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p-CHANDRASEKHAR INTEGRAL EQUATION

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ABSTRACT. Chandrasekhar integral equation has many applications in radiative transfer and some other fields in applied mathematics. Here, we define the p-Chandrasekhar integral equation (p-Ch) via the Urysohn-Stiltjes (U-S) one. Existence of solutions will be studied. The continuous dependence will be proved. Chandrasekhar quadratic (Ch-Q) and cubic (CH-C) integral equations are given.

1. INTRODUCTION

The integral equations of (U-S) type are studied in [2]- [7] and the Chandrasekhar integral equations are also studied [1]. Let $p \in R^+,\ p \geq 1.$ Define the U-S p-integral equation

$$v(t) \ = \ a(t) \ + \ v^p(t) \ \int_0^1 \ f(t,s,v(s))\, d_s\alpha(t,s),\ t \in \ I \tag{1.1}$$

and the p-Ch integral equation

$$v(t) \ = \ a(t) \ + \ v^p(t) \ \int_0^1 \frac{t}{t+s} k(t,s)v(s)ds,\ t \in I. \tag{1.2}$$

Existence of solutions in the class of continuous functions $C \ = \ C(I) = C[0,1]$ will be studied. Sufficient condition for the uniqueness of the solution of (1.2) will be given. Continuous dependence of on $k(t,s)$ will be proved.

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For $p = 1, 2$ we obtain the quadratic and cubic U-S integral equations

$$\upsilon(t) \ = \ a(t) \ + \ \upsilon(t) \ \int_0^1 \ f(t, s, \upsilon(s)) \ d_s\alpha(t, s), \ t \in \ I,$$

$$\upsilon(t) \ = \ a(t) \ + \ \upsilon^2(t) \ \int_0^1 \ f(t, s, \upsilon(s)) \ d_s\alpha(t, s), \ t \in \ I$$

and Ch-q and Ch-C equations

(1.3) $$\upsilon(t) \ = \ a(t) \ + \ \upsilon(t) \ \int_0^1 \frac{t}{t+s} k(t, s)\upsilon(s)ds, \ t \in I$$

and

(1.4) $$\upsilon(t) \ = \ a(t) \ + \ \upsilon^2(t) \ \int_0^1 \frac{t}{t+s} k(t, s)\upsilon(s)ds, \ t \in I.$$

2. Existence of solutions

Assume that the functions $a, \ f, \ b, \ k, \ \alpha$ are continuous and

(i) $\sup_{t\in I} |a(t)| \leq a.$

(ii) $f : I \times I \times R \rightarrow R, \ |f(t, s, x)| \leq b(t, s) + k(t, s)|\upsilon|, \ b, \ k : I \times I \rightarrow R_+$ and

$$b = \sup_t\{b(t, s) : t, s \in I\} \text{ and } \kappa = \sup_t\{k(t, s) : t, s \in I\}.$$

(iii) $\alpha : I \times I \rightarrow R, \ \alpha(0, s) = 0, \ s \in I$ with $\mu = \max\{\sup |\alpha(t, 1)|, \ \sup |\alpha(t, 0)| : t \in I\}.$

(iv) $\forall t_1, t_2 \in I, \ t_1 < t_2 \ s \rightarrow \alpha_2(t_2, s) - \alpha_2(t_1, s)$ is non decreasing on $[0, 1]$ and

$$\forall \ \epsilon > 0, \ \exists \ \delta > 0, \ t_2 - t_1 < \delta \ \Rightarrow \ \bigvee_{s=0}^{1}[\alpha(t_2, s) - \alpha(t_1, s)] \leq \epsilon.$$

Lemma 2.1. *(see [2]) Let* α *satisfies* (iv)*.* $t \rightarrow \alpha(t, s_2) - \alpha(t, s_1)$ *and* $s \rightarrow \alpha(t, s)$ *are non decreasing on* I*.*

Theorem 2.1. *Let* $(i) - (iv)$ *be satisfied and there is a positive root* r *of* $a \ + \ r^p \ (b + kr) \ \mu = r$*. Then (1.1) has a solution* $x \in C$*.*

Proof. Define $\digamma \upsilon(t) = a(t) + \upsilon^p(t) \int_0^1 f(t,s,\upsilon(s))\, d_s\alpha(t,s)$ and $Q = \{\upsilon \in C[0,1] : |\upsilon| \leq r\}$, r is a positive root of $(a + r^p\ (b+\kappa r)\ \mu = r)$. Set Q is a nonempty, bounded, closed, and convex. Let $x \in C$, then

$$\begin{aligned} |\digamma \upsilon(t)| &= |a(t) + \upsilon^p(t) \int_0^1 f(t,s,\upsilon(s))\, d_s\alpha(t,s)| \\ &\leq a + |\upsilon(t)|^p \int_0^1 |f(t,s,\upsilon(s))|\, d_s\alpha(t,s)| \\ &\leq a + |\upsilon(t)|^p \int_0^1 (b(t,s) + k(t,s)|\upsilon|)\, d_s\alpha(t,s)) \\ &\leq a + r^p\ (b+kr) \int_0^1 d_s\alpha(t,s) \\ &\leq a + r^p\ (b+kr)\ (\alpha(t,1) - \alpha(t,0)) \leq a + r^p\ (b+kr)\ \mu = r. \end{aligned}$$

Hence, $\digamma x \in Q$, $\digamma : Q \to Q$ and $\{\digamma \upsilon\}$ is uniformly bounded in Q . Let $\upsilon \in Q$, $\theta(\delta) = \max\{\ \theta_1(\delta),\ \theta_2(\delta)\ \}$ where

$$\theta_1(\delta) = \sup_{x\in Q} \{|f(t_2,s,\upsilon(s)) - f(t_1,s,\upsilon(s))| :\ t_1,\ t_2 \in I,\ t_1 < t_2,\ |t_2 - t_1| < \delta,\ s \in I\},$$

$$\theta_2(\delta) = \sup_{x\in Q} \{|\upsilon(t_2)^p - \upsilon(t_1)^p| :\ t_1,\ t_2 \in I,\ t_1 < t_2,\ |t_2 - t_1| < \delta,\ s \in I\},$$

and from our assumptions $\theta(\delta) \to 0$, as $\delta \to 0$ independently on $\upsilon \in Q$. For $\upsilon \in Q$, t_1, $t_2 \in I$, $|t_2 - t_1| < \delta$, we have

$$\begin{aligned} &|\digamma \upsilon(t_2) - \digamma \upsilon(t_1)| = |a(t_2) + \upsilon^p(t_2) \int_0^1 f(t_2,s,\upsilon(s))\, d_s\alpha(t_2,s) \\ &- a(t_1) - \upsilon^p(t_1) \int_0^1 f(t_1,s,\upsilon(s))\, d_s\alpha(t_1,s)| \\ &\leq |a(t_2) - a(t_1)| + |\upsilon^p(t_2) \left\{ \int_0^1 f(t_2,s,\upsilon(s))\, d_s\alpha(t_2,s) \right. \\ &\quad \left. - \int_0^1 f(t_1,s,\upsilon(s))\, d_s\alpha(t_2,s) \right\} \\ &\quad + \upsilon^p(t_2) \int_0^1 f(t_1,s,\upsilon(s))\, d_s\alpha(t_2,s) - \upsilon^p(t_1) \int_0^1 f(t_1,s,\upsilon(s))\, d_s\alpha(t_1,s)| \\ &\leq |a(t_2) - a(t_1)| + |\upsilon^p(t_2)\, \{ \int_0^1 f(t_2,s,\upsilon(s)) - f(t_1,s,\upsilon(s))\, d_s\alpha(t_2,s)\} \end{aligned}$$

$$+ \ \upsilon^p(t_2) \int_0^1 f(t_1, s, \upsilon(s)) \, d_s(\alpha(t_2, s) \ - \ \alpha(t_1, s))$$

$$+ \ (\upsilon^p(t_2) - \upsilon^p(t_1)) \int_0^1 f(t_1, s, \upsilon(s)) \, d_s\alpha(t_1, s)$$

$$\leq |a(t_2) - a(t_1)| + r^p \ \theta_1(\delta) \int_0^1 d_s\alpha(t_2, s)$$

$$+ \ r^p \ (b + kr) \int_0^1 d_s(\alpha(t_2, s) \ - \ \alpha(t_1, s)) \ + \ \theta_2(\delta) \ (b + kr) \int_0^1 d_s\alpha(t_1, s)$$

$$\leq |a(t_2) - a(t_1)| + r^p \ \theta_1(\delta) \int_0^1 d_s\alpha(t_2, s)$$

$$+ \ r^p \ (b + kr) \bigvee_{s=0}^{1} [\alpha(t_2, s) - \alpha(t_1, s)] \ + \ \theta_2(\delta) \ (b + kr) \int_0^1 d_s\alpha(t_1, s)$$

$$\leq |a(t_2) - a(t_1)| + r^p \ \theta_1(\delta) \int_0^1 d_s\alpha(t_2, s)$$

$$+ \ r^p \ (b + \kappa r) N(\epsilon) \ + \ \theta_2(\delta) \ (b + \kappa r) \int_0^1 d_s\alpha(t_1, s),$$

$$N(\epsilon) = \sup\{\bigvee_{s=0}^{1} (\alpha(t_2, s) - \alpha(t_1, s)) : t_1, t_2 \in I, \ t_1 < t_2, \ t_2 - t_1 \leq \epsilon\}.$$

Then $\{\digamma \upsilon\}$ is equicontinuous on Q and $\digamma Q$ is compact.

Let $\{\upsilon_n\} \subset Q$, and $\{\upsilon_n\} \to \upsilon$, in $Q \subseteq R$, then Lebesgue Theorem (see [8]) implies

$$\begin{aligned}
\digamma \upsilon_n(t) &= a(t) \ + \ \upsilon_n^p(t) \int_0^1 f(t, s, \upsilon_n(s)) \, d_s\alpha(t, s), \\
\lim_{n\to\infty} \digamma \upsilon_n(t)) &= a(t) \ + \ \upsilon^p(t) \lim_{n\to\infty} \int_0^1 f(t, s, \upsilon_n(s)) \, d_s\alpha(t, s) \\
&= a(t) \ + \ \upsilon^p(t) \int_0^1 f(t, s, \lim_{n\to\infty} \upsilon_n(s)) \, d_s\alpha(t, s) \\
&= a(t) \ + \ \upsilon^p(t) \int_0^1 f(t, s, \upsilon_0(s)) \, d_s\alpha(t, s) = \digamma \upsilon(t).
\end{aligned}$$

Hence $\digamma$ is continuous and has a fixed point $\upsilon \in Q$ and (1.1) has a solution $\upsilon \in C$ ([8]). □

3. p- CHANDRASEKHAR INTEGRAL EQUATIONS

Let

$$\alpha(t,s)=\begin{cases} t\ ln\ \frac{t+s}{t}, t\in(0,1],\ s\in I,\\ 0,\ t=0,\ s\in I.\end{cases}$$

Then α satisfies (iv), (v), (vi) (see [5] and [6]) and we obtain

$$\upsilon(t)=a(t)+\upsilon^P(t)\cdot\int_0^1\frac{t}{t+s}f(t,s,\upsilon(s))ds.$$

Let $f(t,s,\upsilon)=k(t,s)\upsilon(s)$, then we obtain (1.2). Also, at $p=1,\ 2$ we obtain (1.3) and (1.4).

4. UNIQUENESS OF THE SOLUTION AND CONTINUOUS DEPENDENCE

Theorem 4.1. *Let the assumptions of Theorem 2.1 be satisfied. If* $kr^p(1+r)\ <1$*, then (1.2) has unique solution.*

Proof. Let υ and ν be two solutions of (1.2), then we have

$$\begin{aligned}&\upsilon(t)-\nu(t)\\ &=\ \upsilon^p(t)\int_0^1\frac{t}{t+s}k(t,s)\upsilon(s)ds-\nu^p(t)\int_0^1\frac{t}{t+s}k(t,s)\nu(s)ds\\ &=\ (\upsilon^p(t)-\nu^p(t))\int_0^1\frac{t}{t+s}k(t,s)\upsilon(s)ds\\ &\quad+\nu^p(t)\int_0^1\frac{t}{t+s}k(t,s)(\upsilon(s)-\nu(s))ds,\end{aligned}$$

further,

$$||\upsilon-\nu||\le||\upsilon-\nu||\ kr^{p+1}+||\upsilon-\nu||r^pk,$$

and

$$||\upsilon-\nu||\ (1-\kappa r^p(1+r))\ \le 0.$$

Thus, $\upsilon(t)=\nu(t)$ and the solution is unique. □

Theorem 4.2. *Let the assumptions of Theorem 4.1 be satisfied. The solution of (1.2) depends continuously on the function* $\kappa(t,s)$ *in the sense that for all* $\epsilon>0,$

exists $\delta(\epsilon) > 0$ *such that* $||k - k^*|| = \sup\{|k(t,s) - k^*(t,s)| < \delta,\, t, s \in [0,1]\}$, *then* $||\upsilon - \upsilon^*|| < \epsilon$ *where* $\upsilon^*(t)$ *is the solution of*

$$\upsilon^*(t) \;=\; a(t) \;+\; \upsilon^{*p}(t) \int_0^1 \frac{t}{t+s} k^*(t,s)\upsilon^*(s)ds. \tag{4.1}$$

Proof. Let υ and υ^* be the solutions of (1.2) and (4.1). Then we can get

$$\begin{aligned}
&|\upsilon(t) - \upsilon^*(t)| \\
&= \;|\upsilon^p(t) \int_0^1 \frac{t}{t+s} k(t,s)\upsilon(s)ds - \upsilon(t)^{*p}(t) \int_0^1 \frac{t}{t+s} k^*(t,s)\upsilon^*(s)ds| \\
&= |\upsilon^P(t)| \int_0^1 \frac{t}{t+s} (k(t,s) - k^*(t,s))\upsilon(s)ds \\
&\quad + (\upsilon^p(t) - \upsilon^{*p})(t) \Bigg| \int_0^1 \frac{t}{t+s}\, k^*(t,s)\upsilon(s)ds \\
&\quad + \upsilon^{*p}(t) \int_0^1 \frac{t}{t+s}\, k(t,s)(\upsilon(s) - \upsilon^*(s))ds \Bigg| \\
&\le\; r^{1+p}||k - k^*|| \;+\; pr^p k||\upsilon - \upsilon^*|| \;+\; r^p\kappa||\upsilon - \upsilon^*||.
\end{aligned}$$

This implies that

$$||\upsilon - \upsilon^*|| \le \frac{r^{1+p}}{(1 - \kappa r^p(1+r))} ||k - k^*|| \le \frac{r^{1+p}}{(1 - kr^p(1+r))} \delta < \epsilon.$$

□

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