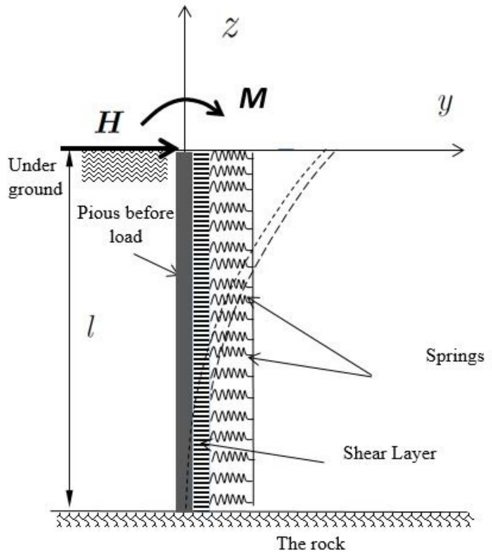


FIGURE 1. Pile under combined loading (axial and lateral).

First, we are interested in the existence and uniqueness of the solutions to the problem (2.1).

2.1. **Existence of a Hilbert basis of $\mathbb{L}^2(\Omega)$.** Since the (2.1) problem can be associated with an eigenvalue problem, we will solve it using a Hilbert basis of $\mathbb{L}^2(\Omega)$. Thus demonstrating the existence of a Hilbert basis of $\mathbb{L}^2(\Omega)$ amounts to verifying the hypotheses of the [9, Theorem 7.2.8].

Consider the following problem:

$$\begin{cases} E_p I_p \dfrac{d^4 u(z)}{dz^4} + K_p u(z) = P(z)\ , \forall z \in]0, l[, \\ u(0) = \dfrac{du(0)}{dz} = 0, \\ \dfrac{d^2 u(l)}{dz^2} = \frac{M}{E_p I_p}, \\ \dfrac{d^3 u(l)}{dz^3} = \frac{H}{E_p I_p}. \end{cases} \tag{2.2}$$

We pose $w(z) = u(z) - \frac{H}{6E_pI_p}z^3 + (\frac{Hl}{2E_pI_p} - \frac{M}{2E_pI_p})z^2$ and (2.2) becomes the following homogenous boundary problem:

$$\text{(2.3)} \qquad \begin{cases} E_pI_p\dfrac{d^4w(z)}{dz^4} + K_pw(z) = G(z)\ , \forall z \in]0, l[, \\ w(0) = \dfrac{dw(0)}{dz} = 0, \\ \dfrac{d^2w(l)}{dz^2} = 0, \\ \dfrac{d^3w(l)}{dz^3} = 0. \end{cases}$$

with $G(z) = P(z) - K_p\left(\frac{H}{6E_pI_p}z^3 - (\frac{Hl}{2E_pI_p} - \frac{M}{2E_pI_p})z^2\right)$. First we prove that (2.3) admits a unique solution.

Lemma 2.1. *According to the Lax-Milgram theorem the problem* (2.3) *admits a unique solution* $w \in V = \{v \in \mathbf{H^2}(\Omega); v(0) = v'(0) = 0\}$ *verifying the following variational formulation:*

$$a(w, v) = L(v)\ \forall\ v \in\ V$$

with

$$a(w, v) = E_pI_p\int_0^l \frac{d^2w(z)}{dz^2}\frac{d^2v(z)}{dz^2}dz\ + K_p\int_0^l w(z)v(z)dz$$

and

$$L(v) = \int_0^l G(z)v(z)dz.$$

Proof. We define the space $\mathbb{L}^2(\Omega)$ provided with the scalar product:

$$\langle f, v\rangle = \int_0^l f(z)v(z)dz\ \text{ for all } f, v \in \mathbb{L}^2(\Omega).$$

and the space V with the reduced norm

$$\|w\|_V = \|w^{(2)}\|_{\mathbb{L}^2(\Omega)} \text{ for all } w \in V.$$

Space V is a closed subspace of $\mathbf{H^2}(\Omega)$ therefore it is a sobolev space in addition

$$|a(u, v)| \leq (E_pI_p + K_pl^4)\|u\|_V\|v\|_V$$

implies a is continuous, we also have

$$a(u, u) \geq E_pI_p\|u\|_V^2$$

then a is coercive and L is linear by definition and the inequality

$$|L(v)| \leq l^2 \|G\|_{\mathbb{L}^2(\Omega)} \|v\|_V$$

shows it is continuous. Therefore, the problem (2.3) admits a unique solution according to Lax-Milgram theorem. □

So we can define our operator as follows:

$$\begin{array}{rccc} \mathcal{L}: & \mathbb{L}^2(\Omega) & \to & V \\ & g & \mapsto & \mathcal{L}g \end{array} \tag{2.4}$$

with $\mathcal{L}g$ the solution of the equation (2.3). In other words, the operator $\mathcal{L}$ is defined by:

$$\mathcal{L}g \in V \text{ such that } a(\mathcal{L}g, v) = \langle g, v \rangle_{\mathbb{L}^2(\Omega)} \text{ for all } v \in V. \tag{2.5}$$

Now, the objective is to show that the operator $\mathcal{L}$ thus defined is linear continuous, self-adjoint, compact and definite-positive.

Lemma 2.2. *The operator $\mathcal{L}$ defined in* (2.4) *is continuous linear, self-adjoint, compact and definite-positive.*

Proof.

I) **Linearity of $\mathcal{L}$**
The linearity of $\mathcal{L}$ defined in (2.4) is a consequence of lemma 2.1.

II) **Continuity of $\mathcal{L}$**
By taking $v = \mathcal{L}g$ in (2.5), we obtain thanks to the coercivity of a and the continuous injection of V into $\mathbb{L}^2(\Omega)$ and from $\mathbb{L}^2(\Omega)$ to $\mathbb{L}^1(\Omega)$:

$$E_p I_p \|\mathcal{L}g\|_V^2 \leq a(\mathcal{L}g, \mathcal{L}g) = \langle g, \mathcal{L}g \rangle_{\mathbb{L}^2(\Omega)}$$

now

$$\langle g, \mathcal{L}g \rangle_{\mathbb{L}^2(\Omega)} \leq \|g\|_{\mathbb{L}^2(\Omega)} \|\mathcal{L}g\|_{\mathbb{L}^2(\Omega)} \leq l^2 \|g\|_{\mathbb{L}^2(\Omega)} \|\mathcal{L}g\|_V$$

and so we get

$$E_p I_p \|\mathcal{L}g\|_V^2 \leq l^2 \|g\|_{\mathbb{L}^2(\Omega)} \|\mathcal{L}g\|_V$$

$$\Longrightarrow \|\mathcal{L}g\|_V \leq \frac{l^2}{E_p I_p} \|g\|_{\mathbb{L}^2(\Omega)}$$

hence the continuity of $\mathcal{L}$.

III) **Self-adjoint of $\mathcal{L}$**

By taking $v = \mathcal{L}h$ with $g, h \in \mathbb{L}^2(\Omega)$ in (2.5), we obtain thanks to the symmetry of a:

$$\begin{aligned} \langle g, \mathcal{L}h\rangle_{\mathbb{L}^2(\Omega)} &= a(\mathcal{L}g, \mathcal{L}h) \\ &= a(\mathcal{L}h, \mathcal{L}g) = \langle h, \mathcal{L}g\rangle_{\mathbb{L}^2(\Omega)}. \end{aligned}$$

IV) **Compactness of $\mathcal{L}$**

Let $\mathcal{I} : V \to \mathbb{L}^2(\Omega), g \mapsto \mathcal{I}g = g$ the injection operator and $\mathcal{L}g$ the operator defined in (2.4). So we have: $\mathcal{I} \circ \mathcal{L}$ defined from $\mathbb{L}^2(\Omega)$ to value in $\mathbb{L}^2(\Omega)$.

$$\mathcal{L}g \in V,\ \forall g \in \mathbb{L}^2(\Omega) \Rightarrow \mathcal{L}g = (\mathcal{I} \circ \mathcal{L})g,\ \forall g \in \mathbb{L}^2(\Omega)$$

and since $\mathcal{I}$ is compact then $\mathcal{L}$ is compact as a compound of compact and continuous operator.

V) **$\mathcal{L}$ is definite-positive**

It comes from the coercivity of a, indeed:

$$\langle g, \mathcal{L}g\rangle_{\mathbb{L}^2(\Omega)} = a(\mathcal{L}g, \mathcal{L}g) \geq E_p I_p \|\mathcal{L}g\|_V^2 > 0, \forall 0 \neq g \in \mathbb{L}^2(\Omega).$$

□

The hypotheses of the [9, Theorem 7.2.8] are verified therefore the eigenvalues of $\mathcal{L}$ form a sequence $(\lambda_k)_{k\geq 1}$ of real numbers strictly positive which tend to 0, and there exists a Hilbertian basis $(u_k)_{k\geq 1}$ of V formed by eigenvectors of $\mathcal{L}$. Therefore, we get the spectral decomposition of any element v of V.

3. Variational formulation of the problem

We obtain the following variational formulation of the problem (2.1), find $u(t) :]0, T[\to V$ such that:

$$\left\{ \begin{array}{l} \frac{d^2}{dt^2}\langle u(t), v\rangle_{\mathbb{L}^2(\Omega)} + a(u(t), v) = \langle P(t), v\rangle_{\mathbb{L}^2(\Omega)}, \forall v \in V,\ 0 < t < T \\ u(t=0) = u_0;\ \frac{du}{dt}(t=0) = u_1 \end{array} \right. \tag{3.1}$$

with

$$V = \{v \in H^2(\Omega); v(0) = v'(0) = 0\};$$

$$P(t) :]0, T[\to \mathbb{L}^2(\Omega);$$

$$a(u(t), v) = E_p I_p \int_\Omega u''(z,t) v''(z)\, dz + K_p \int_\Omega u(z,t) v(z)\, dz;$$

$$L(v) = \langle P(t), v\rangle_{\mathbb{L}^2(\Omega)} = \int_\Omega P(z,t)v(z)\,dz - \frac{H}{E_pI_p}v(l) + \frac{M}{E_pI_p}v'(l).$$

Remark 3.1. *We denote by* $u(z,t)$ *the value* $u(t)(z)$*,* $P(z,t)$ *the value* $P(t)(z)$*.*

3.1. **Semi-discretization in space.** Let N_h be the number of interior points of the discretization and $h = \frac{l}{N_h+1}$ the discretization step. We construct an internal variational approximation by introducing a subspace V_h of V of finite dimension. V_h will be a finite element subspace $\mathbb{P}_3$ on the discretization. The semi-discretization of (3.1) is therefore the following variational approximation: We look for $u_h(t)$ function of $]0,T[$ with values in V_h such that:

$$(3.2)\quad \begin{cases} \frac{d^2}{dt^2}\langle u_h(t), v_h\rangle_{\mathbb{L}^2(\Omega)} + a(u_h(t), v_h) = \langle P_h(t), v_h\rangle_{\mathbb{L}^2(\Omega)}, \forall v_h \in V_h,\ 0 < t < T \\ u_h(t=0) = u_{0,h};\ \frac{du_h}{dt}(t=0) = u_{1,h} \end{cases},$$

where $u_{0,h} \in V_h$ is an approximation of the initial data u_0 and $u_{1,h} \in V_h$ is also an approximation of the initial data u_1.

We introduce the basis $(w^{(i)}, z^{(j)})$ of V_h [7] for all $1 \le i,j \le N_h+1$. We are looking for $u_h(t)$ in the form

$$u_h(t) = \sum_{i=1}^{N_h+1} U_i^h w^{(i)}(z) + \sum_{j=1}^{N_h+1} (U_j^h)' z^{(j)}(z).$$

We denote by U^h the vector of coordinates of u_h in the same way we have:

$$u_{0,h}(t) = \sum_{i=1}^{N_h+1} U_i^{0,h} w^{(i)}(z) + \sum_{j=1}^{N_h+1} (U_j^{0,h})' z^{(j)}(z),$$

$$u_{1,h}(t) = \sum_{i=1}^{N_h+1} U_i^{1,h} w^{(i)}(z) + \sum_{j=1}^{N_h+1} (U_j^{1,h})' z^{(j)}(z),$$

where, $U^{0,h}$ denotes the vector of coordinates of $u_{0,h}$ and $U^{1,h}$ denotes the vector of coordinates of $u_{1,h}$ and (3.2) becomes for all $1 \le i,j \le N_h+1$

$$(3.3)\quad \begin{cases} \frac{d^2}{dt^2}\langle u_h(t), w^{(i)}\rangle_{\mathbb{L}^2(\Omega)} + a(u_h(t), w^{(i)}) = \langle f_h(t), w^{(i)}\rangle_{\mathbb{L}^2(\Omega)}, \forall 0 < t < T \\ \frac{d^2}{dt^2}\langle u_h(t), z^{(j)}\rangle_{\mathbb{L}^2(\Omega)} + a(u_h(t), z^{(j)}) = \langle f_h(t), z^{(j)}\rangle_{\mathbb{L}^2(\Omega)}, \forall 0 < t < T \end{cases}.$$

Hence, according to (3.3) the variational approximation (3.2) is equivalent to the following linear system of ordinary differential equations with constant coefficients:

$$\left\{ \begin{array}{l} \mathcal{M}_h \frac{d^2U^h}{dt^2} + \mathcal{K}_h U^h = B^h \\ U^h(0) = U^{0,h}; \frac{dU^h}{dt}(0) = U^{1,h} \end{array} \right. . \tag{3.4}$$

The mass matrix is defined by:

$$\mathcal{M}_h = \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix}$$

with

$$M^{11} = (\langle w^{(i)}, w^{(j)} \rangle)_{1 \le i,j \le N_h+1}; M^{12} = (\langle z^{(i)}, w^{(j)} \rangle)_{1 \le i,j \le N_h+1};$$

$$M^{21} = (\langle w^{(i)}, z^{(j)} \rangle)_{1 \le i,j \le N_h+1}; M^{22} = (\langle z^{(i)}, z^{(j)} \rangle)_{1 \le i,j \le N_h+1}.$$

The stiffness matrix is defined by:

$$\mathcal{K}_h = \begin{pmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{pmatrix}$$

with

$$K^{11} = (a(w^{(i)}, w^{(j)}))_{1 \le i,j \le N_h+1}; K^{12} = (a(z^{(i)}, w^{(j)}))_{1 \le i,j \le N_h+1};$$

$$K^{21} = (a(w^{(i)}, z^{(j)}))_{1 \le i,j \le N_h+1}; K^{22} = (a(z^{(i)}, z^{(j)}))_{1 \le i,j \le N_h+1};$$

and the matrix of the second member is defined by:

$$B^h = \begin{pmatrix} B^1 \\ B^2 \end{pmatrix}$$

with

$$B^1 = (L(w^{(i)}))_{1 \le i \le N_h+1} \text{and} B^2 = (L(z^{(j)}))_{1 \le j \le N_h+1}.$$

3.2. **Total discretization in space-time.** We decompose the time interval $[0,T]$ into N time step $\Delta t = \frac{T}{N}$, we set $t_n = n\Delta t$ for $n \in \{0, 1 \ldots, N\}$ and we denote by U_n^h the approximation of $U^h(t_n)$. To calculate numerically approximate solutions of (3.4) we use the following Newmark time-stepping method:

$$\begin{cases} \mathcal{M}_h \ddot{U}_{n+1}^h + \mathcal{K}_h U_{n+1}^h = B_{n+1}^h \\ \dot{U}_{n+1}^h = \dot{U}_n^h + \Delta t \left((1-\delta)\ddot{U}_n^h + \delta \ddot{U}_{n+1}^h \right) \\ U_{n+1}^h = U_n^h + \Delta t \dot{U}_n^h + \frac{(\Delta t)^2}{2} \left((1-2\theta)\ddot{U}_n^h + 2\theta \ddot{U}_{n+1}^h \right) \end{cases} . \tag{3.5}$$

Here, the real parameters δ and θ will be fixed as follows $0 \le \delta \le 1$; $0 \le \theta \le \frac{1}{2}$ [9], Δt is the time-step fixed later. So inserting the formula for U_{n+1} into $\mathcal{M}_h \ddot{U}_{n+1}^h + \mathcal{K}_h U_{n+1}^h = B_{n+1}^h$ at time t_{n+1} we obtain from (3.5) the following scheme

$$\begin{cases} \ddot{U}_{n+1}^h = (\mathcal{M}_h + \theta(\Delta t)^2 K_h)^{-1} \left(B_{n+1}^h - \mathcal{K}_h [U_n^h + \Delta t \dot{U}_n^h + \frac{(\Delta t)^2}{2}(1-2\theta)\ddot{U}_n^h] \right) \\ \dot{U}_{n+1}^h = \dot{U}_n^h + \Delta t \left((1-\delta)\ddot{U}_n^h + \delta \ddot{U}_{n+1}^h \right) \\ U_{n+1}^h = U_n^h + \Delta t \dot{U}_n^h + \frac{(\Delta t)^2}{2} \left((1-2\theta)\ddot{U}_n^h + 2\theta \ddot{U}_{n+1}^h \right) . \end{cases}$$

The acceleration

$$\ddot{U}_n^h = \mathcal{M}_h^{-1}(B_n^h - \mathcal{K}_h U_n^h)$$

follows from the equation $\mathcal{M}_h \ddot{U}_n^h + \mathcal{K}_h U_n^h = B_n^h$. Knowing $U_n^h, \dot{U}_n^h, \ddot{U}_n^h$ we find $U_{n+1}^h, \dot{U}_{n+1}^h, \ddot{U}_{n+1}^h$.

4. NUMERICAL SIMULATION

the calculation of the coefficients of the matrices $\mathcal{M}_h$, $\mathcal{K}_h$ and B^h is carried out in the same way as in [7]. The parameters of simulation [8, 10] are as follows:

TABLE 1. Parameters of simulation

l(m)	$E_pI_p(MN.m^2)$	$K_p(kN/m^2)$	$P(kN/m)$	$m(kg/m)$
20	3000	100	200	48.2
20	3000	552	200	48.2
20	3000	1103	200	48.2
40	3000	87	200	48.2
40	3000	437	200	48.2
40	3000	874	200	48.2

And we choose $T = 1s$, $dt = 0.01s$, $\delta = 0.6$ and $\theta = 0.4$ for the Newmark Scheme. We obtain the following responses of the pile:

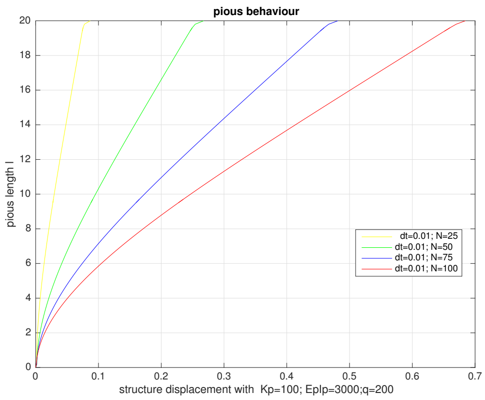


FIGURE 2
Behaviour of the pile of length $l = 20m$ at t_{25},t_{50}, t_{75} and t_{100} for $K_p = 100$

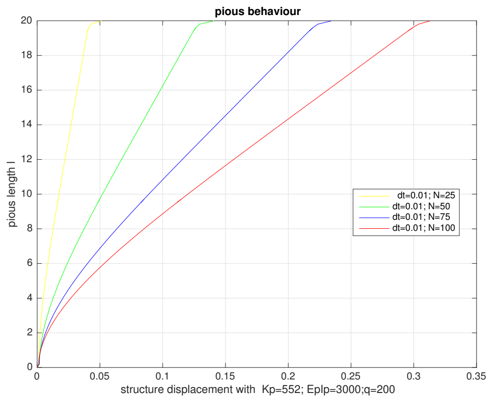


FIGURE 3
Behaviour of the pile of length $l = 20m$ at t_{25},t_{50}, t_{75} and t_{100} for $K_p = 552$

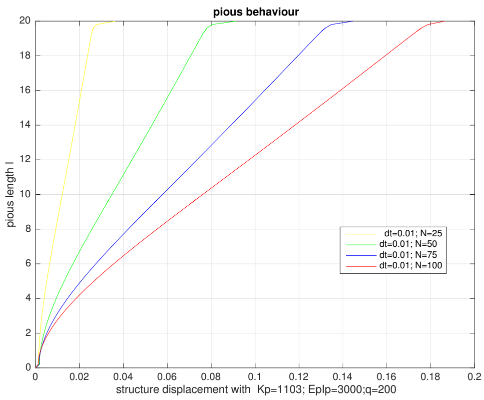


FIGURE 4
Behaviour of the pile of length $l = 20m$ at t_{25},t_{50}, t_{75} and t_{100} for $K_p = 1103$

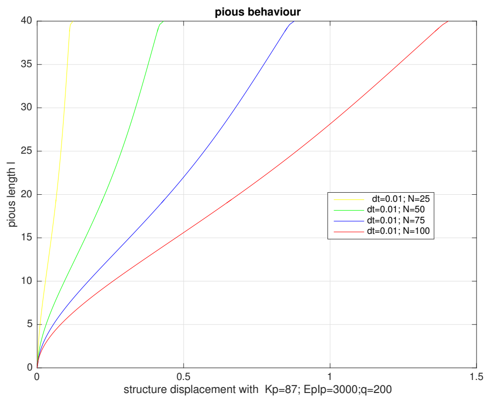
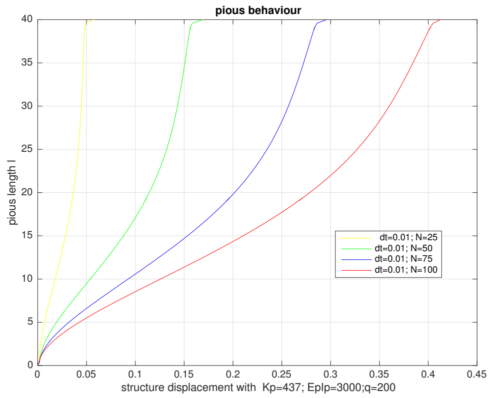


FIGURE 5
Behaviour of the pile of length $l = 40m$ at t_{25},t_{50}, t_{75} and t_{100} for $K_p = 87$



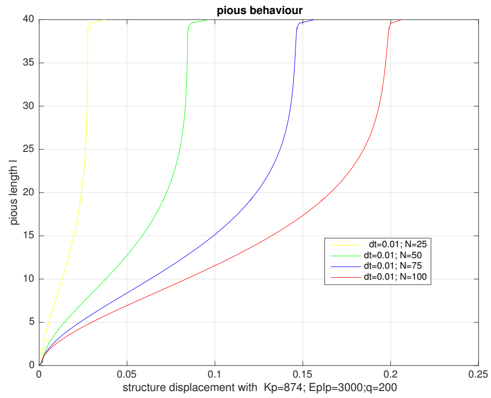


FIGURE 6
Behaviour of the pile of length $l = 40m$ at t_{25},t_{50}, t_{75} and t_{100} for $K_p = 437$

FIGURE 7
Behaviour of the pile of length $l = 40m$ at t_{25},t_{50}, t_{75} and t_{100} for $K_p = 874$

5. RESULTS AND DISCUSSIONS

We observe from figures 2,3 and 4 that the deformation of the pile depends on soil parameter K_p at any time t_n. And the deformation at the head of the pile is more important than inside. Figures 5,6 and 7 show that if l increases and K_p decreases the shape of the deflection changes slightly but we still note a decrease in displacement when the number of iterations N in time increases. In addition, the deformation is significant throughout the pile. In the previous studies [6,7] we have established in the case of the stationary model of pile under axial and lateral load coupling that to reduce the deformations of the pile it is necessary to increase the parameter K_p of the soil. In this work, we also establish that at each moment by increasing the parameter of the soil the deformation of the pile decreases.

6. CONCLUSION

In this work, we use on the one hand the tools of spectral theory and Lax-Milgram theorem to prove the existence and uniqueness of the solution and on the other hand we use finite element method to determine an approximate solution to partial differential equation. We observe that when the soil parameter K_p increases then the displacement of pile decreases even if the number of iterations N in time increases. This work confirms and reinforces the previous results obtained in the analysis of the deflection in the stationary case.

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