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BAYESIAN ESTIMATION BASED ON FIRST FAILURE CENSORED DATA

N. M. YHIEA

ABSTRACT. In this paper, we discuss the problem of estimating the parameters of the generalized linear exponential distribution based on progressive first- failure censoring scheme. The maximum likelihood and Bayes methods of estimation are used. The Markov Chain Monte Carlo technique is used for computing the Bayes estimates under informative and non-informative priors. The Bayes estimates of the parameters are compared with their corresponding maximum likelihood estimates. A numerical example is provided to illustrate the proposed methods. A real data sets are used to show the performance of the censoring schemes using maximum likelihood estimator and Bayes estimator.

1. INTRODUCTION

The probability density function (pdf) of the generalized linear exponential distribution (GLED) is given by

$$\begin{aligned} f(x_i) &= \alpha\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}(\lambda+\theta x_i)\, e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}} \\ \alpha,\theta &> 0 \text{ and } \lambda \geq 0, \end{aligned} \tag{1.1}$$

and the corresponding cumulative distribution function (cdf) is given by

$$F(x_i) = 1 - e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}, \alpha,\theta > 0 \text{ and } \lambda \geq 0.$$

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The GLED was originally proposed by [1], they derived some statistical properties such as moments, modes and quantiles. They presented some special cases are as well as distributions related to this generalization. They derived Maximum likelihood estimators of the parameters. This distribution can be used for modeling bathtub, increasing and decreasing hazard rate behavior. This distribution is important because it contains some widely known distributions like exponential distribution, Rayleigh distribution, linear exponential distribution, Weibull distribution and Generalized Rayleigh distribution.

Censoring is utilized in the statistical analysis of reliability characteristics of a system or device even with a loss in efficiency. There are several types of censoring schemes which are utilized in life-testing experiments.the two most popular censoring schemes are conventional type-I and type-II censoring schemes. These censoring schemes with different lifetime models have been studied extensively by many authors such as [2] and [3]. Authors in [4] proposed a life test in which the experimenter might decide to group the test units into several sets, each as an assembly of test units, and then run all the test units simultaneously until the first failure in each group. [5] showed that when the lifetime of an item is very high and test facilities are limited but test material is relatively less expensive, one can test $k \times n$ units by testing n sets or groups, each containing k units.

The life test is then conducted by testing each of these sets of units separately until the occurrence of first failure in each set. Such a censoring scheme is called first-failure censoring scheme. Under this scheme, one can save a considerable amount of time as well as money. The first-failure censoring schemes were also studied by [6] and [7]. But these censoring schemes do not allow the removal of units from the test at points other than the final termination point. Another type of censoring called progressive censoring allows removal of units from the test at times other than the final termination point. The progressive type-II censoring has become very popular in life-testing experiments because it saves time and cost of the experiment.Progressive type-II censoring was introduced in the literature by [8]. [9] presented an overview on progressive censoring. Some recent studies on progressive censoring can be found in [10], [11] and [12].

Paper [13] combined the concepts of first-failure censoring and progressive censoring to develop a new life-test plan called progressive first-failure censoring scheme. They obtained maximum likelihood interval estimation (MLEs), and expected time on test for the parameters of the Weibull distribution based on the progressive first-failure censored sample. Some recent studies on progressive first failure censoring scheme can be found in [14], [15] and [16].

Paper [17] discussed point and interval estimation for Lindley distribution based on progressive first failure censoring by two methods: Maximum likelihood estimation (Mle) and Bayesian estimation. Also, they made a comparison between Bayesian estimation under Symmetric and Asymmetric Loss Functions. Highest Posterior Density (HPD) interval and Approximate Confidence Interval (CI) are obtained. The lifetime performance index CL for the three-parameter power Lomax distribution(POLO) under progressive first-failure type II right censoring sample with respect to a lower specification limit (L) has been evaluated by [18]. Also, he discussed the statistical inference concerning CL via obtaining the maximum likelihood of CL on the base of progressive first-failure censoring.

Recently, [19] studied the problem of statistical estimation and optimal censoring from three-parameter inverted generalized linear exponential distribution under progressive first failure censored samples, [20] discussed estimation and prediction for the two-parameter Chen distribution on the basis of progressive first failure censoring.

The progressive first-failure censoring scheme can be described as follows: suppose that n independent groups with k items within each group are put on a life-testing experiment at time zero and the progressive censoring scheme $R = (R_1, R_2, \dots, R_m)$ is pre-fixed such that R_1 groups and the group in which the first failure is observed are randomly removed from the experiment as soon as the first failure has occurred. R_2 groups and the group in which the second failure is observed are randomly removed from the remaining live $(n - R_1 - 1)$ groups at the time of the second failure, and so on. This procedure continues until all remaining live R_m groups and the group in which the mth failure has occurred are removed at the time of the mth failure. It is clear that $\sum_{i=1}^{m} R_i + m = n$. Note that if $R_1 = R_2 =\cdots= R_m = 0$, the progressive first-failure censoring is reduced to a first-failure censoring scheme and if $R_1 = R_2 = \cdots = R_m - 1 = 0$ and $R_m = n - m$, this scheme reduces to first-failure type-II censoring.With $k = 1$ in each group, the progressive first-failure censoring scheme becomes the progressive type-II censoring scheme. Thus, progressive first-failure censoring is a generalization of progressive censoring.

Let $x_{1:m:n:k}, x_{2:m:n:k}, \ldots, x_{m:m:n:k}$ be a progressive first-failure type-II censored sample from population with pdf f $(\cdot)$ and distribution function F$(\cdot)$ with progressive censoring scheme R. On the basis of a progressive first-failure censored sample, the likelihood function is given by [9] and [13] as follows:

$$
\begin{aligned}
& L\left(x_{1:m:n:k}, x_{2:m:n:k}, \ldots, x_{m:m:n:k}\right) \\
&= A k^m \prod_{i=1}^{m} f\left(x_{i:m:n:k}\right)\left[1-F\left(x_{i:m:n:k}\right)\right]^{k(R_i+1)-1}, \\
& 0 < x_{1:m:n:k} < x_{2:m:n:k} < \ldots < x_{m:m:n:k} < \infty
\end{aligned} \tag{1.2}
$$

where

$$A = n(n-R_1-1)(n-R_1-R_2-1)\ldots(n-R_1-R_2-\ldots R_{m-1}-m+1). \tag{1.3}$$

In this paper, we describe the maximum likelihood and Bayes estimators of the unknown parameters of GLED under progressive first-failure type-II censoring. In Section 2, we obtain MLEs of the parameters. The asymptotic confidence intervals of parameters are obtained using observed Fisher information matrix. Bayes estimators are obtained using the MCMC. Also, highest posterior density (HPD) credible intervals of the parameters are derived in Section 3. In Section 4, a Monte Carlo simulation study is performed to compare the effects of group sizes, number of groups, effective sample sizes and different censoring schemes on derived estimates. Section 5 deals with a real data set for illustration purposes.

2. Maximum Likelihood Estimation (MLE)

In this section, we estimate the parameters α, θ and λ, by constructing the maximum likelihood and we compute the observed Fisher information based on a likelihood equations.

Let $x_{1:m:n:k}, x_{2:m:n:k}, \ldots, x_{m:m:n:k}$ be a progressive first-failure type-II censored sample from GLED with pre-fixed progressive censoring scheme R, effective sample size m, no.of groups n and group size k. Hereafter, we shall use $x = (x_1, x_2, \ldots, x_m)$ in place of $(x_{1:m:n:k}, x_{2:m:n:k}, \ldots, x_{m:m:n:k})$. Then, the likelihood function using Equations (1.1) and (1.2) is given by

$$
\begin{aligned}
L(\alpha,\theta,\lambda|x) &= A k^m \alpha^m \prod_{i=1}^{m}\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}(\lambda+\theta x_i) \\
& e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}\left[e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}\right]^{k(R_i+1)-1},
\end{aligned} \tag{2.1}
$$

where A is defined in (1.3). The log-likelihood function can be written as

$$l(\alpha,\theta,\lambda|x)=\log A+m\log k+m\log\alpha+(\alpha-1)\sum_{i=1}^{m}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right) \tag{2.2}$$
$$+\sum_{i=1}^{m}\log\left(\lambda+\theta x_i\right)-\sum_{i=1}^{m}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)-\sum_{i=1}^{m}[k(R_i+1)-1]\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}.$$

By differentiating (2.2) with respect to α,θ,and λ, and equating each result to zero, three equations must be simultaneously satisfied to obtain MLE of $\hat{\alpha},\hat{\theta}$ and $\hat{\lambda}$. The maximum likelihood equations can be obtained as the solution of

$$\begin{aligned}\frac{\partial l(\alpha,\theta,\lambda|x)}{\partial\alpha} &= \frac{m}{\alpha}+\sum_{i=1}^{m}\left(\lambda x_i+\frac{1}{2}\theta x_i^2\right)\\ &-\sum_{i=1}^{m}[k(R_i+1)-1]\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}\ln\left(\lambda x_i+\frac{1}{2}\theta x_i^2\right),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\frac{\partial l(\alpha,\theta,\lambda|x)}{\partial\theta} &= (\alpha-1)\sum_{i=1}^{m}\frac{1}{2}x_i^2+\sum_{i=1}^{m}\frac{x_i}{\lambda+\theta x_i}\\ &-\sum_{i=1}^{m}\frac{1}{2}x_i^2-\sum_{i=1}^{m}\frac{\alpha}{2}x_i^2[k(R_i+1)-1]\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1},\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\frac{\partial l(\alpha,\theta,\lambda|x)}{\partial\lambda} &= (\alpha-1)\sum_{i=1}^{m}x_i+\sum_{i=1}^{m}\frac{1}{\lambda+\theta x_i}\\ &-\sum_{i=1}^{m}x_i-\sum_{i=1}^{m}\alpha x_i[k(R_i+1)-1]\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}.\end{aligned} \tag{2.5}$$

Solving $\frac{\partial l(\alpha,\theta,\lambda|x)}{\partial\alpha}=0$, $\frac{\partial l(\alpha,\theta,\lambda|x)}{\partial\theta}=0$,and $\frac{\partial l(\alpha,\theta,\lambda|x)}{\partial\lambda}=0$,we obtain $\hat{\alpha},\hat{\theta}$ and $\hat{\lambda}$.

Since, the Equations (2.3), (2.4) and (2.5) are nonlinear equations in three parameter $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\lambda}$.The exact solution is not easy to compute.Therefore a some numerical methods must be employed.

2.1. **Approximate confidence interval.** The asymptotic variance-covariance matrix of the estimators of the parameters $\varphi=(\varphi_1,\dots\varphi_n)$ is obtained by inverting the Fisher information matrix (given by taking the expectation of the second derivative of the log- likelihood functions) in which elements are negatives. In the present situation, ie seems appropriate to approximate the expected values by their MLE. Accordingly, the approximate variance-covariance matrix is given by [21].

$$\begin{pmatrix} -\frac{\partial^2 l}{\partial^2 \varphi_1^2} & \cdots & -\frac{\partial^2 l}{\partial^2 \varphi_1 \varphi_n} \\ \vdots & \vdots & \vdots \\ -\frac{\partial^2 l}{\partial^2 \varphi_n} & \cdots & -\frac{\partial^2 l}{\partial^2 \varphi_n^2} \end{pmatrix}^{-1}_{(\hat{\varphi_1},\ldots,\hat{\varphi_n})}$$

From the log-likelihood equation (2.2), we get

$$\frac{\partial^2 l(\alpha,\theta,\lambda|x)}{\partial\alpha^2} = -\frac{m}{\alpha^2} - \sum_{i=1}^{m}[k(R_i+1)-1]\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha}[\ln\left(\lambda x_i + \frac{1}{2}\theta x_i^2\right)]^2, \tag{2.6}$$

$$\begin{aligned} &\frac{\partial^2 l(\alpha,\theta,\lambda|x)}{\partial\theta^2} \\ &= \sum_{i=1}^{m}\frac{-x_i^2}{(\lambda+\theta x_i)^2} - \sum_{i=1}^{m}\frac{\alpha(\alpha-1)x_i^4}{4}[k(R_i+1)-1]\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-2}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\frac{\partial^2 l(\alpha,\theta,\lambda|x)}{\partial\lambda^2} \\ &= \sum_{i=1}^{m}\frac{-1}{(\lambda+\theta x_i)^2} - \sum_{i=1}^{m}\alpha(\alpha-1)x_i^2[k(R_i+1)-1]\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-2}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{\partial^2 l(\alpha,\theta,\lambda|x)}{\partial\alpha\partial\theta} &= \sum_{i=1}^{m}\frac{1}{2}x_i^2 - \sum_{i=1}^{m}[k(R_i+1)-1]\left(\frac{1}{2}x_i^2\frac{\left(\lambda x_i+\frac{1}{2}\theta x_i\right)^{\alpha}}{\lambda x_i+\frac{1}{2}\theta x_i^2}\right. \\ &\quad \left. + \frac{1}{2}\alpha x_i^2\left(\ln\left(\lambda x_i+\frac{1}{2}\theta x_i^2\right)\right)\left(\lambda x_i+\frac{1}{2}\theta x_i^2\right)^{\alpha-1}\right), \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 l(\alpha,\theta,\lambda|x)}{\partial\alpha\partial\lambda} &= \sum_{i=1}^{m}x_i - \sum_{i=1}^{m}[k(R_i+1)-1]\left(\frac{x_i\left(\lambda x_i+\frac{1}{2}\theta x_i\right)^{\alpha}}{\lambda x_i+\frac{1}{2}\theta x_i^2}\right. \\ &\quad \left. + \alpha x_i\left(\ln\left(\lambda x_i+\frac{1}{2}\theta x_i^2\right)\right)\left(\lambda x_i+\frac{1}{2}\theta x_i\right)^{\alpha-1}\right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\frac{\partial^2 l(\alpha,\theta,\lambda|x)}{\partial\theta\partial\lambda} \\ &= \sum_{i=1}^{m}\left(\frac{-x_i}{(\lambda+\theta x_i)^2}\right) - \sum_{i=1}^{m}[k(R_i+1)-1]\frac{\alpha\,(\alpha-1)\,x_i^3}{2}\left(\lambda x_i+\frac{1}{2}\theta x_i\right)^{\alpha-2}. \end{aligned} \tag{2.10}$$

Then, the asymptotic variance-covariance matrix of the estimators of the parameters α, θ and λ is obtained by inverting the Fisher information matrix given by taking the expectation of Equations (2.6)- (2.10), in which elements are negatives. In the present situation, it seems appropriate to approximate the expected values by their MLE. Accordingly, the approximate variance -covariance matrix is given as

$$\begin{pmatrix} \hat{\sigma_{\alpha\alpha}} & \hat{\sigma_{\alpha\theta}} & \hat{\sigma_{\alpha\lambda}} \\ \hat{\sigma_{\alpha\theta}} & \hat{\sigma_{\theta\theta}} & \hat{\sigma_{\theta\lambda}} \\ \hat{\sigma_{\alpha\lambda}} & \hat{\sigma_{\theta\lambda}} & \hat{\sigma_{\lambda\lambda}} \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2 \alpha^2} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2 \alpha\theta} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2 \alpha\lambda} \\ -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2 \alpha\theta} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2 \theta^2} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2 \theta\lambda} \\ -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2 \alpha\lambda} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2 \theta\lambda} & -\frac{\partial^2 l(\alpha,\lambda,\theta)}{\partial^2 \lambda^2} \end{pmatrix}^{-1}_{(\hat{\alpha},\hat{\theta},\hat{\lambda})}.$$

The approximate confidence interval for the parameters α, θ and λ are respectively given as:

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{\sigma}_{\alpha\alpha}}, \quad \hat{\theta} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{\sigma}_{\theta\theta}} \text{ and } \quad \hat{\lambda} \pm Z_{\frac{\gamma}{2}} \sqrt{\hat{\sigma}_{\lambda\lambda}},$$

where $Z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\gamma}{2}$.

3. Bayesian Estimation

In this section, we describe the Bayesian estimators of the unknown parameters of GLED based on a progressive first-failure type-II censored sample. In Bayesian estimation, we consider three types of loss functions. The first is squared error loss function (SEL) which is classified as a symmetric function and associates equal importance to losses for overestimation and underestimation of equal magnitude. The second is LINEX loss function which is asymmetric, was introduced by [23]. These loss functions were widely used by several authors, among of them [24], [25], [26] and [27]. The third is the Entropy loss used by several authors [28] and [29]. From the likelihood function in Equation (2.1), the joint conjugate priors for the unknown parameters are not easy to obtain. Here, we assume the following independent gamma priors for the parameters α, θ and λ as

$$g_1(\alpha) \propto \alpha^{a_2-1} e^{-\alpha a_1}, \quad a_1 > 0, a_2 > 0 \quad \alpha > 0,$$

$$g_2(\theta) \propto \theta^{a_4-1} e^{-\theta a_3}, \quad a_3 > 0, a_4 > 0, \ \theta > 0,$$

and

$$g_3(\lambda) \propto \lambda^{a_6-1} e^{-\lambda a_5}, \quad a_6 > 0, a_5 > 0, \ \lambda \geq 0,$$

where $a_1, a_2, a_3, a_4, a_5, a_6$ are chosen to reflect prior knowledge about α, θ and λ. When $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$, there are non-informative priors of α, θ and λ.

The joint prior distribution of α, θ and λ can be written as

$$g(\alpha,\theta,\lambda) \propto \alpha^{a_2-1}e^{-\alpha a_1}\ \theta^{a_4-1}e^{-\theta a_3}\ \lambda^{a_6-1}e^{-\lambda a_5}\quad ,\ \alpha,\theta>0,\ \lambda\geq 0,\tag{3.1}$$

$$a_1, a_2, a_3, a_4, a_5, a_6 > 0.$$

Thus, using the likelihood function in Equation (2.1) and joint prior distribution in Equation (3.1), the joint posterior distribution of α, θ and λ is given by

$$\begin{aligned}\pi(\alpha,\theta,\lambda \mid x) &= \frac{L(x|\alpha,\theta,\lambda)g(\alpha,\theta,\lambda)}{\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty L(x|\alpha,\theta,\lambda)g(\alpha,\theta,\lambda)\,\alpha\theta\lambda},\\ &= \frac{1}{T}k^m\alpha^{a_2+m-1}e^{-\alpha a_1}\ \theta^{a_4-1}e^{-\theta a_3}\ \ \lambda^{a_6-1}e^{-\lambda a_5}\\ &\times\prod_{i=1}^{m}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}(\lambda+\theta x_i)\,e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^\alpha}\left[e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^\alpha}\right]^{k(R_i+1)-1},\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}T=&\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty\left[\alpha^{a_2+m-1}e^{-\alpha a_1}\ \theta^{a_4-1}e^{-\theta a_3}\ \lambda^{a_6-1}e^{-\lambda a_5}\right.\\ &\times\prod_{i=1}^{m}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}(\lambda+\theta x_i)\,e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^\alpha}\left[e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^\alpha}\right]^{k(R_i+1)-1}.\end{aligned}$$

In Bayesian statistics the posterior distribution $\pi(\alpha,\theta,\lambda \mid x)$ contains all information on the unknown parameters given the observed data. All statistical inference can be deduced from the posterior distribution. We observe that equation (3.2) can not solved explicitly, so we use Markov Chain Monte Carlo (MCMC) technique to obtain the Bayes estimator for α, θ and λ and the corresponding HPD credible intervals . There are several conventional methods to define such Markov Chains exist, including Gibbs sampling, Metropolis-Hastings (MH) and reversible jump. Using these algorithms it is possible to implement posterior simulation in essentially any issue which allow point wise evaluation of the prior distribution and likelihood function. From Equation (3.2), the conditional posterior of α, is proportional to

$$g_1^*(\alpha \mid \theta, \lambda,\ x) \ \propto \ \alpha^{a_2+m-1}e^{-\alpha a_1} \prod_{i=1}^{m}\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}$$

$$(\lambda+\theta x_i)\, e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}\left[e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}\right]^{k(R_i+1)-1}, \tag{3.3}$$

Similarly, the conditional posterior distribution for θ and λ are respectively

$$g_2^*(\theta \mid \alpha, \lambda,\ x) \ \propto \ \theta^{a_4+m-1}e^{-\theta a_3} \prod_{i=1}^{m}\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1}$$

$$(\lambda+\theta x_i)\, e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}\left[e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}\right]^{k(R_i+1)-1}, \tag{3.4}$$

and

$$g_3^*(\lambda \mid \alpha, \theta,\ x) \ \propto \ \theta^{a_6+m-1}e^{-\theta a_5} \prod_{i=1}^{m}\left(\lambda x_i + \frac{\theta}{2}x_i^2\right)^{\alpha-1} \tag{3.5}$$

$$(\lambda+\theta x_i)\, e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}\left[e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}\right]^{k(R_i+1)-1}.$$

We can see the conditional posteriors distributions for α, θ and λ are log-concave and can not be reduced analytically to well known distributions. So, as suggested by [22], a common way to solve this problem is to use the hybrid algorithm by combined a MH sampling with Gibbs sampling scheme using normal distribution. There for the algorithm works as follow:

(i) Set the initial values of α, θ and λ say (α_0, θ_0,λ_0).
(ii) Set $j = 1$.
(iii) Using MH, generate α_1^j from $g_1^*(\alpha^{j-1} \mid \theta^{j-1}$,$\lambda^{j-1}$,$x$) with normal distribution, $N(\alpha^{j-1}$,$\beta\alpha V\alpha)$.
(iv) Using MH, generate θ_1^j from $g_2^*(\theta^{j-1} \mid \alpha^{j-1}$,$\lambda^{j-1}$,$x$) with normal distribution, $N(\theta^{j-1}$,$\beta_\theta V_\theta)$.
(v) Using MH, generate λ_1^j from $g_3^*(\lambda^{j-1} \mid \alpha^{j-1}$,$\theta^{j-1}$,$x$) with normal distribution, $N(\lambda^{j-1}$,$\beta_\lambda V_\theta)$,where $\beta_\alpha, \beta_\theta$ and β_λ are scaling factor and $V\alpha$, V_θ and V_θ are variances-covariance matrix.
(vi) Set $j = j + 1$.
(vii) Repeat steps from 1to 5 N times and obtain $f_1(\alpha_j, \theta_j, \lambda_j), j = 1, \ldots, N$.
(viii) The Bayes estimators of $u(\alpha$, θ, $\lambda)$ can be approximated as:

$$\hat{u}_{Mc} \approx \frac{\frac{1}{N}\sum_{j=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)} \tag{3.6}$$

where

$$f_1(\alpha,\theta,\lambda)=\prod_{i=1}^{m}\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha-1}(\lambda+\theta x_i)\,e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}[e^{-\left(\lambda x_i+\frac{\theta}{2}x_i^2\right)^{\alpha}}]^{k(R_i+1)-1},$$

(ix) Ordered α_j , θ_j and λ_j ,$j=1,\dots N$ and suppose that we would like to construct the HPD credible intervals of α,θ and λ Now, we construct all $100(1-\eta)\%$ credible intervals of α say $(\alpha_{[1]},\alpha_{[N(1-\alpha)]}),\dots(\alpha_{[N\alpha]},\alpha_{[N]})$.Here [X] denotes the largest integer less than or equal to X. Then the HPD credible interval of α is that interval which has the shortest length. Similarity, the HPD credible interval of θ and λ can also be constructed.

3.1. **Bayes Estimate Based on MCMC under LINEX Loss Function.**

(i) For estimating α, consider $u\,(\alpha_j,\theta_j,\lambda_j)=\exp[-h\alpha_j]$, then

$$\hat{\alpha}_{BL}=\frac{-1}{h}\log\left[\frac{\frac{1}{N}\sum_{j=1}^{N}u\,(\alpha_j,\theta_j,\lambda_j)\;f_1(\alpha_j,\theta_j,\lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\;f_1(\alpha_j,\theta_j,\lambda_j)}\right]. \tag{3.7}$$

(ii) For estimating θ, consider $u\,(\alpha_j,\theta_j,\lambda_j)=\exp[-h\theta_j]$, then

$$\hat{\theta}_{BL}=\frac{-1}{h}\log\left[\frac{\frac{1}{N}\sum_{j=1}^{N}u\,(\alpha_j,\theta_j,\lambda_j)\;f_1(\alpha_j,\theta_j,\lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\;f_1(\alpha_j,\theta_j,\lambda_j)}\right]. \tag{3.8}$$

(iii) For estimating λ, consider $u\,(\alpha_j,\theta_j,\lambda_j)=\exp[-h\lambda_j]$, then

$$\hat{\lambda}_{BL}=\frac{-1}{h}\log\left[\frac{\frac{1}{N}\sum_{i=1}^{N}u\,(\alpha_j,\theta_j,\lambda_j)\;f_1(\alpha_j,\theta_j,\lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\;f_1(\alpha_j,\theta_j,\lambda_j)}\right]. \tag{3.9}$$

where h represents the direction and degree of symmetry, (if $h>0$, the overestimation is more serious than underestimation, and vice-versa). For h close to zero, the LINEX loss is approximately SEL and therefore almost symmetric. The estimators $\hat{\alpha}_{BL},\hat{\theta}_{BL}$ and $\hat{\lambda}_{BL}$ are Bayes estimators obtained by using the approximation for LINEX loss function.

3.2. **Bayes Estimate Based on MCMC under Squared Error Loss Function.**

(i) For estimating α, consider $u\,(\alpha_j,\theta_j,\lambda_j)=\alpha_j$, then

$$\hat{\alpha}_{SL}=\frac{\frac{1}{N}\sum_{j=1}^{N}u\,(\alpha_j,\theta_j,\lambda_j)\;f_1(\alpha_j,\theta_j,\lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\;f_1(\alpha_j,\theta_j,\lambda_j)}. \tag{3.10}$$

(ii) For estimating θ, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \theta_j$, then

$$\hat{\theta}_{SL} = \frac{\frac{1}{N}\sum_{j=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)}. \tag{3.11}$$

(iii) For estimating λ, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \lambda_j$, then

$$\hat{\lambda}_{SL} = \frac{\frac{1}{N}\sum_{i=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)}. \tag{3.12}$$

The estimators $\hat{\alpha}_{SL}, \hat{\theta}_{SL}$ and $\hat{\lambda}_{SL}$are Bayes estimators obtained by using the approximation for SEL function.

3.3. **Bayes estimate based on MCMC under Entropy Function.**

(i) For estimating α, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \alpha_j^{-h}$, therefore

$$\hat{\alpha}_{BE} = \left(\frac{\frac{1}{N}\sum_{j=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)}\right)^{-\frac{1}{h}}. \tag{3.13}$$

(ii) For estimating θ, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \theta_j$, then

$$\hat{\theta}_{SL} = \frac{\frac{1}{N}\sum_{j=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)}. \tag{3.14}$$

(iii) For estimating λ, consider $u\ (\alpha_j, \theta_j, \lambda_j) = \lambda_j^{-h}$, then

$$\hat{\lambda}_{BE} = \left(\frac{\frac{1}{N}\sum_{j=1}^{N} u\ (\alpha_j, \theta_j, \lambda_j)\ f_1(\alpha_j, \theta_j, \lambda_j)}{\frac{1}{N}\sum_{j=1}^{N}\ f_1(\alpha_j, \theta_j, \lambda_j)}\right)^{-\frac{1}{h}}. \tag{3.15}$$

The estimators $\hat{\alpha}_{BE}, \hat{\theta}_{BE}$ and $\hat{\lambda}_{BE}$ are Bayes estimators obtained by using the approximation for Entropy function.

4. Simulation Study

This section deals with obtaining some numerical results. We simulated 1000 progressively first-failure censored samples from GLED $(\alpha, \theta, \lambda)$. The samples were simulated by using the algorithm described in [30]. We used different sample sizes n, different effective sample of sizes m, different k, and different of sampling schemes to compare the MLEs and different Bayes estimators. First we used the non informative gamma priors for the three parameters, that is when hyper parameters are 0 ($a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$). Second, we used informative prior, $a_1 = 2$, $a_2 = 0.05, a_3 = 1, a_4 = 2, a_5 = 0.05$ and $a_6 = 1$. In two cases, we used the SEL, LINEX loss function and Entropy function to compute the Bayes estimates. We also computed the Bayes estimates and 95% credible intervals based on 10000 MCMC samples and discard the first 1000 values as burn-in. Also, we compute the MLEs and 95% confidence intervals based on the observed Fisher information matrix. The different censoring schemes used in this article have been represented by short notations such as (0*3) denotes (0,0,0) and ((1,0)*3) denotes (1,0,1,0,1,0). We are giving results only for two different group sizes $k = 2, 5$ and for each group size, two different no. of groups $n = 30, 50$ and for each n, there are different failure information m and $h = 2$. The results of simulations are presented in Tables 1-4. From these tables the following conclusions are made:

(i) In case of maximum likelihood estimation, as n increases the MSE of estimates decrease as expected. Moreover, as m increases, the MSE of estimates decrease. Also, as the value of the group size k increases, the MSE of most parameter almost decrease.

(ii) Bayes estimates are also very good in respect of MSE. Bayes estimates are better than MLEs in respect of MSE as they include prior information. It is also observed that as the failure proportion m/n increases, the point estimates become even better. Also, as group size k increases, the MSE increase in most of the cases.

(iii) Average length of confidence/HPD credible intervals narrow down as n increases. HPD credible intervals are better than confidence intervals in respect of average length. Also, as the group size k increases, the average length of confidence intervals increases while the average length of HPD credible intervals narrows down almost in all cases.

(iv) The MSE of Bayesian estimators (SEL, LINEX and Entropy) is always similar in most cases.

TABLE 1. Average values of the different estimators and the corresponding MSEs when $\alpha = 2, \theta = 3$ and $\lambda = 1$ with informative priors

				MLE			Bayes SEL			Bayes LINEX			Bayes Entropy		
k	n	m	*Scheme*	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$
2	30	15	$(15, 0^{*14})$	2.2762	3.3637	1.0016	2.265	3.069	0.994	2.265	3.013	0.994	2.265	3.029	0.994
				(0.19924)	(0.14726)	(0.01015)	(0.18494)	(0.31988)	(0.00739)	(0.18491)	(0.40873)	(0.00738)	(0.18492)	(0.38268)	(0.00738)
			(1^{*15})	1.9912	2.7535	1.0145	1.99	2.813	1.027	1.99	2.802	1.027	1.99	2.806	1.027
				(0.10463)	(0.94217)	(0.14202)	(0.1031)	(0.84647)	(0.15307)	(0.1031)	(0.85048)	(0.15302)	(0.13031)	(0.85507)	(0.15303)
			$(0^{*14}, 15)$	1.4648	3.238	0.4938	1.464	3.205	0.495	1.464	3.305	0.495	1.464	3.305	0.495
				(0.31705)	(0.6293)	(0.26476)	(0.31712)	(0.5682)	(0.26339)	(0.31712)	(0.5687)	(0.2634)	(0.31712)	(0.5686)	(0.2634)
		20	$(10, 0^{*19})$	1.7546	2.9554	0.9343	1.755	2.959	0.934	1.755	2.959	0.934	1.755	2.959	0.934
				(0.06097)	(0.1267)	(0.04957)	(0.06082)	(0.00759)	(0.05085)	(0.06082)	(0.00767)	(0.05085)	(0.06082)	(0.00763)	(0.05085)
			$(0^{*5}, 1^{*10}, 0^{*5})$	2.0397	2.7224	0.7961	2.04	2.77	0.795	2.04	2.769	0.794	2.04	2.769	0.794
				(0.0674)	(0.5798)	(0.05734)	(0.06792)	(0.6887)	(0.05767)	(0.06791)	(0.6883)	(0.05768)	(0.06791)	(0.68969)	(0.0577)
			$(0^{*19}, 10)$	1.7612	3.2911	0.7323	1.761	3.191	0.736	1.761	3.191	0.736	1.761	3.192	0.736
				(0.14992)	(0.24008)	(0.08772)	(0.14916)	(0.29629)	(0.08725)	(0.14916)	(0.29696)	(0.08727)	(0.14916)	(0.29688)	(0.08729)
	50	30	$(20, 0^{*29})$	1.8584	2.8134	0.8816	1.858	2.822	0.883	1.858	2.821	0.883	1.858	2.821	0.883
				(0.03423)	(0.12167)	(0.02192)	(0.03409)	(0.11209)	(0.02122)	(0.03409)	(0.11213)	(0.02123)	(0.03409)	(0.11212)	(0.02123)
			$(0^{*5}, 1^{*20}, 0^{*5})$	2.0679	2.895	1.1247	2.07	2.853	1.129	2.07	2.851	1.129	2.07	2.852	1.129
				(0.01862)	(0.46402)	(0.03061)	(0.01904)	(0.48501)	(0.03091)	(0.01904)	(0.48625)	(0.03092)	(0.01904)	(0.48594)	(0.03091)
			$(0^{*29}, 20)$	2.0159	3.2862	0.812	2.018	3.108	0.825	2.018	3.091	0.825	2.018	3.1	0.825
				(0.03127)	(0.191603)	(0.05248)	(0.03478)	(0.27936)	(0.04627)	(0.03478)	(0.25764)	(0.04628)	(0.03478)	(0.27107)	(0.0463)
		40	$(10, 0^{*39})$	2.022	2.8267	1.0852	2.026	2.803	1.079	2.026	2.798	1.079	2.026	2.8	1.079
				(0.02516)	(0.07966)	(0.2159)	(0.0254)	(0.08354)	(0.02064)	(0.0254)	(0.08628)	(0.02063)	(0.0254)	(0.08508)	(0.02062)
			$(0^{*15}, 1^{*10}, 0^{*15})$	2.0249	3.1337	0.8623	2.024	3.231	0.863	2.024	3.23	0.863	2.024	3.231	0.863
				(0.013215)	(0.12719)	(0.02024)	(0.013306)	(0.20162)	(0.02044)	(0.013305)	(0.20007)	(0.02044)	(0.013306)	(0.20097)	(0.02044)
			$(0^{*39}, 10)$	2.0227	2.8267	1.0852	2.026	2.798	1.079	2.026	2.808	1.079	2.026	2.8	1.079
				(0.02516)	(0.07966)	(0.02159)	(0.0254)	(0.08628)	(0.02063)	(0.0254)	(0.08079)	(0.02064)	(0.0254)	(0.08508)	(0.02062)
5	30	15	$(15, 0^{*14})$	1.7295	2.7162	0.8432	1.729	2.63	0.842	1.729	2.629	0.842	1.729	2.629	0.842
				(0.14071)	(0.8919)	(0.04229)	(0.14063)	(0.81326)	(0.04204)	(0.14063)	(0.81302)	(0.04204)	(0.14063)	(0.81331)	(0.04204)
			(1^{*15})	1.3129	2.7655	0.6107	1.313	2.773	0.614	1.313	2.773	0.614	1.313	2.773	0.614
				(0.47216)	(0.05499)	(0.15158)	(0.47198)	(0.0514)	(0.14898)	(0.47198)	(0.0516)	(0.14898)	(0.47198)	(0.05151)	(0.14898)
			$(0^{*14}, 15)$	1.7779	2.5209	0.8032	1.777	2.513	0.805	1.777	2.501	0.805	1.777	2.506	0.805
				(0.04935)	(0.22951)	(0.03875)	(0.04991)	(0.23744)	(0.03797)	(0.04991)	(0.24891)	(0.03801)	(0.04991)	(0.24437)	(0.03805)
		20	$(10, 0^{*19})$	1.9472	2.4868	0.9224	1.947	2.435	0.92	1.947	2.426	0.92	1.947	2.429	0.90
				(0.08386)	(0.76673)	(0.03933)	(0.08455)	(0.77016)	(0.0372)	(0.08455)	(0.78221)	(0.0371)	(0.08455)	(0.77828)	(0.037)
			$(0^{*5}, 1^{*10}, 0^{*5})$	2.2574	3.1229	1.0388	2.254	3.127	1.036	2.254	3.124	1.036	2.254	3.126	1.036
				(0.06624)	(0.01511)	(0.0015)	(0.06443)	(0.0161)	(0.00133)	(0.06443)	(0.01547)	(0.00133)	(0.06443)	(0.0158)	(0.00133)
			$(0^{*19}, 10)$	2.2086	3.2167	0.9231	2.21	3.2167	0.933	2.21	3.269	0.933	2.209	3.263	0.933
				(0.04353)	(0.04698)	(0.00592)	(0.04389)	(0.07027)	(0.00452)	(0.04389)	(0.07233)	(0.00452)	(0.04389)	(0.06938)	(0.00453)
	50	30	$(20, 0^{*29})$	2.0531	2.9322	0.9685	2.05	3.006	0.968	2.05	3.004	0.968	2.05	3.005	0.968
				(0.05444)	(0.70911)	(0.01117)	(0.05304)	(0.66491)	(0.0105)	(0.05304)	(0.62081)	(0.0105)	(0.05304)	(0.65061)	(0.0105)
			$(0^{*5}, 1^{*20}, 0^{*5})$	2.1811	2.9366	0.9486	2.179	2.827	0.949	2.179	2.827	0.949	2.179	2.823	0.949
				(0.03279)	(0.00402)	(0.00145)	(0.03202)	(0.0161)	(0.00135)	(0.03203)	(0.016)	(0.00132)	(0.03202)	(0.01026)	(0.00132)
			$(0^{*29}, 20)$	1.8508	2.8953	1.0762	1.849	2.861	1.067	1.849	2.861	1.067	1.849	2.861	1.067
				(0.02226)	(0.01096)	(0.00581)	(0.02284)	(0.01928)	(0.00452)	(0.02284)	(0.0194)	(0.00452)	(0.02284)	(0.01934)	(0.00452)
		40	$(10, 0^{*39})$	1.9418	2.9558	0.9613	1.94	2.986	0.91	1.94	2.985	0.91	1.94	2.985	0.91
				(0.03818)	(0.29319)	(0.01114)	(0.03758)	(0.27358)	(0.0032)	(0.03758)	(0.27348)	(0.0033)	(0.03758)	(0.27358)	(0.003)
			$(0^{*15}, 1^{*10}, 0^{*15})$	2.1273	3.04	1.024	2.123	3.03	1.02	2.123	3.03	1.02	2.123	3.03	1.02
				(0.01621)	(0.00182)	(0.00142)	(0.01503)	(0.00904)	(0.00116)	(0.01503)	(0.00904)	(0.00116)	(0.01503)	(0.00115)	(0.00116)
			$(0^{*39}, 10)$	2.0579	2.8958	1.0746	2.061	3.046	1.082	2.061	3.042	1.052	2.061	3.044	1.052
				(0.00335)	(0.01085)	(0.00517)	(0.00377)	(0.00211)	(0.00317)	(0.00377)	(0.0018)	(0.00432)	(0.00377)	(0.00195)	(0.00432)

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TABLE 2. Average values of the different estimators and the corresponding MSEs when

$\alpha = 2, \theta = 3$ and $\lambda = 1$ with non informative priors

				MLE			Bayes SEL			Bayes LINEX			Bayes Entropy		
k	n	m	Scheme	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$
2	30	15	$(15, 0^{*14})$	2.6030	2.3234	1.3433	2.626	2.508	1.342	2.626	2.501	1.342	2.626	2.504	1.3422
				(0.36379)	(0.45777)	(0.11785)	(0.39221)	(0.2422)	(0.11681)	(0.39193)	(0.24936)	(0.11679)	(0.39205)	(0.24647)	(0.11679)
			(1^{*15})	2.1099	3.6459	0.6685	2.11	3.989	0.668	2.117	3.956	0.668	2.117	3.976	0.667
				(0.01208)	(0.41713)	(0.1099)	(0.01362)	(0.97772)	(0.10991)	(0.0136)	(0.91458)	(0.11026)	(0.01361)	(0.95308)	(0.11071)
			$(0^{*14}, 15)$	1.2114	3.2655	0.6567	1.209	3.192	0.656	1.209	3.19	0.656	1.209	3.191	0.656
				(0.62182)	(0.07052)	(0.11783)	(0.62526)	(0.03699)	(0.11857)	(0.62526)	(0.03616)	(0.11857)	(0.62526)	(0.0366)	(0.11858)
		20	$(10, 0^{*19})$	2.5357	2.7446	0.8801	2.536	2.641	0.88	2.536	2.645	0.88	2.536	2.639	0.88
				(0.28698)	(0.06523)	(0.01438)	(0.287)	(0.12871)	(0.01442)	(0.28703)	(0.12631)	(0.01442)	(0.28698)	(0.1301)	(0.01443)
			$(0^{*5}, 1^{*10}, 0^{*5})$	1.9296	2.3884	0.8739	1.929	2.378	0.873	1.929	2.378	0.873	1.929	2.378	0.873
				(0.00496)	(0.37408)	(0.01591)	(0.00509)	(0.38629)	(0.0161)	(0.00509)	(0.38653)	(0.0161)	(0.00509)	(0.38644)	(0.0161)
			$(0^{*19}, 10)$	1.7057	2.969	0.8189	1.702	2.935	0.8291	1.702	2.936	0.821	1.702	2.9357	0.821
				(0.08664)	(0.049439)	(0.0328)	(0.08857)	(0.041105)	(0.03215)	(0.08857)	(0.041476)	(0.03215)	(0.08857)	(0.041342)	(0.03215)
	50	30	$(20, 0^{*29})$	2.0949	2.9393	1.1123	2.092	2.943	1.1	2.092	2.944	1.1	2.092	2.942	1.1
				(0.00901)	(0.00369)	(0.01261)	(0.00852)	(0.0033)	(0.00992)	(0.00852)	(0.00314)	(0.00993)	(0.00852)	(0.00338)	0.00991)
			$(0^{*5}, 1^{*20}, 0^{*5})$	2.0682	3.5751	1.0237	2.072	3.534	1.018	2.072	3.534	1.018	2.072	3.534	1.018
				(0.00465)	(0.33076)	(0.00056)	(0.0052)	(0.28541)	(0.00034)	(0.00519)	(0.28513)	(0.00034)	(0.0052)	(0.28529)	(0.00034)
			$(0^{*29}, 20)$	2.04	3.0689	0.8568	2.037	3.028	0.838	2.037	3.026	0.838	2.037	3.027	0.838
				(0.0016)	(0.00475)	(0.02051)	(0.00136)	(0.00077)	(0.0216)	(0.00136)	(0.00069)	(0.02617)	(0.00136)	(0.00073)	(0.02619)
		40	$(10, 0^{*39})$	2.0399	3.0648	1.087	2.038	3.092	1.075	2.038	3.09	1.075	2.038	3.092	1.075
				(0.00159)	(0.0042)	(0.00772)	(0.00147)	(0.00851)	(0.00561)	(0.00147)	(0.0085)	(0.00561)	(0.00147)	(0.00851)	(0.0056)
			$(0^{*15}, 1^{*10}, 0^{*15})$	2.0043	2.7718	1.022	2.004	2.732	1.0224	2.004	2.731	1.021	2.004	2.732	1.021
				(0.00002)	(0.05207)	(0.00027)	(0.00002)	(0.07196)	(0.00029)	(0.00002)	(0.0721)	(0.00029)	(0.00002)	(0.07204)	(0.000292)
			$(0^{*39}, 10)$	2.033	2.9483	0.8566	2.03	2.993	0.851	2.03	2.993	0.851	2.03	2.993	0.851
				(0.0014)	(0.00268)	(0.0201)	(0.00092)	(0.00005)	(0.02131)	(0.00092)	(0.00005)	(0.02134)	(0.00092)	(0.00005)	(0.02134)
5	30	15	$(15, 0^{*14})$	1.5405	3.4182	0.5184	1.544	3.385	0.518	1.544	3.383	0.518	1.544	3.384	0.518
				(0.21114)	(0.17505)	(0.23193)	(0.20837)	(0.14845)	(0.2326)	(0.20838)	(0.1466)	(0.23261)	(0.20837)	(0.14763)	(0.23262)
			(1^{*15})	2.1018	3.7249	0.7556	2.108	3.782	0.755	2.108	3.782	0.755	2.108	3.781	0.755
				(0.01036)	(0.5255)	(0.05971)	(0.01157)	(0.61145)	(0.06009)	(0.01157)	(0.60897)	(0.06009)	(0.01157)	(0.61046)	(0.0601)
			$(0^{*14}, 15)$	1.8609	3.6636	0.674	1.857	3.618	0.67	1.857	3.614	0.67	1.857	3.617	0.67
				(0.01934)	(0.4404)	(0.10626)	(0.02041)	(0.38218)	(0.10902)	(0.02041)	(0.37726)	(0.10903)	(0.02041)	(0.38012)	(0.10903)
		20	$(10, 0^{*19})$	1.9537	3.3627	0.7819	1.953	3.411	0.784	1.953	3.41	0.784	1.953	3.41	0.784
				(0.00214)	(0.13154)	(0.04756)	(0.00217)	(0.14685)	(0.04663)	(0.00217)	(0.1414)	(0.04663)	(0.00217)	(0.1414)	(0.04663)
			$(0^{*5}, 1^{*10}, 0^{*5})$	1.9491	2.391	1.0816	1.955	2.452	1.086	1.955	2.451	1.086	1.955	2.451	1.086
				(0.00259)	(0.37089)	(0.00666)	(0.00207)	(0.30024)	(0.0074)	(0.00207)	(0.30126)	(0.00739)	(0.00207)	(0.30087)	(0.00738)
			$(0^{*19}, 10)$	2.0596	2.7411	0.8913	2.062	2.701	0.84	2.062	2.67	0.84	2.062	2.684	0.84
				(0.00355)	(0.06703)	(0.01181)	(0.00382)	(0.08945)	(0.02559)	(0.00382)	(0.1087)	(0.02564)	(0.00382)	(0.09983)	(0.02568)
	50	30	$(20, 0^{*29})$	1.975	3.2785	1.0047	1.969	3.255	1.01	1.969	3.253	1.01	1.969	3.254	1.01
				(0.00062)	(0.07755)	(0.00002)	(0.00094)	(0.06513)	(0.0001)	(0.00094)	(0.06414)	(0.00009)	(0.00094)	(0.06467)	(0.00009)
			$(0^{*5}, 1^{*20}, 0^{*5})$	1.9811	3.1124	0.9386	1.986	3.242	0.944	1.986	3.242	0.944	1.986	3.242	0.944
				(0.00036)	(0.01264)	(0.00377)	(0.00019)	(0.05863)	(0.00316)	(0.00019)	(0.05844)	(0.00316)	(0.00019)	(0.05854)	(0.00316)
			$(0^{*29}, 20)$	1.9775	3.2091	0.9077	1.981	3.025	0.908	1.981	3.024	0.908	1.981	3.024	0.908
				(0.00051)	(0.04372)	(0.00852)	(0.00037)	(0.00061)	(0.00849)	(0.00037)	(0.00057)	(0.00849)	(0.00037)	(0.00059)	(0.0085)
		40	$(10, 0^{*39})$	2.0135	2.7933	1.0042	2.013	2.768	1.001	2.013	2.767	1.001	2.013	2.767	1.001
				(0.00018)	(0.04273)	(0.00001)	(0.00016)	(0.05376)	(0.0000)	(0.00016)	(0.05437)	(0.000)	(0.00016)	(0.05409)	(0.000)
			$(0^{*15}, 1^{*10}, 0^{*15})$	1.986	3.0426	0.9522	1.982	3.208	0.952	1.982	3.208	0.952	1.982	3.208	0.952
				(0.00019)	(0.00182)	(0.00228)	(0.00031)	(0.0495)	(0.00226)	(0.00031)	(0.0495)	(0.00226)	(0.00031)	(0.0495)	(0.00226)
			$(0^{*39}, 10)$	1.9773	3.1023	0.9224	1.971	3.002	0.714	1.974	3.002	0.914	1.973	3.002	0.914
				(0.00037)	(0.01046)	(0.00602)	(0.000)	(0.00031)	(0.00747)	(0.00037)	(0.00031)	(0.00747)	(0.0007)	(0.00031)	(0.00747)

5. REAL DATA

In this section, we consider a real data set to illustrate the estimation methods developed in this paper. We consider a real data set given by The [31]. These data

TABLE 3. The 95% ACI's and HPD credible intervals and the corresponding length of α, θ and λ for non informative priors

				ACI						HPD					
k	n	m	scheme	$\hat{\alpha}$		$\hat{\theta}$		$\hat{\lambda}$		$\hat{\alpha}$		$\hat{\theta}$		$\hat{\lambda}$	
				interval	length	interval	length	interval	length	interval	length	interval	length	interval	length
2	30	15	$(15,0^{*14})$	$\{-0.6048,5.8111\}$	6.41592	$\{0.326,5.9728\}$	5.6468	$\{0.1717,4.8583\}$	4.6866	$\{2.5996,2.645\}$	0.0454555	$\{2.3527,2.637\}$	0.284276	$\{1.3331,1.3505\}$	0.017336
			(1^{*15})	$\{-2.4977,6.7174\}$	9.2151	$\{0.9441,5.5358\}$	4.5939	$\{0.7629,6.0999\}$	5.337	$\{2.1032,2.1346\}$	0.0314526	$\{3.6741,4.3867\}$	0.712552	$\{0.6294,0.7033\}$	0.0738712
			$(0^{*14},15)$	$\{0.7431,3.5542\}$	2.8111	$\{0.5443,5.1245\}$	4.5802	$\{0.5461,5.3254\}$	4.7793	$\{1.2058,1.2119\}$	0.00607397	$\{3.1138,3.2614\}$	0.147595	$\{0.6514,0.6602\}$	0.00876122
		20	$(10,0^{*19})$	$\{0.0126,5.0588\}$	5.04618	$\{-1.3581,3.8473\}$	5.2054	$\{-1.5099,3.27\}$	4.93728	$\{2.5268,2.5441\}$	0.017244	$\{2.5421,2.7211\}$	0.179078	$\{0.8754,0.8867\}$	0.011332
			$(0^{*5},1^{*10},0^{*5})$	$\{0.264,3.5951\}$	3.33114	$\{0.1506,8.9273\}$	8.7767	$\{-1.1249,2.8726\}$	3.2314	$\{1.9257,1.932\}$	0.00633213	$\{2.346,2.402\}$	0.0560248	$\{0.8688,0.8792\}$	0.0103267
			$(0^{*19},10)$	$\{0.73215,5.1287\}$	4.39655	$\{0.3698,4.6782\}$	4.3084	$\{-1.0253,3.4552\}$	4.4805	$\{1.6995,1.7049\}$	0.00536944	$\{2.2952,3.4442\}$	1.149	$\{0.8168,0.8246\}$	0.0078561
	50	30	$(20,0^{*29})$	$\{0.048,4.1419\}$	4.09391	$\{-0.4931,3.3717\}$	3.8648	$\{-1.6414,3.866\}$	6.34582	$\{2.0895,2.0982\}$	0.00871099	$\{2.8695,3.0134\}$	0.143955	$\{1.0936,1.1091\}$	0.0154608
			$(0^{*5},1^{*20},0^{*5})$	$\{0.4414,3.6949\}$	3.25352	$\{-0.1218,3.7272\}$	3.849	$\{-1.185,3.2324\}$	3.83023	$\{2.07,2.0748\}$	0.00482677	$\{3.5021,3.5654\}$	0.0633026	$\{1.0166,1.0206\}$	0.0039782
			$(0^{*29},20)$	$\{-0.3213,4.4012\}$	4.72247	$\{0.3754,5.5132\}$	5.1378	$\{-2.1052,3.8188\}$	8.03929	$\{2.031,2.0426\}$	0.0115413	$\{2.9413,3.092\}$	0.150677	$\{0.8289,0.8498\}$	0.0209028
		40	$(10,0^{*39})$	$\{0.9028,3.1769\}$	2.27402	$\{-1.4912,7.6208\}$	9.11198	$\{-0.4924,2.1448\}$	1.05621	$\{2.0371,2.0392\}$	0.00203615	$\{3.0802,3.1048\}$	0.0246265	$\{0.8244,1.8283\}$	1.00389
			$(0^{*15},1^{*10},0^{*15})$	$\{0.8295,3.179\}$	2.34951	$\{-2.7455,5.2891\}$	8.0346	$\{-0.4529,2.5573\}$	1.15819	$\{2.0034,2.0061\}$	0.00267866	$\{2.7072,2.7683\}$	0.0610953	$\{1.0514,1.0557\}$	0.00431681
			$(0^{*39},10)$	$\{0.5039,3.5635\}$	3.05967	$\{-0.304,5.2005\}$	5.5045	$\{-0.9997,2.6279\}$	2.62698	$\{2.0289,2.0337\}$	0.00482107	$\{2.9654,3.0236\}$	0.0582215	$\{0.812,1.818\}$	1.0698
5	30	15	$(15,0^{*14})$	$\{0.4055,2.8618\}$	2.45626	$\{-0.2996,6.6886\}$	6.9882	$\{-1.0515,3.0484\}$	3.20551	$\{1.5301,1.6341\}$	0.104	$\{2.9086,3.8134\}$	0.8274	$\{0.3977,1.0043\}$	0.6066
			(1^{*15})	$\{-0.5833,3.742\}$	4.32526	$\{-0.5935,6.3921\}$	6.9856	$\{-2.8386,1.4204\}$	4.30426	$\{1.5736,2.5793\}$	1.0057	$\{3.0955,3.8482\}$	0.7527	$\{0.726,0.8066\}$	0.0806
			$(0^{*14},15)$	$\{0.7609,4.5523\}$	3.7917	$\{-2.1547,5.2312\}$	7.3859	$\{-0.10215,3.5872\}$	3.68935	$\{1.8534,1.8606\}$	0.00719791	$\{3.5281,3.7272\}$	0.199146	$\{0.6647,0.6787\}$	0.014056
		20	$(10,0^{*19})$	$\{0.737,3.1704\}$	2.43339	$\{-0.6829,5.4083\}$	6.0912	$\{-0.616,2.1799\}$	1.34286	$\{1.9517,1.9557\}$	0.00397833	$\{3.3809,3.4483\}$	0.0674126	$\{0.7823,0.7859\}$	0.00358573
			$(0^{*5},1^{*10},0^{*5})$	$\{0.2652,3.633\}$	3.36779	$\{-0.9918,3.7738\}$	4.7656	$\{-1.3486,3.5118\}$	4.73585	$\{1.9524,1.9569\}$	0.00452203	$\{2.4032,2.5335\}$	0.130268	$\{1.0756,1.0999\}$	0.0243698
			$(0^{*19},10)$	$\{0.4375,3.9982\}$	3.5607	$\{-0.67421,5.3235\}$	5.99771	$\{-0.1289,2.3945\}$	2.5234	$\{2.0559,2.0696\}$	0.0136657	$\{2.3289,2.9556\}$	0.626709	$0.8203,0.8717\}$	0.0513903
	50	30	$(20,0^{*29})$	$\{0.0501,3.9\}$	3.84995	$\{-0.8639,4.4208\}$	5.2847	$\{-1.8455,3.8549\}$	7.11412	$\{1.9668,1.9733\}$	0.00644604	$\{3.1822,3.3338\}$	0.151675	$\{1.0003,1.0192\}$	0.0188343
			$(0^{*5},1^{*20},0^{*5})$	$\{0.2259,3.7363\}$	3.51033	$\{-0.2201,3.4449\}$	3.665	$\{-1.5115,3.3887\}$	5.12189	$\{1.9838,1.988\}$	0.00426579	$\{3.2032,3.2797\}$	0.0764709	$\{0.9368,0.9536\}$	0.016881
			$(0^{*29},20)$	$\{-0.0553,4.0103\}$	4.06558	$\{-0.7876,3.2958\}$	4.0834	$\{-1.8151,3.6305\}$	6.58978	$\{1.9777,1.9825\}$	0.00479792	$\{2.939,3.0655\}$	0.126497	$\{0.9002,0.9138\}$	0.0135919
		40	$(10,0^{*39})$	$\{0.3276,3.6993\}$	3.3717	$\{-0.6859,3.8725\}$	4.5584	$\{-1.3462,3.5329\}$	4.75582	$\{2.0111,2.014\}$	0.00297002	$\{2.7167,2.834\}$	0.117325	$\{1.0886,1.0976\}$	0.00893318
			$(0^{*15},1^{*10},0^{*15})$	$\{0.2752,4.1142\}$	3.83897	$\{-0.1567,4.2874\}$	4.4441	$\{-1.4344,3.4285\}$	4.91775	$\{1.1872,2.1936\}$	1.0064	$\{3.032,3.2662\}$	0.234282	$\{0.9263,1.0067\}$	0.0804
			$(0^{*39},10)$	$\{0.3368,3.3054\}$	2.96868	$\{-0.6739,3.1187\}$	3.7926	$\{-1.3908,3.6057\}$	5.01497	$\{1.8176,2.8222\}$	1.0046	$\{2.7116,3.8569\}$	1.1453	$\{0.9098,1.1045\}$	0.1947

TABLE 4. The 95% ACI's and HPD credible intervals and the corresponding length of α, θ and λ for informative priors

				ACI						HPD					
k	n	m	scheme	$\hat{\alpha}$		$\hat{\theta}$		$\hat{\lambda}$		$\hat{\alpha}$		$\hat{\theta}$		$\hat{\lambda}$	
				interval	length	interval	length	interval	length	interval	length	interval	length	interval	length
2	30	15	$(15,0^{*14})$	$\{-1.695,7.1307\}$	8.8257	$\{-1.0521,5.5222\}$	6.5743	$\{-3.5334,1.7956\}$	5.329	$\{2.0784,2.7053\}$	0.6269	$\{1.7593,3.0884\}$	1.3291	$\{0.9849,1.1254\}$	0.1405
			(1^{*15})	$\{-0.2177,4.7468\}$	4.96453	$\{-1.7848,4.4581\}$	6.2429	$\{-2.0054,3.6778\}$	7.37559	$\{1.2592,2.2682\}$	1.009	$\{2.6946,4.0663\}$	1.3717	$\{0.8274,1.841\}$	1.0136
			$(0^{*14},15)$	$\{0.1752,2.2599\}$	2.08478	$\{-0.757,7.3617\}$	8.11861	$\{-0.9986,1.73\}$	1.7275	$\{1.2178,1.8196\}$	0.6018	$\{3.0839,3.3077\}$	0.2238	$\{0.3638,0.566\}$	0.2022
		20	$(10,0^{*19})$	$\{0.061,3.5248\}$	3.46381	$\{-0.5526,4.5593\}$	5.1119	$\{-1.6762,4.1351\}$	6.93137	$\{1.7504,1.7981\}$	0.0477	$\{2.8454,2.9753\}$	0.129894	$\{0.225,1.2424\}$	1.0174
			$(0^{*5},1^{*10},0^{*5})$	$\{0.6174,3.9607\}$	3.34331	$\{-0.384,4.9877\}$	5.3717	$\{-0.9333,2.7867\}$	2.60092	$\{2.0162,2.2906\}$	0.2744	$\{2.7064,3.8434\}$	1.137	$\{0.7239,0.9275\}$	0.2036
			$(0^{*19},10)$	$\{0.2267,3.7943\}$	3.56761	$\{-0.6434,4.0688\}$	4.7122	$\{-1.3504,2.8013\}$	3.7828	$\{1.0101,2.0142\}$	1.0041	$\{3.0712,3.8103\}$	0.7391	$\{0.7211,0.7775\}$	0.0564
	50	30	$(20,0^{*29})$	$\{0.6613,3.0249\}$	2.3636	$\{-0.1039,6.0881\}$	6.192	$\{-0.6127,2.619\}$	1.60464	$\{1.8435,1.8768\}$	0.0333	$\{2.4945,2.834\}$	0.3386	$\{0.7898,1.0058\}$	0.216
			$(0^{*5},1^{*20},0^{*5})$	$\{0.4382,4.0209\}$	3.58277	$\{-0.5192,5.4305\}$	5.9497	$\{-1.0998,3.316\}$	3.64711	$\{2.0272,2.234\}$	0.2068	$\{2.8069,2.9884\}$	0.181418	$\{1.1123,1.1307\}$	0.0183728
			$(0^{*29},20)$	$\{-0.5367,4.384\}$	4.92072	$\{-0.9219,5.64\}$	6.5619	$\{-2.8124,4.1012\}$	6.9136	$\{1.9111,2.9239\}$	1.0128	$\{2.4254,5.659\}$	3.2696	$\{0.651,1.6741\}$	1.0231
		40	$(10,0^{*39})$	$\{0.5952,3.1986\}$	2.60343	$\{-0.582,6.7886\}$	7.3706	$\{-0.8368,2.8012\}$	2.34401	$\{1.8068,1.8997\}$	0.08317	$\{2.0103,3.0937\}$	1.0834	$\{0.6729,0.9814\}$	0.3085
			$(0^{*15},1^{*10},0^{*15})$	$\{0.5981,2.488\}$	1.88992	$\{-0.2235,7.6496\}$	7.8731	$\{-0.6402,2.3028\}$	1.47419	$\{1.541,2.5436\}$	1.0026	$\{2.7156,3.743\}$	1.0274	$\{0.8249,0.931\}$	0.1061
			$(0^{*39},10)$	$\{0.5952,3.1986\}$	2.60343	$\{-0.582,5.7886\}$	6.3706	$\{-0.8368,2.8012\}$	2.34401	$\{1.8968,2.8997\}$	1.0029	$\{2.0103,3.0937\}$	1.0834	$\{0.9729,1.9814\}$	1.0085
5	30	15	$(15,0^{*14})$	$\{0.166,2.6532\}$	2.48724	$\{-0.435,5.3466\}$	5.7816	$\{-1.411,3.2933\}$	4.64673	$\{1.4099,2.4115\}$	1.0016	$\{1.4401,3.5043\}$	2.0642	$\{0.7312,0.9415\}$	0.2103
			(1^{*15})	$\{0.0509,2.998\}$	2.94716	$\{-0.5586,3.6458\}$	4.2044	$\{-1.667,3.2405\}$	5.40175	$\{1.2228,1.5292\}$	0.3064	$\{2.441,2.8243\}$	0.3833	$\{0.5928,0.8077\}$	0.2149
			$(0^{*14},15)$	$\{-0.5182,4.0739\}$	4.59212	$\{-0.847,4.8889\}$	5.7359	$\{-2.5082,3.1145\}$	5.6225	$\{1.7742,1.7789\}$	0.0047181	$\{2.3753,2.748\}$	0.372679	$\{0.7872,0.8224\}$	0.0352424
		20	$(10,0^{*19})$	$\{0.5711,3.077\}$	2.50587	$\{-0.2314,5.0757\}$	5.3071	$\{-0.8508,2.3321\}$	1.98418	$\{1.7212,2.8258\}$	1.1046	$\{2.2685,3.4105\}$	1.142	$\{0.7317,0.9374\}$	0.2064
			$(0^{*5},1^{*10},0^{*5})$	$\{0.1513,4.3634\}$	4.21214	$\{-0.1554,3.4013\}$	3.5567	$\{-1.5766,3.6541\}$	5.76103	$\{2.2504,2.2573\}$	0.00689895	$\{3.0447,3.2137\}$	0.169031	$\{1.0322,1.042\}$	0.00986966
			$(0^{*19},10)$	$\{-0.1943,4.6115\}$	4.80583	$\{-0.9924,4.4259\}$	5.4179	$\{-2.0105,3.8566\}$	7.75357	$\{2.2035,2.2134\}$	0.00983745	$\{3.1806,3.3917\}$	0.21107	$\{0.9273,0.9397\}$	0.0123965
	50	30	$(20,0^{*29})$	$0.5854,2.9599\}$	2.37444	$\{-0.0711,7.5531\}$	7.6242	$\{-0.783,2.44\}$	1.91052	$\{1.7699,2.7732\}$	1.0033	$\{2.2432,3.2739\}$	1.0307	$\{0.8288,1.9373\}$	1.1085
			$(0^{*5},1^{*20},0^{*5})$	$\{0.2677,4.0945\}$	3.82685	$\{-0.8726,4.7458\}$	5.6184	$\{-1.4163,3.3135\}$	4.69292	$\{2.1715,2.1834\}$	0.0118756	$\{2.7512,2.9107\}$	0.159487	$\{0.9452,0.9531\}$	0.00792964
			$(0^{*29},20)$	$\{0.5199,3.1818\}$	2.66189	$\{-0.9494,3.74\}$	4.6894	$\{-1.0727,3.3008\}$	3.54075	$\{1.8476,1.8504\}$	0.00273587	$\{2.8158,2.9002\}$	0.0844068	$\{1.0144,1.1281\}$	0.11369
		40	$(10,0^{*39})$	$\{0.7337,3.2346\}$	2.50094	$\{-0.2068,6.7142\}$	6.921	$\{-0.6417,2.8283\}$	1.81495	$\{1.9228,1.9847\}$	0.0619	$\{2.2395,3.309\}$	1.0695	$\{0.0829,1.0919\}$	1.00899
			$(0^{*15},1^{*10},0^{*15})$	$\{0.3728,3.8818\}$	3.50893	$\{-0.6536,3.7388\}$	4.3924	$\{-1.3547,3.7628\}$	5.09756	$\{2.1206,2.1242\}$	0.00352191	$\{3.028,3.4166\}$	0.3886	$\{0.1959,1.2069\}$	1.011
			$(0^{*39},10)$	$\{0.0729,4.5069\}$	4.43394	$\{-0.7027,3.4943\}$	4.197	$\{-1.6571,3.8143\}$	6.32059	$\{2.0327,2.2956\}$	0.2629	$\{2.947,3.143\}$	0.196012	$\{1.0758,1.0873\}$	0.0114277

represents the relief times of twenty patients receiving an analgesic.1.1, 1.4 ,1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7,4.1, 1.8 ,1.5 ,1.2 ,1.4, 3.0 ,1.7, 2.3, 1.6, 2.0 Before progressing, first we would like to check whether the GLED fit this data or not.

The calculated value of the K-S test is 0.18497 for the GLE distribution and this value is smaller than their corresponding values expected at 5% significance level, which is 0.29407 at n = 20 .We have just plotted the empirical survival function and the fitted survival functions in Figure 1. Observe that the GLE distribution can be a good model fitting this data. Figure 2 shows that a Q-Q plot for the data .

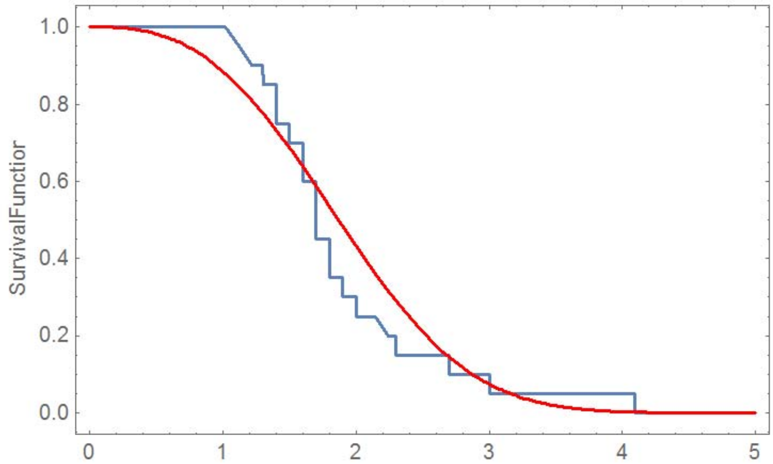

FIGURE 1. Empirical and fitted distribution function for completed data set.

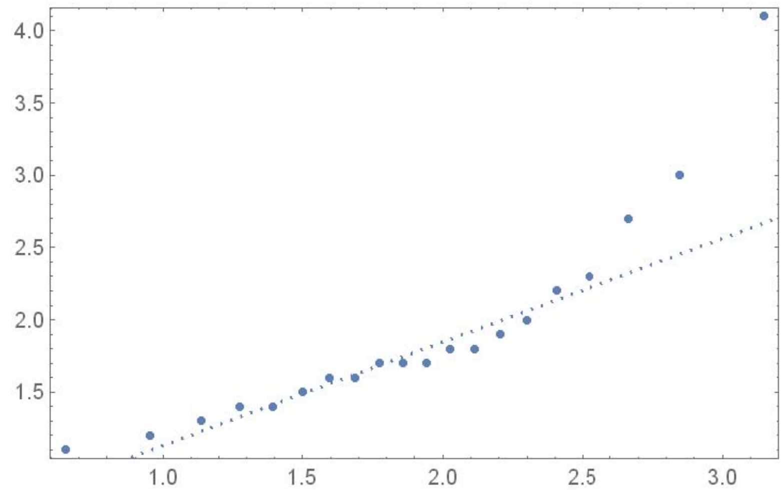

FIGURE 2. Q-Q plot compare data to a specific distribution.

Second, we generate a first-failure censored sample after randomly grouping this data set into $n = 15$ groups with $k = 2$ items within each group Finally, the following first-failure censored sample is obtained: 1.1, 1.3, 1.4, 1.4, 1.6, 1.7, 1.7, 1.7, 1.8, 1.8, 1.9, 2.3, 2.7, 3., 4.1

Now, we generate three progressive first-failure censored samples using three different censoring schemes from the above first-failure censored sample with $m = 10$. The different censoring schemes and the corresponding progressive first-failure censored samples are presented in Table 5. In all the three cases we calculate the ML and Bayes estimates of the parameters. In Bayes estimation we use non-informative priors as we have no prior information about the parameters. For importance sampling procedure, we take $M = 1000$. Also, we obtain 95% confidence and HPD credible intervals for the parameters. The estimates are listed in Table 6 and Table 7.

TABLE 5. Different progressive first-failure censored data sets

k	n	m	censoring scheme	Progressive first failure censored sample
2	15	10	$(5, 0^{*9})$	$1.2, 1.4, 1.7, 1.8, 1.9, 2., 2.2, 2.3, 3., 4.1$
			$(0^{*4}, 2, 3, 0^{*4})$	$1.2, 1.3, 1.4, 1.6, 1.7, 1.8, 1.9, 2.7, 3., 4.1$
			$(0^{*9}, 5)$	$1.2, 1.3, 1.4, 1.6, 1.7, 1.7, 1.7, 1.8, 2.2, 2.7$

TABLE 6. The MLE and Bayes estimates of the parameters for the real data set

				MLE			Bayes SEL			Bayes LINEX			Bayes Entropy		
k	n	m	Scheme	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$
2	15	10	$(5, 0^{*9})$	2.23912	0.0956745	0.149963	2.23537	0.095677	0.150153	2.23537	0.095677	0.150153	2.23537	0.095677	0.150153
				(0.0708)	(0.00245)	(0.04084)	(0.07196)	(0.00245)	(0.0408)	(0.07196)	(0.00245)	(0.0408)	(0.07196)	(0.00245)	(0.0408)
			$(0^{*4}, 2, 3, 0^{*4})$	2.13244	0.0976703	0.151228	2.1379	0.0976692	0.151345	2.13446	0.0976747	0.151042	2.13446	0.0976747	0.151041
				(0.10738)	(0.00256)	(0.04055)	(0.10532)	(0.00256)	(0.04052)	(0.10661)	(0.00256)	(0.04059)	(0.10661)	(0.00256)	(0.04059)
			$(0^{*9}, 5)$	1.97263	0.110446	0.138139	1.97337	0.110444	0.138367	1.9713	0.110448	0.138237	1.9713	0.110448	0.138237
				(0.17636)	(0.00336)	(0.04365)	(0.176)	(0.00336)	(0.04359)	(0.177)	(0.00336)	(0.04362)	(0.177)	(0.00336)	(0.04362)

From Table 6 and Table 7, it is quite clear that all the estimates for the parameters α, θ and λ are quite close to each other. HPD credible intervals are better than confidence intervals in respect of average length.

TABLE 7. The 95% ACI's and HPD credible intervals and the corresponding length of the parameters for the real data set

				AC I						HPD					
k	n	m	scheme	$\hat{\alpha}$		$\hat{\theta}$		$\hat{\lambda}$		$\hat{\alpha}$		$\hat{\theta}$		$\hat{\lambda}$	
				interval	length	interval	length	interval	length	interval	length	interval	length	interval	length
2	15	10	$(5, 0^{*9})$	$\{0.7354, 3.7429\}$	3.00753	$\{-0.1221, 0.3135\}$	0.435574	$\{-0.2038, 0.5037\}$	0.10264	$\{2.2338, 2.2387\}$	0.00495864	$\{0.0957, 0.09578\}$	1.00222×10^{-5}	$\{0.1499, 0.1503\}$	0.000402991
			$(0^{*4}, 2, 3, 0^{*4})$	$\{0.81, 3.4549\}$	2.64484	$\{-0.1119, 0.3072\}$	0.419115	$\{-0.1853, 0.4878\}$	0.0904108	$\{2.1329, 2.1385\}$	0.00561	$\{0.0975, 0.0977\}$	2.90316×10^{-4}	$\{0.1509, 0.15135\}$	0.00045844
			$(0^{*9}, 5)$	$\{0.5801, 3.3652\}$	2.78507	$\{-0.1375, 0.3584\}$	0.495851	$\{-0.2623, 0.5386\}$	0.141307	$\{1.9703, 1.9755\}$	5.2×10^{-3}	$\{0.1104, 0.1105\}$	5.6×10^{-5}	$\{0.138, 0.1384\}$	0.000394016

6. CONCLUSION

The problem of estimating the unknown parameters of GLED is discussed in this article under progressive first-failure censoring. We computed MLEs and Bayes estimators of the parameters and with square error loss function, LINEX loss function and Entropy function. These estimates cannot be obtained in closed forms, but can be derived numerically. Also, in the case of MLEs Asymptotic confidence intervals are constructed using observed Fisher information matrix. Bayes estimates are obtained using the MCMC technique.

We use of the importance sampling procedure to obtain point estimates and HPD credible intervals of the parameters. The method of selecting appropriate lifetime models are studied. This work is mainly associated with progressive first-failure censoring case, and the same methods can be extended for other censoring schemes also.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
UNIVERSITY OF SUEZ CANAL
Email address: nashwa_mohamed@science.suez.edu.eg