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# ESSENTIALLY MULTIPLICATION MODULES

YAHYA TALEBI[1] AND MOTAHAREH IRANI

ABSTRACT. In commutative ring theory, the concept of multiplication modules had been studied extensively. By the way, in this paper we shall introduce a new generalization of multiplication modules, namely *essentially multiplication modules*. We say that over a commutative ring $R$, a module $M$ is essentially multiplication, provided that for every essential submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = MI$.

## 1. INTRODUCTION AND PRELIMINARIES

Throughout this paper $R$ denotes an arbitrary commutative ring with identity and all modules are unitary $R$-modules. Let $M$ be an $R$-module and $N$ a submodule of $M$. We use $N \leq_e M$ and $N \leq_d M$ to denote that $N$ is essential in $M$ and $N$ is a direct summand of $M$, respectively. Moreover we use End$(M_R)$ and $r_R(m)$ to denote the ring of endomorphism of $M$ and the right annihilator in $R$ of an element of $M$. For any unexplained terminology we refer to [1], [2], [3], [4], [5] and [6].

A nonzero submodule $N$ of a module $M$ is said to be essential in $M$ if $N \cap K \neq 0$ for every nonzero submodule $K$ of $M$. Dually a proper submodule of $M$ is small in $M$ in case $M = N + K$ implies that $K = M$.

Multiplication modules and ideals have been investigated in El-Bast and Smith [2] and S. Ebrahimi Athani and S. Khojasteh, G. Ghaleh [3] and others.

[1]*corresponding author*

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Let $R$ be a ring and $M$ an R-module the $M$ is called a multiplication module provided for every submodule $N$ of $M$ there exist an ideal $I$ of $R$ such that $N = IM$.

Our objective is to investigate essential multiplicate.

## 2. Essentially Multiplication Modules

In this section we will introduce a new generalization of multiplication modules using essential submodules.

**Definition 2.1.** *Let $M$ be a module. Then we call $M$, essentially multiplication module in case for every essential submodule $N$ of $M$ There is an ideal $I$ of $R$ that $N = MI$.*

**Example 1.** *Consider the $\mathbb{Z}$-module $M = \mathbb{Q}$. Since for every ideal $I$ of $\mathbb{Z}$ we have $MI = M$, we conclude that $M$ can not be essentially multiplication.*

In general, a divisable $\mathbb{Z}$-module can not be essentially multiplication since for every $n \in \mathbb{Z}$, $nM = M$. As a consequence:

(1) An injective $\mathbb{Z}$-module can not be essentially multiplication.
(2) Let $n \in \mathbb{N}$ does not be square-free. Then the $\mathbb{Z}$-module $M = \mathbb{Z}_n$ is essentially multiplication. To show this, let $N \leq_e M$. Then there is $x \in M$ such that $N = (x)$ (note that every submodule of $M$ is cyclic). It can easily verified that $N = (x\mathbb{Z})M$.
(3) Every semisimple module is essentially multiplication module.

The following example introduces a large class of essentially multiplication modules while they are not multiplication.

**Example 2.** *Let $R$ be a local ring and $M$ a finitely generated semisimple $R$-module. Let $N$ be a non-trivial submodule of $M$. If there is an ideal of $R$ such that $IM = N$, then $N = 0$. To verify this, note that $I \subseteq J(R)$ and $N$ is finitely generated. Also $N = IM = IN \oplus IN'$ for a non-trivial submodule $N'$ of $M$ since $M$ is semisimple. Therefore, $N = IN$. Now the Nakayama's Lemma implies that $N = 0$, a contradiction. It follows that $M$ can not be multiplication, but $M$ is an essentially multiplication module by Example 2.*

**Lemma 2.1.** *Let $M$ be a finitely generated $R$-module and $I \leq_e R$. Then $IM \leq_e M$.*

*Proof.* Let $M =< x_1, \ldots, x_n >$ and $I$ be an essential ideal of $R$. Suppose that $0 \neq x \in M$. Now, there are $r_1, \ldots, r_n$ in $R$ such that $x = \sum_{i=1}^{n} r_i x_i$. Without loss of generality we can assume that each $r_i$ is nonzero. Since $I \leq_e R$, there exists $s_i \in R$ such that $s_i r_i \in I$. Set $s = \prod_{i=1}^{n} s_i$. It is easy to verify that $sx \in IM$. This completes the proof. □

Note that every ideal of $\mathbb{Z}$ is essential in $\mathbb{Z}$.

For a subset $X$ of a module $M$ over a ring $R$, the ideal $\{r \in R | X.r = 0\}$ is called the annihilator of $X$ in $R$; it is denoted by $r_R(X)$ or $r(X)$.

**Corollary 2.1.** *Let $M$ be an $\mathbb{Z}$-module which is not essentially multiplication. Then $M$ is not finitely generated.*

**Lemma 2.2.** *If $M$ is an essential multiplication module, then for every ideal $I$ of $R$ such that $I \subseteq r(M)$, the $\frac{R}{I}$ module M is an essential multiplication module.*

*Proof.* Let $N$ be an essential submodule of $M$ For every $x \in M$ there exists $r \in R$ such that $r.x \in N$. Since for every ideal $I \subseteq r(M)$such that $r+I \in \frac{R}{I}$ then $(r+I)x = r.x + Ix$ and finally $(r + I)x \in N$ so $\frac{R}{I}$-module $M$ is a essential multiplication module. □

For two subsets $X$ and $Y$ of a module $M$ over a ring $R$, the subset $\{r \in R | X.r \subseteq Y\}$ of $R$ is denoted by $(Y : X)$. If $Y$ is a submodule of $M$, then it is directly verified that for any subset $X$ of $M$, the set $(Y : X)$ is a ideal of $R$. For any two submodule $X$ and $Y$ of $M$, it is directly verified $(Y : X)$is an ideal of $R$.

**Proposition 2.1.** *For a module $M$, the following coditions are equivalent:*

*1) $M$ is an essential multiplication module*

*2) $N \subseteq M(N : M)$ for every essential submodule $N$ of M*

*3) $N = M(N : M) = Mr(\frac{M}{N})$ for every essential submodule $N$ of $M$.*

*Proof.* (1) $\Longrightarrow$ (2) Since $M$ is an essential multiplication module, for every essential submodule $N$ of $M$ there exist an ideal $I$ of $R$ such that $N = IM$. Hence also $I \subseteq (N : M)$, so $N = IM \subseteq (N : M)M$. □

**Proposition 2.2.** *For a right module $M$ over a ring $R$, the following conditions are equivalent.*

*1) $M$ is an essentially multiplication module.*

*2) For every ideal $I$ of $R$ such that $I \subseteq r(M)$, the $\frac{R}{I}$-module $M$ is an essential* *multiplication module.*

*3) There exist an ideal $I$ of $R$ such that $I \subseteq r(M)$ and $M$ is an essential multiplication $\frac{R}{I}$-module.*

Let $R$ be a ring and $M$ be an R-module. Recall that a submodule $L$ of $M$ is called fully invariant provided $\varphi(L) \subseteq L$ for every endomorphism $\varphi$ of $M$. Clearly $0$ and $M$ are fully invariant submodules of $M$. Every submodule of the R-module $R$ is fully invariant.

**Proposition 2.3.** *Every homomorphic image of an essential multiplication module is an essential multiplication module.*

*Proof.* Let $M$ be an essential multiplication module over a ring $R$, $h : M \to \overline{M}$ be am epimorphism and $\overline{N}$ be an essential submodule of $\overline{M}$. Then there exists an essential submodule $N$ of $M$ with $h(N) = \overline{N}$. By assumption, there exists an ideal $I$ of the ring $R$ such that $N = MI$. Then $\overline{N} = h(N) = h(MI) = h(M)I = \overline{M}I$ and $\overline{M}$ is an essential multiplication module. □

**Proposition 2.4.** *For an essential multiplication module $M$ over $R$ the following assertions hold:*

*1) Every essential submodule of $M$ is a fully invariant submodule of $M$.*

*2) if $N$ is an essential submodule of $M$ such that $N \bigcap MI = NI$ for every ideal $I$ of $R$, then $N$ is an essential multiplication module.*

*Proof.*

1) Let $N$ be a essential submodule of $M$ and f be an endomorphism of $M$. There exist an ideal $I$ of $R$ such that $N = MI$. then $f(N) = f(IM) = f(M)I \subseteq MI = N$.

2) let $L$ be an essential submodule of $N$, since $M$ is essential multiplication module, there exists an ideal $I$ of $R$ such that $L = MI$. Therefore, $L = MI = L \bigcap MI \subseteq N \bigcap MI = NI \subseteq MI = L$. □

**Proposition 2.5.** *Every endomorphic image of an essential multiplication module is a fully invariant essential multiplication submodule.*

*Proof.* The proof follows from Proposition 2.3 and Proposition 2.4(1). □

**Proposition 2.6.** *Every direct summand of an essential multiplication module is a fully invariant essential multiplication submodule of the module.*

*Proof.* Since every direct summand of an essential multiplication module is an endomorphic image of the module, the assertion follows from Proposition 2.5. □

Let $R$ be a ring, then its center is Cen $R = \{r \in R \mid rx = xr(x \in R)\}$. We may say that an element $r \in R$ is central in case $r \in cenR$.

Note that if $A \subseteq CenR$, then the subring generated by $A$ is also in the center of $R$.

An idempotent $e$ of $R$ is a central idempotent in case it is in the center of $R$.

**Proposition 2.7.** *Let $M$ be an essential multiplication module and $f$ be an endomorphism of $M$ with kerf$\leq_e M$. Then idempotent of End(Imf) is central.*

*Proof.* Let $M$ be an essential multiplication module over a ring $R$, $\overline{M}$ be a homomorphic image of $M$, $R = End(\overline{M})$, and f be an idempotent of the ring $R$. By Proposition 2.3, $\overline{M}$ is an essential multiplication module. By Proposition 2.5, $(1-f)R\,f(\overline{M}) \subseteq f(\overline{M}) \bigcap (1-f)(\overline{M}) = 0$ and $fR(1-f)(\overline{M}) \subseteq (1-f)(\overline{M}) \bigcap f(\overline{M}) = 0$. Therefore, $(1-f)R\,f = f\,R(1-f) = 0$ and $f$ is a central idempotent of $R$. □

**Corollary 2.2.** *If $M$ is an essential multiplication module over a ring $R$ and $P$ is an ideal of $R$ such that $M \neq MP$, then there exists a cyclic submodule $X$ of $M$ such that $P$ does not contain the annihilator of the module $\frac{X}{M}$.*

*Proof.* Since $M \neq MP$, there exist a cyclic submodule $X$ of $M$ that is not contained in the module $MP$. Since $M$ is an essential multiplication module, there exists an ideal $I$ of $R$ such that $X = MI$. Then $I$ is not contained in $P$, since $X$ is not contained in $MP$. Since $X = IM$ then $I$ is contained in $r(X/M)$. It follows that $P$ can not be contained in $r_M(X/M)$. □

**Definition 2.2.** *A non empty set L together with two binary operations $\vee$ and $\wedge$ (read join and meet) on $L$ is called a lattice if it satisfies the following identities:*

$L1: (a) x \vee y \approx y \vee x$

$(b)\ x \wedge y \approx y \wedge x$

*L2:* $(a)\ x \vee (y \vee z) \approx (x \vee y) \vee z$

$(b)\ x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$

*L3:* $(a)\ x \vee x \approx x$

$(b)\ x \wedge x \approx x$

$L4: (a) x \approx x \vee (x \wedge y)$

$(b)\ x \approx x \wedge (x \vee y)$

## 3. General properties of essential multiplication module

In this section we study some properties of essential multiplication module.

**Proposition 3.1.** *For a R-module $M$ over a ring $R$, the following conditions are equivalent.*

*(1) $M$ is an essential multiplication module.*

*(2) For every cyclic essential submodule $X$ of $M$, there exists a right ideal $B$ of the ring $R$ such that $X = MB$.*

*(3) For every essential submodule $X$ of $M$, there exists a set $\{X_i\}_{i\in I}$ of submodules of $X$ and a set $\{B_i\}_{i\in I}$ of ideals of $R$ such that $X = \sum_{i\in I} X_i$ and $X_i = MB_i$ for each $i \in I$.*

*Proof.* The implication $(1) \Rightarrow (2)$ is obvious. $(2) \Rightarrow (3)$ let $X$ be an essential submodule of $M$, $\{X_i\}_{i\in I}$ be the set of all essential cyclic submodule of $X$, and $B_i = (X_i : M)(i \in I)$. By assumption, $X_i \subseteq MB_i \subseteq X_i$ for all i. Since $X = \sum_{i\in I} X_i$ we have that $\{X_i\}$ and $B_i$ are the required sets.

$(3) \Rightarrow (1)$ let $X$ be an essential submodule of $M$ and a set $\{B_i\}_{i\in I}$ of ideals of $R$ such that $X = \sum_{i\in I} X_i$ and $X_i = MB_i$ for each $i \in I$. we denote by B the ideal $\sum_{i\in I} B_i$ of $R$. Then $X = \sum_{i\in I} X_i = \sum_{i\in I} MB_i = M(\sum_{i\in I} B_i) = MB$, and $M$ is an essential multiplication module. □

**Proposition 3.2.** *Let $M$ be an essential multiplication module and $K \leq_e M$ then $M/K$ is essential multiplication.*

*Proof.* let $N/K$ be an essential submodule of $M/K$. Since $K \leq_e M$, we conclude that $N \leq_e M$(see [1,proposition 5.16 (1)]). Now there is an ideal $I$ of $R$ such that $IM = N$. It is not hard to check that $I(M/K) = N/K$ which completes the proof. □

**Remark 3.1.** *Let $M$ be an essential multiplication module and $N$ an essential submodule of $M$. Then $IM = N$ for an ideal $I$. If $I$ is nilpotent, then $N$ is small in $M$. Generally if $R$ is a ring with all ideals nilpotent. Then every essential submodule of an essentially multiplication module is small. Recall from [6] that a module $M$ is uniform, provided that each submodule of $M$ is essential in $M$. Examples of uniform modules include Z-modules $Z$ and $Z_{p^\infty}$. It is clear that, for an uniform module two concepts multiplication and essentially multiplication coincide.*

**Proposition 3.3.** *Every direct summand of an essential multiplication module is essential multiplication.*

*Proof.* Let $M = N \bigoplus N'$ and K be an essential submodule of N. then by [1, proposition 5.20(2)], $K \bigoplus N'$ is an essential submodule of M. now by assumption, there is an ideal I of R such that $K \bigoplus N' = IM$. Hence $(K \bigoplus N') \bigcap N = IM \bigcap N = I(N \bigoplus N') \bigcap N$. By modularity, $K = IN$, as required. □

The following gives an easy characterization of essentially multiplication modules.

**Lemma 3.1.** *An R-module $M$ is essentially multiplication if and only if for every essential submodule $N$ of $M$ and each $m \in M$, there is an ideal $I$ of $R$ such that $N + R_m = IM$.*

*Proof.* let $M$ be an essentially multiplication, $N \leq_e M$ and $m \in M$. Then $N + Rm$ is an essential submodule of $M$ by [1, proposition 5.16(1)]. So that $IM = N + R_m$ for an ideal of $R$. For the converse, let $N \leq_e M$. Then for each $m \in N$, there is an ideal $I$ of $R$ such that $IM = N + Rm = N$. This completes the proof. □

**Remark 3.2.** *Let $R$ be a ring with just one non-trivial ideal $I$. Let $M$ be an essentially multiplication R-module. Then every essential submodule of $M$ is maximal in $M$. To show this, let $N \leq_e M$, then for every $m \in M\mathbb{N}$, we have $N + Rm \leq_e M$. Therefore, either $N + Rm = IM$ or $N + Rm = M$. Fist one implies that $N + Rm = N$ which is a contradiction, otherwise $N + Rm = M$. It follows that $N$ is maximal submodule. Recall that a submodule $N$ of $M$ fully invariant in $M$ if for every endomorphism f of $M$, f(N)$\subseteq N$. A module $M$ is called a duo module, in case every submodule of $M$ is fully invariant in $M$. It is well-known that if N is fully invariant and $M = M_1 \bigoplus M_2$, then $N = (N \bigcap M_1) \bigoplus (N \bigcap M_2)$.*

**Theorem 3.1.** *Let $M$ be a right module over a ring $R$ and let $M = \bigoplus_{i \epsilon I} M_i$ then the following conditions are equivalent:*

*1) $M$ is an essential multiplication module.*

*2) Evry essential submodule of $M$ is fully invariant in $M$, and all modules $M_i$ are essential multiplication modules such that there exist ideals $B_i$ of $R$ with $M_i = MB_i (i \epsilon I)$.*

*3) $N = \bigoplus_{i \epsilon I} (N \cap M_i)$ for every essentiol submodule $N$ of $M$, and all modules $M_i$ are essential multiplication modules such that there exist ideals $B_i$ of $R$ with $M_i = MB_i (i \epsilon I)$.*

*4) For every finite subset $J$ of $I$ the module $\bigoplus_{i\epsilon J} M_j$ is a essential multiplication module such that $\bigoplus_{j\epsilon I} M_j = MB_j$ for some ideal $B_j$ of $R$.*

*Proof.* The implication $(1) \Rightarrow (2)$ follows from corollary $2.9$ (1), corollary $(2.12)$.

$(2) \Rightarrow (3)$ let $N$ be an essential submodule of $M$ and let $\varphi_i : M \to M_i$ be natural projections. Since $N$ is a fully invariant essential submodule of $M$, we have $\varphi_i(N) \subseteq N$ for all $i\epsilon I$. Therefore, $N \subseteq \bigoplus_{i\epsilon I} \varphi_i(N) \subseteq N$. Thus, $N = \bigoplus_{i\epsilon I} \varphi_i(N)$ and $N = \bigoplus_{i\epsilon I} N \bigcap M_i$.

$(3) \Rightarrow (1)$ let $N$ be an essential submodule of $M$ and $N_i = N \bigcap M_i (i\epsilon I)$. By assumption, all modules $M_i$ are essential multiplication modules, $N = \bigoplus_{i\epsilon I} N_i$ and for every $i\epsilon I$, there exist ideals $B_i$ and $C_i$ of such that $M_i = MB_i$ and $N_i = M_i C_i$. Since $N_i = MB_i C_i$ and $N = \sum_{i\epsilon I} N_i$ it follows from Proposition 3.1 that M is a essential multiplication module.

$(1) \Rightarrow (4)$ since M is a essential multiplication module, $\bigoplus_{j\epsilon J} M_j = MB_j$ for some ideal $B_j$ of $R$. By Corollary 2.2, the direct summand $\bigoplus_{j\epsilon J}$ of the essential multiplication module M is an essential multiplication module.

$(4) \Rightarrow (1)$ let $N$ be an essential cyclic submodule of $M$. There exists a finite subset $J$ of $I$ such that $N \subseteq \bigoplus_{j\epsilon J} M_J$. By assumption $\bigoplus_{j\epsilon J}$ is an essential multiplication module such that $\bigoplus_{j\epsilon J} M_j = MB_j$ for some ideal $B_j$ of $R$. Since $\bigoplus_{j\epsilon J} M_j$ is an essential multiplication, there exists an ideal $Cj$ of $R$ such that $(\bigoplus_{j\epsilon J} M_j)C_j = N$. then $N = MB_j C_j$. By Proposition 3.1. $M$ is an essential multiplication module. □

**Lemma 3.2.** *Let $R$ be a ring, $M$ be an essential multiplication module and $M = X \bigoplus Y$. Then the following assertions hold.*

*(1) $Z = X \bigcap Z \bigoplus Y \bigcap Z$ for any essential submodule $Z$ of the module $M$.*

*(2) if there exist a module $P$ and epimorphisms $\alpha : P \to X$ and $\beta : P \to Y$, then the homomorphism $\alpha + \beta : P \to M$ is an epimorphism.*

*(3) If the modules $X$ and $Y$ are cyclic then $M$ is a cyclic*

*Proof.*

(1) The proof follows from Theorem 3.1.

(2) We denote by Z the essential submodule $(\alpha + \beta)(P)$ of $M = X \bigoplus Y$. Let $\pi_X : M \to X$ and let $\pi_Y : M \to Y$be natural projections. We have $\pi(Z) = \alpha(P) = X$ and $\pi_Y(Z) = \beta(\rho) = Y$; in addition, $Z = X \bigcap Z \bigoplus Y \bigcap Z$ by (1).

Therefore, $X = \pi_X(Z) = X \bigcap Z \subseteq Z$ and $Y = \pi_Y(Z) = Y \bigcap Z \subseteq Z$. Therefore, $M = X \bigoplus Y = Z$ and $\alpha + \beta$ is an epimorphism.

(3) Since X and Y are cyclic essential multiplication modules, there exist epimorphisms $R_R \to X$ and $R_R \to Y$. By $(2)$, there exists an epimorphism $R_R \to M$; therefore $M$ is a cyclic essential multiplication module. □

**Corollary 3.1.** *Let $M$ be an essential multiplication module that is a direct sum of finitely many cyclic essential multiplication modules, then $M$ is a cyclic essential multiplication module.*

*Proof.* The proof follows from Lemma 3.2.(3). □

A submodule $N$ of a module $M$ is called a superfluous submodule if $N + M \neq M$ for every proper submodule $X$ of $M$. $A$ ring $R$ is said to be semilocal if the factor ring $R/J(R)$ is an Artinian ring.

A ring is semiprimitive if and only if it has a faithful semisimple left module.

A ring $R$ is said to be quasi-invariant if each of its maximal ideals is an ideal of $R$. A module $M$ is called an invariant (resp.quasi-invariant) if each of its submodules (resp.each of its maximal submodules) is a fully invariant submodule of $M$. It is directly verified that a ring $R$ is invariant (resp.quasi-invariant) if and only if $R$ is an invariant (quasi-invariant) R-module.

**Theorem 3.2.** *Let $M$ be an artinian essential multiplication module and $J$ is jacobson radical. Thn the following assertions hold.*

*(1) The module $M/JM$ is cyclic module.*

*(2) If $JM$ is supperfluous essential submodule of $M$, then $M$ is a cyclic module.*

*(3) If $M$ is a finitely generated module, then $M$ is a cyclic module.*

*Proof.*

(1) The homomorphic image $M/JM$ of the essential multiplication module $M$ is an essential multiplication module by Proposition 2.3. Since $M/JM$ is a smiprimitve Artinian module, $M/JM$ is a finitely generated semisimple module. By Corollary 3.1. The factor module $M/JM$ is a cyclic module.

(2) Since $M/JM$ is cyclic R-module, there exists a cyclic submodule $X$ of $M$ such that $M = X + JM$. By assumption $JM$ is a superfluous submodule of $M$. therefor, $M = X$.

(3) Since $M$ is a finitely generated module, $J(M)$ is a supperflues submodule of $M$. By(2), $M$ is a cyclic module. □

**Proposition 3.4.** *For a ring $R$, the following conditions are equivalent.*

*1) $R$ is an invariant ring.*

*2) All cyclic R-modules are essential multiplication module.*

*3) The free principal ideal $I$ of $R$ is an essential multiplication ideal.*

*Proof.*

$(1) \Rightarrow (2)$ Let M be a cyclic R-module with generator m, $N$ be an essential submodule of $M = mR$ and $I = (N : m)$. Since $N \subseteq mR$, we have $N \subseteq m(N : m) \subseteq N$; therefore, $N = mI$. In addition, $I$ is an idea of $R$, since the ring $R$ is an invariant. Therefore, $N = mI = m(RI) = (mR)I = MI$ and M is an essential multiplication module.

The implication $(2) \Rightarrow (3)$ is obvious.

$(3) \Rightarrow (1)$ Let $N$ be an ideal of the ring $R$. Since $R_R$ is an essential multiplication module, there exist an ideal $I$ of $R$ such that $N = RI$. Therefore, $N = RI = R(RI) = RN$ and $N$ is an ideal ring $R$. □

**Lemma 3.3.** *For a ring $R$, the following assertion hold.*

*1) If $B_1, ..., B_u$, $C_1, ..., C_v$ are ideals of $R$ such that $R = B_s + C_t$ for all s and t, then $R = (\cap_{s=1}^{u} B_i) + (\cap_{y=1}^{v} C_j)$.*

*2) If B and C are ideals of $R$ and $M$ is a R-module such that $M/MB$ and $M/MC$ are finitely generated modules, then $M/M(BC)$ is a finitely generated module.*

*3) If $B_1, ..., B_n$ are ideals of $R$ and $M$ is a R-module such that all modules $M/MB_i$ are finitely generated, then $M/M(B_1, ..., B_n)$, $M/M(B_1 \cap ... \cap B_n)$ and $M(MB_1 \cap ... \cap MB_n)$ are finitely generated modules.*

Note that in the next theorem $M$ is R-module where $R$ is invariant ring.

**Theorem 3.3.** *For a module $M$ over an invariant ring $R$, the following conditions are equivalent.*

*1) $M$ is an essential multiplication module which is direct sum of finitely many cyclic modules.*

*2) $M$ is a cyclice module.*

*3) There exist elements $m_1, ..., m_n$ of $M$ such that $M = \bigoplus_{i=1}^{n} m_i R$ and $R = r(m_i) + r(mj)$ for all $i \neq j$.*

*Proof.* The implication $(1) \Rightarrow (2)$ follows from Corollary 3.9.

The implication $(2) \Rightarrow (1)$ follows from Proposition 3.4.

$(2) \Rightarrow (3)$ Let $M = m_1 R$ we set $m_2 = 0$. Then $M = m_1 R \bigoplus m_2 R$ and $R = r(m_1) + r(m_2)$.

$(3) \Rightarrow (2)$ we set $m = m_1 + ... + m_n \in M$. Let $i \in \{1, 2, ..., n\}$. Since $r(m_i)$ is an ideal of the invariant ring $R$ and $R = r(m_i) + r(m_j)$ for all $j \neq i$, Lemma 3.3(1) implies $R = r(m_i) + \bigcap_{j \neq i} r(mj)$.

Therefore, there exist an element $a_i \in r(m_i)$ such that $1 - a_i \in \bigcap_{j \neq i} r(m_j)$. Therefore, $m(1 - a_i) = (m_1 + ... + m_n)(1 - a_i) = m_i(1 - a_i) = m_i$.

Thus $m_i R \subseteq mR$ for every i. Therefore, $M = mR$. □

**Corollary 3.2.**

*(1) Every simple module is an essential multiplication module.*

*(2) Every nonzero essential multiplication module over a simple ring is an simple module.*

**Corollary 3.3.** *Let $M$ be an essential multiplication module, $P$ be an maximal ideal of $R$.*

*1)If $M \neq MP$, then the module $\frac{M}{MP}$ is simple and there exists a cyclic essential submodule $N$ of $M$ such that $R = P + r(\frac{M}{N})$. is a cyclic module with at most two essential submodules of $M$. 2) $\frac{M}{MP}$ is a cyclic essential multiplication module with at most two submodules and either $M = MP$ or $MP$ is a maximal submodule of $M$.*

*Proof.* 1) By Proposition 2.3, $M/MP$ is an essential multiplication module over the simple ring $\frac{R}{P}$. Since $\frac{M}{MP} \neq 0$, it follows from Proposition 3.4(2) that the module $M/MP$ is simple. By Proposition 2.3, there exists a cyclic essential submodule $N$ of $M$ such that $P$ does not contain $r(\frac{M}{N})$; in addition, P is a maximal ideal. Therefore, $R = P + r(\frac{M}{N})$.

2) By Proposition 2.3, $M/MP$ is an essential multiplication module over the simple ring $\frac{R}{P}$. If $\frac{M}{MP} = 0$, then the zero module $\frac{M}{MP}$ is a cyclic module with exactly one submodule. If $M/MP \neq 0$, then by (1), $MP$ is an essential maximal submodule of $M$ and $M/MP$ is a cyclic module with exactly two submodules. □

**Corollary 3.4.** *Let $M$ be an essential multiplication module and $R$ be a ring with commutative multiplication of ideals, P be a maximal ideal of $R$. Then the following conditions are equivalent.*

*1) $M = MP$*

*2) $N = NB$ for every essential submodule $N$ of $M$.*

*3) $X = XP$ for every cyclic essential submodule of $M$.*

*4) $P$ does not condition the annihilator of any cyclic essential submodule of $M$.*

*Proof.* The implication $(1) \Rightarrow (2)$ follows from Lemma 3.3(1) the implications $(2) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ are obvious.

$(3) \Rightarrow (4)$ assume that $P$ contains the annihilator of some essential cyclic submodule $X$ of $M$. By assumption $X = Xp = 0$. Then $R = r(X) \subseteq P$; this is a contradiction.

$(4) \Rightarrow (3)$ Let $X$ be a cyclic essential submodule of $M$. Since $r(X)$ is not contained in $P$ and the ideal $P$ is maximal, $R = P + r(M)$. Therefore, $X = XR = X(P + r(X)) = XP + Xr(X) = XP$. $\square$

**Corollary 3.5.** *Let $M$ be an R-module, $R$ an invariant ring with essential commutative multiplication of ideals. The following conditions are equivalent.*

*1) $M$ is an essential multiplication module.*

*2) For any maximal ideal $P$ of R, either $M = MP$ or $M \neq MP$ and there exists a cyclic essential submodule $N$ of $M$ such that $R = P + r(M|N)$.*

*3) For any maximal ideal $P$ of R, either $P$ does not contain the annihilator of any cyclic essential submodule $N$ of $M$ such that $P$ does not contain $r(M/N)$.*

*4) For any maximal ideal $P$ of R, either $P$ does not contain the annihilator of any element of $M$ or there exist elements $p \in P$ and $x \in M$ such that $M(1 - P) \subseteq xR$.*

*Proof.* $(1) \Rightarrow (2)$ Assume that $M = MP$. Let $N$ be a cyclic essential submodule of $M$. Since $M = MP$, we have $N = NP$ by Theorem 3.3(1). Therefore, $P$ does not contain $r(N)$.

Now assume that $M \neq MP$. By Corollary 2.2(1), there exists a cyclic essential submodule $N$ of $M$ such that $R = P + r(M/N)$. The implication $(2) \Rightarrow (1)$ follows from Corollary 3.2.

$(3) \Rightarrow (4)$ Assum that there exists an element $m \in M$ such that $r(m) \subseteq P$. Since $R$ is an invariant ring, $P$ contains the annihilator of the cyclic essential submodule $mR \in M$. It follows from (3) that there exist a cyclic essential submodule $N$ of $M$ such that the maximal ideal $P$ does not contain $r(M/N)$. Therefore, $R = P + r(M/N)$ and there exists an element $p \in P$ such that $M(1 - p) \subseteq N$.

$(4) \Rightarrow (1)$ Let Y be a cyclic essential submodule of $M$. By Proposition 3.1, it is enough to prove that $Y \subseteq M(Y : M).if(M(Y : M) : Y) = R$, then $Y = Y(M(Y : M) : Y) \subseteq M(Y : M)$. Assume that $(M(Y : M) : Y) \neq R$. Then there exists a maximal ideal $P$ of $R$ such that $(M(Y : M) : Y) \subseteq P$. By assumption, either $P$ doesn't contain the annihilator of any cyclic essential submodule of $M$ or there exist a cyclic essential submodule $N$ of $M$ such that $P$ does not contain $r(M/N)$.

Since $r(Y) \subseteq (M(Y:M):Y) \subseteq P$, it follows from the assumption that there exist an element $p \in P$ and a cyclic essential submodule $N$ of $M$ such that $M(1-p) \subseteq N$. It follows that $Y(1-P)R$ is a submodule of the cyclic module $N$ over the invariant ring $R$. Therefore, there exists an ideal $D$ of $R$ such that $Y(1-P)R = ND$. We have $MD(1-P)R = M(1-P)D \subseteq ND \subseteq Y$. Therefore, $D(1-P)R \subseteq (Y:M)$. It follows that $Y(1-P^2) \subseteq Y(1-P)R(1-P)R = ND(1-P)R \subseteq M(Y:M)$. Therefore, $(1-P^2) \in (M(Y:M):Y) \subseteq P$ and $1 \in pR + P = P$; this is a contradiction. $\square$

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Department of Mathematics
University of Mazandaran
Babolsar, Iran
*Email address*: talebi@umz.ac.ir

Department of Mathematics
University of Mazandaran
Babolsar, Iran
*Email address*: motahareh.irani64@yahoo.com