ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.12, 10431–10436
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.12.32

A STUDY ON FUZZY MONOTONICALLY NORMAL SPACE

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ABSTRACT. The concepts like stratifiable space, semi-stratifiable space and monotonically normal space were studied by Gary Gruenhage, in Hand Book of Set Theoretic Topology. In this paper we establish a relation between fuzzy monotonically normal space and fuzzy stratifiable space. Also we prove a necessary and sufficient condition for a fuzzy topological space to become a fuzzy monotonically normal space and study some of its properties.

1. INTRODUCTION

Generalized metric space is closely related to metrization theory. Metrizability is a very nice but restrictive property of topological spaces. The concept like stratifiable space and monotonically normal space were studied by Gary Gruenhage. For a detailed discussion reference may be made of [3]. In this paper we establish a relation between fuzzy monotonically normal space and fuzzy stratifiable space. Also we prove a necessary and sufficient condition for a fuzzy topological space to become a fuzzy monotonically normal space and study some of its properties.

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2020 *Mathematics Subject Classification.* 54A40, 54E35, 03E72.

Key words and phrases. Fuzzy stratifiable space, fuzzy semi-stratifiable space, fuzzy monotonically normal space.

2. Preliminaries

In this paper we use C. L. Chang's [2] definition of fuzzy topological space. A fuzzy set in X is called a fuzzy point if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is α where $(0 < \alpha \leq 1)$, then we denote this fuzzy point by x_α, where the point x is called its support. The fuzzy set which takes every element in X to 0 is denoted by $\underline{0}$ and which takes every element in X to 1 is denoted by $\underline{1}$. A fuzzy topological space is called T_1 if every fuzzy point in it is a closed fuzzy set. A fuzzy topological space (X, F) is called regular if for each $x \in X, \alpha \in (0,1], U \in F$ with $x_\alpha \leq U$, there exists $V \in F$ such that $x_\alpha \leq V \leq \overline{V} \leq U$ where $\overline{V}$ is the closure of V. All the fuzzy topological spaces considered are assumed to be T_1 and regular. $\mathbb{N}$ denotes the set of all natural numbers. We list some of the definitions which we are using in this paper.

Definition 2.1. *[4] A fuzzy topological space (X, F) is called fuzzy semi-stratifiable if there is a function G which assigns to each $n \in \mathbb{N}$ and fuzzy closed subset H of X, a fuzzy open set $G(n, H)$ with $H \leq G(n, H)$ such that*
(1) $H = \wedge_n G(n, H)$;
(2) $H \leq K \implies G(n, H) \leq G(n, K)$.

Definition 2.2. *[1] A fuzzy topological space (X, F) is called stratifiable if to every $U \in F$, one can assign a sequence of fuzzy sets $U_n \in F$ such that*
(1) $\overline{U_n} \leq U$ *for all* $n \in \mathbb{N}$;
(2) $\vee_n U_n = U$;
(3) $U \leq V (V \in F)$ *implies* $U_n \leq V_n$ *for all* $n \in \mathbb{N}$.

Theorem 2.1. *[4] A fuzzy topological space (X, F) is fuzzy stratifiable if there is a function G which assigns to each $n \in \mathbb{N}$ and fuzzy-closed subset H of X, a fuzzy-open set $G(n, H)$ with $H \leq G(n, H)$ such that*
(1) $H = \wedge_n G(n, H)$;
(2) $H \leq K \implies G(n, H) \leq G(n, K)$;
(3) $H = \wedge_n \overline{G(n, H)}$.

3. Relation between fuzzy stratifiable space and fuzzy monotonically normal space

Definition 3.1. *A fuzzy topological space (X, F) is called fuzzy monotonically normal if to each pair (H, K) of disjoint fuzzy closed subsets of X, one can assign a fuzzy open set D(H, K) such that*

(1) $H \leq D(H, K) \leq \overline{D(H, K)} \leq K^c$;

(2) if $H \leq H'$ and $K' \leq K$, then $D(H, K) \leq D(H', K')$.

The function D is called fuzzy monotone normality operator for X.

Remark 3.1. *Observe that one can always modify D so that $D(H, K) \wedge D(H', K') = \underline{0}$.*

Now fuzzy monotone normality can be thought of as the difference between fuzzy stratifiable and fuzzy semi-stratifiable spaces.

Theorem 3.1. *A space X is fuzzy stratifiable if and only if X is fuzzy semi-stratifiable and fuzzy monotonically normal.*

Proof. Let X be a fuzzy stratifiable space, with G satisfying the conditions of definition of a fuzzy stratifiable space and $G(n + 1, H) \leq G(n, H)$. Clearly X is fuzzy semi-stratifiable. To prove X is fuzzy monotonically normal, we define $D(H, K) = \vee_n G(n, H) - \overline{G(n, K)}$. Clearly D(H,K) is a fuzzy open set with $H \leq D(H, K)$. Suppose $y_\alpha \leq K$ for some $\alpha \in (0, 1]$. Then $y_\alpha \not\leq \overline{G(m, H)}$ for some $m \in \mathbb{N}$. Clearly $y_\alpha \leq G(m, K)$. Therefore $y_\alpha \leq (\underline{1} - \overline{G(m, H)}) \wedge G(m, K)$ which implies $K \leq (\underline{1} - \overline{G(m, H)}) \wedge G(m, K)$. By the definition, $D(H, K) \leq [(\underline{1} - \overline{G(m, H)}) \wedge G(m, K)]^c \leq K^c$. Moreover $\overline{D(H, K)} \leq K^c$. Therefore $H \leq D(H, K) \leq \overline{D(H, K)} \leq K^c$. Suppose $H \leq H'$ and $K' \leq K$. Then $G(n, H) \leq G(n, H')$ and $G(n, K') \leq G(n, K)$. Therefore $D(H, K) = \vee_n G(n, H) - \overline{G(n, K)} \leq G(n, H') - G(n, K') = D(H', K')$. Therefore X is fuzzy monotonically normal.

Conversely assume that X is fuzzy semi-stratifiable and fuzzy monotonically normal, with G and D satisfying the condition of definition of a fuzzy semi-stratifiable space and fuzzy monotonically normal space, respectively. Let $G'(n, H) = D(H, (G(n, H))^c)$. Then $\wedge_n G'(n, H) = \wedge_n D(H, (G(n, H))^c)$. Also $H \leq D(H, (G(n, H))^c) \leq \overline{D(H, (G(n, H))^c)} \leq G(n, H)$. Then $H = \wedge_n H \leq \wedge_n D(H, (G(n, H))^c) = \wedge_n G'(n, H) \leq \wedge_n \overline{D(H, (G(n, H))^c)} = \wedge_n \overline{G'(n, H)} \leq \wedge_n G(n, H) = H$. Therefore $\wedge_n G'(n, H) = H$. Also $\wedge_n \overline{G'(n, H)} = H$. Suppose $H \leq K$.Then $G(n, H) \leq G(n, K)$

that is $G(n,K)^c \leq G(n,H)^c$ which implies $D(H, G(n,H)^c) \leq D(K, G(n,K)^c)$ and hence $G'(n,H) \leq G'(n.K)$. Therefore X is fuzzy stratifiable. □

4. Some properties of fuzzy monotonically normal space

Definition 4.1. *A map $f : X \to Y$ is called fuzzy open (f-open) if and only if f(u) is open fuzzy set in Y for every open fuzzy set u in X.*

Definition 4.2. *A map $f : X \to Y$ is called fuzzy closed (f-closed) if and only if f(v) is closed fuzzy set in Y for every closed fuzzy set v in X.*

Theorem 4.1. *Fuzzy monotonically normal spaces are preserved under fuzzy closed maps.*

Proof. Let X be a fuzzy monotonically normal space. Suppose $f : X \to Y$ is a fuzzy closed map and D_X is a fuzzy monotone normality operator for X. We have to prove that Y is a fuzzy monotonically normal space. Let H and K be disjoint closed fuzzy subsets of Y. Define $D_Y(H,K) = f^*(D_X(f^{-1}(H).f^{-1}(K)))$ where $f^*(U) = \{y : f^{-1}(y) \leq U\}$. Clearly, $f^*(U)$ is open fuzzy set if U is open fuzzy set. Therefore $D_Y(H,K)$ is an open fuzzy set. From the definition of $D_Y(H,K)$, it is easy to check that, $H \leq D_Y(H,K) \leq \overline{D_Y(H,K)}$.

To prove condition (1), it remains to prove that $\overline{D_Y(H,K)} \leq K^c$. Suppose $y_\alpha \leq K$ for some $\alpha \in (0,1]$. Then $y_\alpha \leq \overline{f^*(D_X(f^{-1}(H).f^{-1}(K)))}^c$ which implies $y_\alpha \leq \overline{D_Y(H,K)}^c$. Thus $\overline{D_Y(H,K)} \leq K^c$. Suppose $H \leq H'$ and $K' \leq K$. Then $H \leq D_Y(H,K) \leq \overline{D_Y(H,K)} \leq K^c$ and $H' \leq D_Y(H',K') \leq \overline{D_Y(H',K')} \leq (K')^c$. Clearly $K^c \leq (K')^c$. Thus $D_Y(H,K) \leq D_Y(H',K')$. Therefore Y is fuzzy monotonically normal. □

Theorem 4.2. *A fuzzy topological space (X,F) is fuzzy monotonically normal if and only if for each fuzzy open subset U of X and $x_\alpha \leq U$ for some $\alpha \in (0,1]$, one can assign an open fuzzy subset U_{x_α} with $x_\alpha \leq U_{x_\alpha}$ satisfying the following condition:*

$$U_{x_\alpha} \wedge V_{y_\alpha} \neq \underline{0} \implies x_\alpha \leq V$$

or $y_\alpha \leq U$.

Proof. Suppose (X,F) is a fuzzy monotonically normal space. Let D be a fuzzy monotone normality operator for X such that $D(H,K) \wedge D(K,H) = \underline{0}$. Let $U_{x_\alpha} =$

$D(x_\alpha, U^c_{x_\alpha})$. Suppose $x_\alpha \nleq V$ and $y_\alpha \nleq U$. Then $y_\alpha \leq U^c_{x_\alpha}$. Therefore $D(x_\alpha, U^c_{x_\alpha}) \leq D(x_\alpha, y_\alpha)$. That is $U_{x_\alpha} \leq D(x_\alpha, y_\alpha)$. Similarly $V_{y_\alpha} \leq D(y_\alpha, x_\alpha)$. Therefore $D(x_\alpha, y_\alpha) \wedge D(y_\alpha, x_\alpha) = \underline{0}$ implies $U_{x_\alpha} \wedge V_{y_\alpha} = \underline{0}$. Hence $x_\alpha \nleq V$ and $y_\alpha \nleq U$ implies $U_{x_\alpha} \wedge V_{y_\alpha} = \underline{0}$. Therefore $U_{x_\alpha} \wedge V_{y_\alpha} \neq \underline{0}$ implies $x_\alpha \leq V$ or $y_\alpha \leq U$.

Conversely assume that, the given conditions of the theorem hold. Let H and K be disjoint fuzzy closed subsets of X. Let $D(H, K) = \vee\{U_{x_\alpha} : x_\alpha \leq H and U \wedge K = \underline{0}\}$. Clearly $D(H, K)$ is a fuzzy open set with $H \leq D(H, K)$. Suppose $y_\alpha \leq K$. Hence $y_\alpha \leq H^c$. If $x_\alpha \leq H$ and $U \wedge K = \underline{0}$ then $x_\alpha \nleq H^c$ and $y_\alpha \nleq U$ (since $y_\alpha \leq K$). Therefore by the given condition, $(H^c)_{y_\alpha} \wedge U_{x_\alpha} = \underline{0}$. Therefore $y_\alpha \nleq \overline{D(H, K)}$. Hence $\overline{D(H, K)} \leq K^c$. That D is "monotone" is clear from the definition. □

Theorem 4.3. *Every fuzzy subspace of a fuzzy monotonically normal space is fuzzy monotonically normal.*

Proof. Suppose (X, F) is a fuzzy monotonically normal space. Then for each fuzzy open subset U of X and $x_\alpha \leq U$, we have an open fuzzy subset U_{x_α} satisfying the condition:

$$U_{x_\alpha} \wedge V_{y_\alpha} \nleq \underline{0} \quad \text{implies} \quad x_\alpha \leq V \text{ or } y_\alpha \leq U.$$

Let Y be a fuzzy subspace of X. Let W be a fuzzy open subset of Y and $y_\alpha \leq W$. Let W' be a fuzzy open subset of X with $W' \wedge Y = W$. Let $W_{y_\alpha} \leq W'_{y_\alpha}$. Clearly this assignment satisfies the conditions of a fuzzy monotonically normality for Y. □

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