

DEFENSIVE ALLIANCE DIFFERENCE SECURE SETS OF A PATH

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ABSTRACT. Let $G = (V, E)$ be a simple connected graph. A non-empty subset S of V is called a *defensive alliance* of G if for every $v \in S$, $|N[v] \cap S| \geq |N[v] - S|$. Let $f : V(G) \rightarrow \{1, 2, 3, \dots, |V|\}$ be a bijection. A subset $S \subseteq V$ is called a *difference secure set* of G with respect to f if for all $u, v \in S$, there is a $w \in S$ such that $|f(u) - f(v)| = f(w)$ if and only if $uv \in E$. A defensive alliance S of G which is also a difference secure set is called *defensive alliance difference secure set (ad-set)*. In this paper, we initiate the study of various types of *ad-sets* and compute the minimum cardinality of each set, particularly for paths.

1. INTRODUCTION

All the graph considered in this paper are simple, connected and finite. For a graph $G = (V, E)$, S is a non-empty subset of $V(G)$, $\langle S \rangle$ denotes the subgraph of G induced by S and $\overline{S} = V - S$. Let a_1 be the graph property satisfied by atleast one subset S among the varieties of subsets of $V(G)$. Using the a_1 graph property 4 different types of sets are obtained and is defined in Table 1:

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TABLE 1. Varieties of sets with a_1 property

Sets	Condition for the sets
a_1 -set	S should satisfies the a_1 -property.
a_1^* -set	Both S and \overline{S} should satisfies the a_1 -property.
A_1 -set	S should satisfies the a_1 -property and \overline{S} should not satisfies the a_1 - property.
A_1^* -set	Both S and \overline{S} should not satisfies the a_1 -property.

Let a_2 be one more property satisfied by any subset of varieties of subsets of $V(G)$. If any subset has to satisfy both a_1 and a_2 property then we get 4^2 varieties of a_1a_2 -sets of G as shown in Table 1:

TABLE 2. Varieties of sets with a_1 and a_2 properties

a_1a_2 -set	$a_1a_2^*$ -set	a_1A_2 -set	$a_1A_2^*$ -set.
$a_1^*a_2$ -set	$a_1^*a_2^*$ -set	$a_1^*A_2$ -set	$a_1^*A_2^*$ -set
A_1a_2 -set	$A_1a_2^*$ -set	A_1A_2 -set	$A_1A_2^*$ -set
$A_1^*a_2$ -set	$A_1^*a_2^*$ -set	$A_1^*A_2$ -set	$A_1^*A_2^*$ -set.

Similarly, with a_1, a_2 and a_3 properties we will have 4^3 different types of $a_1a_2a_3$ -sets of G . In general, with $a_1, a_2, a_3, \dots, a_k$ properties we will get 4^k different types of $a_1a_2a_3 \dots a_k$ -sets of G . The properties are studied for 2^n subsets of $V(G)$ of G having order n . We make the study more meaningful by excluding null set and whole set from the subsets of $V(G)$. Hence, we study the properties for $2^n - 2$ subsets of $V(G)$. A a_1a_2 -set is said to be a minimal a_1a_2 -set of G if none of its proper subsets are a_1a_2 -set of G . The minimum cardinality of a minimal a_1a_2 -set of G is called *lower a_1a_2 number* and is denoted by $l_{a_1a_2}(G)$.

Let $G = (V, E)$ be a graph. If $v \in V$ and $S \subseteq V$, then $N(v) = \{u \in V : uv \in E\}$, $N[v] = N(v) \cup \{v\}$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. A *defensive alliance* is a subset S of V such that $v \in S$ implies $|N[v] \cap S| \geq |N[v] - S|$. The vertices of $N[v] - S$ as attackers of v and those of $N[v] \cap S$ as defenders of v . Thus, for any v in a defensive alliance, there are atleast as many defenders as there are attackers, and any attack on a single vertex can be defended. The studies on defensive alliance are included in [2, 3, 5]. We recall the following for the immediate reference:

Theorem 1.1. [5] *The subgraph induced by a minimal defensive alliance set of a connected graph G is connected.*

Theorem 1.2. [5] *For any graph G , $a(G) = 1$ if and only if there exists a vertex $v \in V$ such that $\deg(v) \leq 1$.*

Remark 1.1. *For a path P_n , any vertex $v_i \in V(G)$ with $\deg(v_i) = 2$ and a subset $S \subset V(G)$, then $S = \{v_i\}$ is not defensive alliance.*

A graph G is a *difference graph* if there is a bijection f from V to a set of positive integers S such that $xy \in E$ if and only if $|f(x) - f(y)| \in S$. The more research work on difference graphs can be found in [1, 4, 7]. Now we define *difference secure number* as follows: Let $f : V(G) \rightarrow \{1, 2, 3, \dots, |V|\}$ be a bijection. A subset $S \subseteq V$ is called a *difference secure set* of G with respect to f if for all $u, v \in S$, there is a $w \in S$ such that $|f(u) - f(v)| = f(w)$ if and only if $uv \in E$. Among all such f the maximum cardinality of a *difference secure set* is called *difference secure number* of G and it is denoted by $DSN(G)$.

B. Sooryanarayana et.al in [6] introduced the neighbourhood resolving sets of a graph and studied various types of *nr-set*. Likewise, for a graph G we define *ad-set* if S is defensive alliance difference secure set. In alliances the number of defendable members who are able to defend immediately is decided by the codes assigned to them. The difference of the codes assigned to them also occurs if the members of the alliance are neighbours and those members will always be able to defend without delay in time. If not, codes does not exist and members do not defend.

Remark 1.2. *For any graph $G = (V, E)$, the singleton subset $S = \{v\}$, $v \in V$ is always difference secure set.*

Remark 1.3. *For a connected graph G with order $n \geq 2$, the subset $S \subset V(G)$ with $|S| = 2$ is always a difference secure set.*

Proof. Let $S = \{v_1, v_2\}$ be a subset of $V(G)$ and $f : V \rightarrow \{1, 2, 3, \dots, n\}$ be the labeling function. We have two possibilities, either v_1 is adjacent to v_2 or not. If v_1 is adjacent to v_2 , then take $f(v_1) = 2f(v_2)$ or else take $f(v_1) \neq 2f(v_2)$. In both the ways S is a difference secure set. \square

Remark 1.4. *For any $a > 0$, the difference secure set $S \subset V(G)$ for any triangle free graph $G(n \geq 3)$, cannot contain a subset with labels $\{a, 2a, 3a\}$.*

Proof. If possible, let $f(v_i) = a, f(v_j) = 2a$ and $f(v_k) = 3a$ for some $v_i, v_j, v_k \in S$, then $\langle \{v_i, v_j, v_k\} \rangle \cong C_3$, a contradiction to the fact that G is a triangle free graph. \square

Remark 1.5. For any path P_n with difference secure set S , if $d \in f(S)$, then the set $\{x, x + d, x + 2d\} \not\subseteq f(S)$, for $x > d$.

Proof. Let $f : V(P_n) \rightarrow \{1, 2, 3, \dots, n\}$ be the labeling function and $S \subset V(P_n)$. Consider the subset $\{v_i, v_j, v_k, v_l\} \subset S$. Let $f(v_i) = x, f(v_j) = x + d, f(v_k) = x + 2d, f(v_l) = d$. Suppose, if the set $\{x, x + d, x + 2d\} \subseteq f(S)$, then $|f(v_j) - f(v_i)| = d = f(v_l) = |f(v_k) - f(v_j)| \Rightarrow$ the vertices v_i, v_k, v_l are adjacent to v_j ; a contradiction to the fact that $\deg(v_j) \leq 2$ in P_n . \square

Theorem 1.3. [8] For a path P_n of order n , $DSN(P_n) = \lceil \frac{n}{2} \rceil + 1$.

Theorem 1.4. [8] For $n \geq 11$, the graph P_n cannot have both sets S and \bar{S} as difference secure simultaneously.

Observation 1.5. For a graph P_n , $2 \leq n \leq 10$, there exists a set S for which both S and \bar{S} are difference secure.

2. ad -sets OF DIMENSION ONE

Theorem 2.1. For any integer n , $l_{ad}(G) = l_{a^*d}(G) = 1$ if and only if $\delta(G) = 1$.

Proof. Let S be a subset of $V(G)$. From Remark 1.2, the singleton set is always difference secure set and hence a d -set. From Theorem 1.2, $S = \{v\}$ is a defensive alliance whenever $\deg(v) = 1$. Then \bar{S} is a set of remaining $n - 1$ vertices also form a defensive alliance. The minimum cardinality of minimal a -set, a^* -set is one. Hence, $l_{ad}(G) = l_{a^*d}(G) = 1$. On the otherhand, let G be a graph with $l_{ad}(G) = l_{a^*d}(G) = 1$. Then there exist singleton set $S = \{v\}$ which satisfies a -set, a^* -set and d -set properties. If S is defensive alliance then $\{v\}$ must be a pendant vertex. Hence, $\delta(G) = 1$. \square

Theorem 2.2. For any integer n ,

$$l_{aD}(G) = l_{a^*D}(G) = \begin{cases} \text{does not exist,} & \text{for } n = 2 \\ 1, & \text{for } n \geq 3 \end{cases}$$

if and only if G is a triangle free graph with $\delta(G) = 1$.

Proof. Let G be a triangle free graph with $\delta(G) = 1$ and $S \subset V(G)$. From Theorem 2.1, we proved that for any graph G , $S = \{v_1\}$, where v_1 is a pendant vertex is a a -set and a^* -set. Consider a labeling function $f : V(G) \rightarrow \{1, 2, 3, \dots, n\}$. For $n = 2$, $S = \{v_1\}$ with $f(v_1) = 1$ and $f(v_2) = 2$. For $n = 3$, $G \cong P_3$. Take $S = \{v_1\}$ with $f(v_1) = 1$ and $f(\bar{S}) = \{2, 3\}$. For $n \geq 4$, if $S = \{v_1\}$, then from Remark 1.4 and Remark 1.5, $C_3 \subseteq \langle \bar{S} \rangle$. Therefore, \bar{S} is not a difference secure set since G is a triangle free graph. Hence, S is a D -set. Therefore, $l_{aD}(G) = l_{a^*D}(G) = 1$.

Conversely, let G be a graph with $l_{aD}(G) = l_{a^*D}(G) = 1$. Then there exist singleton set $S = \{v\}$ which is a a -set, a^* -set and D -set. If S is a defensive alliance then $\{v\}$ must be a pendant vertex. Hence, $\delta(G) = 1$. Since $|\bar{S}| = n - 1$, it will contain a labels of the form $\{a, 2a, 3a\}$ for any $a > 0$. As \bar{S} is not a difference secure set, the vertices of $\langle \bar{S} \rangle$ cannot be labeled $\{a, 2a, 3a\}$ for any $a > 0$. This implies G must be a triangle free graph. \square

3. ad -sets OF A PATH

From Theorem 2.1 and Theorem 2.2, the path P_n is a triangle free graph and $\delta(P_n) = 1$. Hence, we have the following corollary:

Corollary 3.1. For any integer n , $l_{ad}(P_n) = l_{a^*d}(P_n) = l_{aD}(P_n) = l_{a^*D}(P_n) = 1$.

Theorem 3.1. For integer n ,

$$l_{ad^*}(P_n) = l_{a^*d^*}(P_n) = \begin{cases} 1, & \text{for } 2 \leq n \leq 5 \\ \lfloor \frac{n}{2} \rfloor - 1, & \text{for } 6 \leq n \leq 8 \\ \lfloor \frac{n}{2} \rfloor, & \text{for } 9 \leq n \leq 10 \\ \text{does not exist,} & \text{for } n \geq 11. \end{cases}$$

Proof. From Theorem 2.1, minimum cardinality of a -set, a^* -set is one. Consider the labeling function $f : V(P_n) \rightarrow \{1, 2, 3, \dots, n\}$ and $S \subset V(P_n)$. When $2 \leq n \leq 5$, from Remark 1.2 the set $S = \{v_1\}$ is difference secure set. Consider $f(\bar{S}) = \{2\}$, $f(\bar{S}) = \{1, 2\}$, $f(\bar{S}) = \{1, 2, 4\}$ and $f(\bar{S}) = \{4, 2, 5, 3\}$ for which \bar{S} are difference secure for $n = 2, 3, 4, 5$ respectively. We also observe that, as v_1 is an end vertex of P_n , S is a ad^* -set and a^*d^* -set. Also, S is a minimal set and hence, $l_{ad^*}(P_n) = l_{a^*d^*}(P_n) = 1$.

When $6 \leq n \leq 8$, $S = \{v_1\}$ with $f(v_1) \neq 2$. Then \bar{S} will contain vertices with labels $\{1, 2, 3\}$ or $\{2, 4, 6\}$. By Remark 1.4, \bar{S} is not a difference secure set. Also, if $S = \{v_1\}$ with $f(v_1) = 2$, then \bar{S} will contain vertices with labels $\{1, 3, 4, 5, 6\}$. This

implies from Remark 1.5, \bar{S} is not a difference secure set. Therefore, for $n \geq 6$, the singleton set is not d^* -set. So, we take $|S| \geq 2$. Also, from Theorem 1.3, we have $DSN(P_n) = \lceil \frac{n}{2} \rceil + 1$. If $|S| < \lfloor \frac{n}{2} \rfloor - 1$, then $|\bar{S}| > \lceil \frac{n}{2} \rceil + 1$. Hence, \bar{S} cannot become a difference secure set. Therefore, $\lfloor \frac{n}{2} \rfloor - 1 \leq |S| \leq \lceil \frac{n}{2} \rceil + 1$. When $n = 6, 7$, by Remark 1.3 the subsets $S_1 = \{v_1, v_6\}$ and $S_2 = \{v_2, v_3\}$ respectively, are difference secure. We also observe that the subsets $f(\bar{S}_1) = \{1, 4, 3, 6\}$ and $f(\bar{S}_2) = \{5, 4, 7, 3, 6\}$ are difference secure. For P_8 , consider $S = \{v_1, v_7, v_8\}$ with $f(v_1) = 5$, $f(v_7) = 1$, $f(v_8) = 2$ and $\bar{S} = \{v_2, v_3, v_4, v_5, v_6\}$ with $f(v_2) = 8$, $f(v_3) = 4$, $f(v_4) = 7$, $f(v_5) = 3$ and $f(v_6) = 6$. From Theorem 1.2 and Remark 1.1, S is an a -set and a^* -set. Hence, $l_{ad^*}(P_n) = l_{a^*d^*}(P_n) = \lfloor \frac{n}{2} \rfloor - 1$.

When $n = 9, 10$, the sets $S_1 = \{v_2, v_3, v_4, v_5\}$ with $f(S_1) = \{2, 1, 9, 8\}$ and $S_2 = \{v_1, v_2, v_3, v_4, v_5\}$ with $f(S_2) = \{5, 10, 9, 1, 2\}$ respectively, are difference secure. Simultaneously, the subsets $\bar{S}_1 = \{v_1, v_6, v_7, v_8, v_9\}$ with $f(\bar{S}_1) = \{5, 4, 7, 3, 6\}$ and $\bar{S}_2 = \{v_6, v_7, v_8, v_9, v_{10}\}$ with $f(\bar{S}_2) = \{6, 3, 7, 4, 8\}$ will also becomes difference secure. Since $|N[S_1] \cap S_1| \geq |N[S_1] - S_1|$ and $|N[S_2] \cap S_2| \geq |N[S_2] - S_2|$ holds for S_1 and S_2 , they are a -sets. Also, as the subsets \bar{S}_1 and \bar{S}_2 hold $|N[\bar{S}_1] \cap \bar{S}_1| \geq |N[\bar{S}_1] - \bar{S}_1|$ and $|N[\bar{S}_2] \cap \bar{S}_2| \geq |N[\bar{S}_2] - \bar{S}_2|$, \bar{S}_1 and \bar{S}_2 are a^* -sets. Hence, $l_{ad^*}(P_n) = l_{a^*d^*}(P_n) = \lfloor \frac{n}{2} \rfloor$. When $n \geq 11$, by Theorem 1.4, $l_{ad^*}(P_n)$ does not exist. \square

Theorem 3.2. For any path P_n , $l_{Ad}(P_n) = \begin{cases} \text{does not exist,} & \text{for } n = 2 \\ 2, & \text{for } n = 3 \\ 3, & \text{for } n \geq 4. \end{cases}$

Proof. For any singleton subset $S = \{v\}$ of $V(P_n)$, if v is a pendant vertex, then both S and \bar{S} will be defensive alliance, otherwise S itself will not be defensive alliance. Hence, S is not a A -set. It is obvious from Theorem 1.2 that P_2 does not have a A -set. For $n = 3$, the set $S = \{v_1, v_3\}$ is a defensive alliance, where as $\bar{S} = \{v_2\}$ is not defensive alliance. Hence, S is a A -set. Also, for a labeling function $f : V(P_n) \rightarrow \{1, 2, 3, \dots, n\}$, take $f(v_1) = 1$, $f(v_3) = 3$ and $f(v_2) = 2$. Then $S = \{v_1, v_3\}$ is a d -set. Therefore, $l_{Ad}(P_3) = 2$.

Consider the set $S = \{v_i, v_j\}$, $1 \leq j \leq n$, with v_i as one of the end vertex of P_n . If v_j is adjacent to v_i , then both S and \bar{S} will be defensive alliance. And if v_j is not adjacent to v_i then S itself is not defensive alliance. Hence, we take $|S| > 2$. Let $S = \{v_i, v_j, v_k\}$, $1 \leq j, k \leq n$, with v_i as end vertex. We cannot have $\langle S \rangle$ disconnected, otherwise S will not be a defensive alliance. So, to have S as a

defensive alliance and \bar{S} as a non-defensive alliance, take v_i not adjacent to v_j and v_k , also v_j and v_k adjacent to each other. Now define a labeling function $f : V \rightarrow \{1, 2, 3, \dots, n\}$ as follows: $f(v_j) = 2f(v_k)$ where $f(v_k) = a \in \{1, 2, 3, \dots, \lfloor \frac{n}{2} \rfloor\}$ and $f(v_i) \not\equiv 0 \pmod{a}$. Hence, S is a Ad -set. Therefore, $l_{Ad}(P_n) = 3$. \square

Theorem 3.3. For any path P_n ,

$$l_{Ad^*}(P_n) = \begin{cases} 2, & \text{for } n = 3 \\ 3, & \text{for } 4 \leq n \leq 8 \\ \text{does not exist,} & \text{for } n = 2 \text{ and } n \geq 9. \end{cases}$$

Proof. It is obvious that P_2 does not have a A -set. Let $S \subset V(P_n)$. For P_3 , the subset $S = \{v_1, v_3\}$ is a A -set. But for a path P_n , $n \geq 4$, the set S is the minimal set A -set if it is of the form $\{v_1, v_3, v_4\}$ or $\{v_{n-3}, v_{n-2}, v_n\}$. Now if this set S has to be a d^* -set then there should exist some labeling function f such that both S and \bar{S} are difference secure. Define $f : V(P_n) \rightarrow \{1, 2, 3, \dots, n\}$ as follows:

For $n = 3$, $f(S) = \{1, 3\}$ and $f(\bar{S}) = \{2\}$. Here, S is a A -set and d^* -set. Hence, $l_{Ad^*}(P_3) = 2$. We consider for different n the labeling where both S and \bar{S} are difference secure. In the set $S = \{v_1, v_3, v_4\}$ are adjacent so that we can label $f(v_3) = 2f(v_4)$ and the pendant vertex v_1 can be chosen such that $f(v_1) \neq 2f(v_3)$ and $f(v_1) \neq 2f(v_4)$. Then S is difference secure. Similarly, the set $\{v_{n-3}, v_{n-2}, v_n\}$ is a difference secure. For the above set S and for each $4 \leq n \leq 8$, label \bar{S} as $f(\bar{S}) = \{1\}$, $f(\bar{S}) = \{1, 5\}$, $f(\bar{S}) = \{5, 1, 2\}$, $f(\bar{S}) = \{5, 3, 7, 4\}$ and $f(\bar{S}) = \{5, 6, 3, 7, 4\}$. We observe that the above \bar{S} are difference secure. Hence, S is a d^* -set. Therefore, $l_{Ad^*}(P_n) = 3$.

Consider $S = \{v_2, v_3, v_4, v_5\}$ with $f(S) = \{2, 1, 9, 8\}$ for $n = 9$. Then we get $\bar{S} = \{v_1, v_6, v_7, v_8, v_9\}$ with $f(\bar{S}) = \{5, 4, 7, 3, 6\}$. Similarly, for $n = 10$, consider $S = \{v_1, v_2, v_3, v_4, v_5\}$ with $f(S) = \{5, 10, 9, 1, 2\}$ and $\bar{S} = \{v_6, v_7, v_8, v_9, v_{10}\}$ with $f(\bar{S}) = \{6, 3, 7, 4, 8\}$. This set S is a unique d^* -set but not a A -set (since both S and \bar{S} are defensive alliance). From Theorem 1.4, P_n , $n \geq 11$ has no d^* -set. \square

Theorem 3.4. For any path P_n , $l_{AD}(P_n) = \begin{cases} \text{does not exist,} & \text{for } 2 \leq n \leq 7 \\ 3, & \text{for } n \geq 8. \end{cases}$

Proof. For $n = 2$, it is trivial that $l_{AD}(P_2)$ does not exist. Let $f : V \rightarrow \{1, 2, 3, \dots, n\}$ be the labeling function and S be a subset of $V(P_n)$. For $n = 3$, the set $S = \{v_1, v_3\}$ is a minimal A -set, but there exist a labeling function with $f(S) = \{1, 3\}$ and $f(\bar{S}) = \{2\}$ where both S and \bar{S} becomes difference secure. Hence, S fails to be

a D -set. For a path P_n the set $S = \{v_1, v_3, v_4\}$ is a minimal defensive alliance and $\bar{S} = \{v_2, v_5, v_6, \dots, v_n\}$ is not a defensive alliance. Hence, S is a A -set. By the argument done in the Theorem 3.3, the set $S = \{v_1, v_3, v_4\}$ is difference secure. Also, for each $4 \leq n \leq 7$, the sets with label $f(\bar{S}) = \{1\}$, $f(\bar{S}) = \{1, 5\}$, $f(\bar{S}) = \{5, 1, 2\}$ and $f(\bar{S}) = \{5, 3, 7, 4\}$ respectively, are difference secure. Hence, S is not a D -set. Therefore, $l_{AD}(P_n)$ does not exist. We also observed that, for $n = 8$, the label $f(\bar{S}) = \{6, 3, 7, 4, 8\}$ makes \bar{S} not a difference secure set S (since v_2 is not adjacent to v_5). Further, for $n \geq 9$, $\{3, 6, 9\} \subseteq f(\bar{S})$ from Remark 1.4, \bar{S} is not a difference secure set. Hence, $n \geq 8$, S is a D -set. Therefore, S is a AD -set and $|S| = 3$. \square

Theorem 3.5. For any integer $n \geq 4$, $l_{A^*d}(P_n) = 2$.

Proof. Let S be a subset of $V(P_n)$. We know that singleton set $S = \{v\}$, where v is a pendant vertex, is always a defensive alliance. Also, if v is not a pendant vertex, then \bar{S} becomes defensive alliance. Hence, S is not a A^* -set. Therefore, we must have $|S| \geq 2$. Let $S = \{v_{i-1}, v_{i+1}\}$ for $2 \leq i \leq n-1$ contains one pair of non-adjacent vertices. Then obviously \bar{S} will contain atleast one pair of non-adjacent vertices. From Remark 1.1, both S and \bar{S} are A^* -sets. Let $f : V \rightarrow \{1, 2, 3, \dots, n\}$ be a labeling function. Suppose if $f(v_{i-1}) = 1$ and $f(v_{i+1}) = 3$, then S will be difference secure (since $|f(v_{i-1}) - f(v_{i+1})| = 2 \notin S$). Hence, S is both defensive alliance and difference secure. Therefore, $l_{A^*d}(P_n) = 2$. \square

Theorem 3.6. For any path P_n ,

$$l_{A^*d^*}(P_n) = \begin{cases} 2, & \text{for } n = 4, 5 \\ 3, & \text{for } n = 6, 7, 8 \\ \text{does not exist,} & \text{for } n = 2, 3 \text{ and } n \geq 9. \end{cases}$$

Proof. Let $f : V \rightarrow \{1, 2, 3, \dots, n\}$ be a labeling function and $S \subset V(P_n)$. Consider the set S and \bar{S} which contains atleast one vertex other than pendant vertex which is not adjacent to other vertices. Clearly, S is a A^* -set. From Remark 1.1, for P_2 and P_3 there exists no set S which is a A^* -set. Therefore, $l_{A^*d^*}(P_2)$ and $l_{A^*d^*}(P_3)$ does not exist.

We take a set S which is a A^* -set and define the labeling f so that S is also a d^* -set as below:

Case 1: For $n = 4, 5$, we take,

- (i) $S = \{v_1, v_3\}$ with labeling $f(v_1) = 1, f(v_3) = 4, f(v_2) = 2$ and $f(v_4) = 3$, for $n = 4$.
- (ii) $S = \{v_1, v_3\}$ with labeling $f(v_1) = 1, f(v_3) = 3, f(v_2) = 5, f(v_4) = 2$ and $f(v_5) = 4$, for $n = 5$.

Hence, $l_{A^*d^*}(P_4) = l_{A^*d^*}(P_5) = 2$.

From Theorem 1.3, for $n > 6$, if $|S| = 2$ then, \bar{S} is not a difference secure set. When $n = 6$, for the set $\{v_i, v_{i+2}\}$, $1 \leq i \leq n - 2$, \bar{S} is not a difference secure set. Hence, we take $|S| > 2$ for the following case.

Case 2: For $6 \leq n \leq 8$, we define the labeling as follows:

- (i) $S = \{v_2, v_4, v_5\}$ with labeling $f(v_2) = 6, f(v_4) = 1, f(v_5) = 2, f(v_1) = 3, f(v_3) = 4$ and $f(v_6) = 5$, when $n = 6$.
- (ii) $S = \{v_2, v_4, v_5\}$ with labeling $f(v_2) = 7, f(v_4) = 1, f(v_5) = 2, f(v_1) = 5, f(v_3) = 4, f(v_6) = 3$ and $f(v_7) = 6$, When $n = 7$.
- (iii) $S = \{v_1, v_2, v_4\}$ with labeling $f(v_1) = 1, f(v_2) = 2, f(v_4) = 8, f(v_3) = 5, f(v_5) = 6, f(v_6) = 3, f(v_7) = 7$ and $f(v_8) = 4$, When $n = 8$.

Therefore, $l_{A^*d^*}(P_6) = l_{A^*d^*}(P_7) = l_{A^*d^*}(P_8) = 3$.

Case 3: For $n \geq 9$,

When $n = 9, 10$, there exist unique set S is a d^* -set but fails to be a A^* -set. The only d^* -set S is shown below:

- (i) When $n = 9$, we get a unique set $S = \{v_1, v_6, v_7, v_8, v_9\}$ with labeling $f(v_1) = 5, f(v_6) = 4, f(v_7) = 7, f(v_8) = 3, f(v_9) = 6, f(v_2) = 2, f(v_3) = 1, f(v_4) = 9$ and $f(v_5) = 8$.
- (ii) For $n = 10$, we get a unique set $S = \{v_1, v_7, v_8, v_9, v_{10}\}$ with labeling $f(v_1) = 7, f(v_7) = 5, f(v_8) = 8, f(v_9) = 3, f(v_{10}) = 6, f(v_2) = 9, f(v_3) = 10, f(v_4) = 1, f(v_5) = 2$ and $f(v_6) = 4$.

Also, by Theorem 1.4, for $n \geq 11$, d^* -set does not exist for P_n . Hence, $n \geq 9$, $l_{A^*d^*}(P_n)$ does not exist. \square

Theorem 3.7. For any path P_n , $l_{A^*D}(P_n) = \begin{cases} \text{does not exist,} & \text{for } n = 2, 3 \\ 2 & \text{for } n \geq 4. \end{cases}$

Proof. The set $S = \{v_i, v_{i+2}\}$, $1 \leq i \leq n - 2$, is a minimum subset of $V(P_n)$ which is a A^* -set (since S and \bar{S} are not a defensive alliance). For $n = 2, 3$, we cannot have any A^* -set. Therefore, $l_{A^*D}(P_2)$ and $l_{A^*D}(P_3)$ does not exist.

Consider a labeling function $f : V \rightarrow \{1, 2, 3, \dots, n\}$. For P_n to have a D -set, there must exist atleast one difference secure set S , for which \bar{S} is not a difference secure set. From Remark 1.3, the set $S = \{v_2, v_4\}$ is difference secure. For $n \geq 4$, \bar{S} is not difference secure. When $n = 4, 5$, we label \bar{S} as $f(\bar{S}) = \{2, 4\}$, $f(\bar{S}) = \{2, 4, 5\}$ respectively, which is clearly not difference secure. When $n = 6, 7$, there exist no labeling for \bar{S} which is difference secure and for $n \geq 8$, $S = \{v_2, v_4\}$, we get $|\bar{S}| > \lceil \frac{n}{2} \rceil + 1$ which implies from Theorem 1.3, \bar{S} is not a difference secure set. Hence, $l_{A^*D}(P_n) = 2$. \square

Lemma 3.1. *For any path P_n of order n , D^* -set does not exist.*

Proof. The maximal difference secure set for a path P_n is $\lceil \frac{n}{2} \rceil + 1$. Hence, for n upto $\lceil \frac{n}{2} \rceil + 1$ there exist atleast one set S which is difference secure. When $n > \lceil \frac{n}{2} \rceil + 1$, we get \bar{S} as difference secure. Therefore, both S and \bar{S} cannot be difference secure sets simultaneously. Hence, P_n does not have D^* -set. \square

From Lemma 3.1 we have the following Theorem.

Theorem 3.8. *For integer $n \geq 3$, $l_{aD^*}(P_n) = l_{a^*D^*}(P_n) = l_{AD^*}(P_n) = l_{A^*D^*}(P_n)$ does not exist.*

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