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FIXED POINT THEOREMS FOR SUZUKI TYPE GENERALIZED $\mathcal{Z}$-CONTRACTIONS IN GENERALIZED METRIC SPACES

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ABSTRACT. In this paper, the notion of Suzuki type generalized $\mathcal{Z}$-contractions is introduced and a new fixed point theorem for such contractions is established. An example and an application to integral equation are given to support main result.

1. INTRODUCTION AND PRELIMINARIES

Kannan [7] extended the class of contractive mappings as follows.

Let (X, d) be a metric space. A mapping $T : X \to X$ is called Kannan contraction if there exists a constant $k \in [0, \frac{1}{2})$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)].$$

Kannan [7] then proved that every Kannan contraction mappings defined on complete metric spaces has a unique fixed point.

Afterward, Azam and Arshad [3] extended Kannan's result [7] to generalized metric spaces.

Recently, Khojasteh *et al.* [9] introduced the notion of $\mathcal{Z}$-contractions by defining the concept of simulation functions, and they proved the following theorem.

Theorem 1.1. *Let (X, d) be a complete metric space, and let $T : X \to X$ be a $\mathcal{Z}$-contraction mapping w.r.t. a function ζ, i.e.,*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0 \ \forall x, y \in X,$$

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where function $\zeta : [0,\infty)\times[0,\infty)\to\mathbb{R}$ *satisfies conditions:*

($\zeta 1$) $\zeta(0,0)=0$*;*

($\zeta 2$) $\zeta(t,s)<s-t\ \ \forall s,t>0$*;*

($\zeta 3$) *for any sequence* $\{t_n\},\{s_n\}\subset(0,\infty)$*,*

$$\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n>0\Rightarrow \lim_{n\to\infty}\sup\zeta(t_n,s_n)<0.$$

Then T has a unique fixed point.

They unified the some existing metric fixed point results. Afterward, many authors (for example, [1,5,8,10,11]) obtained generalizations of the result of [9].

Very recently, Isik *et al.* [5] obtained the following theorem introducing the notion of almost $\mathcal{Z}$-contraction mapping w.r.t. a function ζ.

Theorem 1.2. *[5] Let (X,d) be a complete metric space, and let $T:X\to X$ be an almost $\mathcal{Z}$-contraction mapping w.r.t. a function ζ. That is, T satisfies the following condition:*

$$\zeta(d(Tx,Ty),d(x,y)+Ln(x,y))\geq 0\ \forall x,y\in X,$$

where $L\geq 0$ and $n(x,y)=\min\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$ and $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ satisfies conditions ($\zeta 2$) and

($\zeta 4$) *for any sequence* $\{t_n\},\{s_n\}\subset(0,\infty)$ *with* $t_n\leq s_n\ \ \forall n=1,2,3,\cdots$

$$\lim_{n\to\infty} t_n=\lim_{n\to\infty} s_n>0\Rightarrow \lim_{n\to\infty}\sup\zeta(t_n,s_n)<0.$$

Then T has a unique fixed point.

In the paper, we introduce the concept of a new type of contraction mappings, and we establish a new fixed point theorem for such contraction mappings in the setting of generalized metric spaces. We give an example to illustrate main theorem and give an application to integral equation.

Let $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be a function. Then we say that

(1) $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is a *simulation function* [9] in the sene of Khojasteh *et al.* if and only if ($\zeta 1$), ($\zeta 2$) and ($\zeta 3$) hold;

(2) $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is a *simulation function* [2] in the sene of Argoubi et al. if and only if ($\zeta 2$) and ($\zeta 3$) hold;

(3) $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is a *simulation function* [10] in the sene of Roldan Lopez de Hierro *et al.* if and only if ($\zeta 1$), ($\zeta 2$) and ($\zeta 4$) hold;

(4) $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a *simulation function* [5] in the sene of Iisk *et al.* if and only if ($\zeta 2$) and ($\zeta 4$) hold.

From now on, let $\mathcal{Z}$ be the family of all simulation function in the sene of Iisk *et al.*, and $\zeta \in \mathcal{Z}$ is briefly called *simulation function*. Note that $\zeta(t, t) < 0$ for all $t > 0$.

Example 1. *([5, 8, 9]) Let $\zeta_b, \zeta_w, \zeta_\eta, \zeta_I : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be functions defined as follows, respectively:*

(1) $\zeta_b(t, s) = ks - t \ \ \forall t, s \geq 0$*, where* $k \in (0, 1)$*;*
(2) $\zeta_w(t, s) = s - \phi(s) - t \ \ \forall t, s \geq 0$ *where* $\phi : [0, \infty) \to [0, \infty)$ *is continuous such that* $\phi^{-1}(\{0\}) = 0$*;*
(3) $\zeta_\eta(t, s) = \eta(s) - t \ \ \forall t, s \geq 0$ *where* $\eta : [0, \infty) \to [0, \infty)$ *is upper semicontinuous such that* $\eta(t) < t \ \forall t > 0$ *and* $\eta^{-1}(\{0\}) = 0$*;*
(4) $$\zeta_I(t, s) = \begin{cases} 1 & \text{if } (s, t) = (0, 0) \text{ or } s = t, \\ 2(s - t) & \text{if } s < t, \\ \lambda s - t & \text{otherwise,} \end{cases}$$
$\forall s, t \geq 0$*, where* $\lambda \in (0, 1)$.
Then $\zeta_b, \zeta_w, \zeta_\eta, \zeta_I \in \mathcal{Z}$.

Remark 1.1. *Simulation functions $\zeta_b, \zeta_w, \zeta_\eta, \zeta_I$ are non-decresing w.r.t. the second variable, and ζ_I is not satisfied condition ($\zeta 1$) and ($\zeta 3$).*

We recall the following definitions which are in [4].

Let X be a nonempty set, and let $d : X \times X \to [0, \infty)$ be a map such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y

(d1) $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$;
(d3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then d is called a *generalized metric* on X and (X, d) is called a *generalized metric space*.

Note that if triangle inquality holds, then condition (d3) is satisfied. So every metric space is a generalized metric space.

Let (X, d) be a generalized metric space, $\{x_n\} \subset X$ be a sequence and $x \in X$. Then we say that

(1) $\{x_n\}$ is convergent to x (denoted by $\lim_{n\to\infty} x_n = x$) if, and only if, $\lim_{n\to\infty} d(x_n, x) = 0$;
(2) $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$;
(3) (X, d) is complete if and only if every Cauchy sequence in X is convergent to some point in X.

Let (X, d) be a generalized metric space.

A map $T : X \to X$ is called *continuous* at $x \in X$ if and only if for any $V \in \tau$ containg Tx, there exists $U \in \tau$ containg x such that $TU \subset V$, where τ is the topology on X induced by the generalized metric d. That is,

$$\tau = \{U \subset X : \forall x \in U, \exists B \in \beta, x \in B \subset U\},$$
$$\beta = \{B(x, r) : x \in X, \forall r > 0\},$$
$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

If T is continuous at each point $x \in X$, then it is called *continuous*.

Note that T is continuous if and only if it is sequentially continuous, i.e., $\lim_{n\to\infty} d(Tx_n, Tx) = 0$ for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n, x) = 0$.

Lemma 1.1. *[6] Let (X, d) be a generalized metric space, $\{x_n\} \subset X$ be a Cauchy sequence and $x, y \in X$. If there exists a positive integer N such that*

(1) $x_n \neq x_m \ \forall n, m > N$;
(2) $x_n \neq x \ \forall n > N$;
(3) $x_n \neq y \ \forall n > N$;
(4) $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y)$,

then $x = y$.

2. Fixed point theorems

Let (X, d) be a generalized metric space.

A mapping $T : X \to X$ is called *Suzuki type generalized $\mathcal{Z}$-contraction* if and only if for all $x, y \in X$ with $\frac{1}{2}\min\{d(x, Tx), d(y, Tx)\} < d(x, y)$,

$$\zeta(d(Tx, Ty), \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)]\} + Ln(x, y)) \geq 0 \tag{2.1}$$

where $\zeta \in \mathcal{Z}, L \geq 0$ and $n(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y.Tx)\}$.

Now, we prove our main result.

Theorem 2.1. *Let (X, d) be a complete generalized metric space, and let $T : X \to X$ be Suzuki type generalized $\mathcal{Z}$-contraction. Then T has a fixed point, and for every initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to the fixed point.*

Proof. Let $x_0 \in X$ be a point. Define a sequence $\{x_n\} \subset X$ by $x_n = Tx_{n-1} = T^n x_0 \; \forall n = 1, 2, 3 \cdots$.

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T, and the proof is finished.

Assume that

$$x_{n-1} \neq x_n \; \forall n = 1, 2, 3 \cdots . \tag{2.2}$$

We deduce that

$$\begin{aligned} & n(x_{n-1}, x_n) \\ &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}) \\ &= \min\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ &= 0 \; \forall n = 1, 2, 3, \cdots . \end{aligned}$$

Also, we infer that

$$\frac{1}{2} \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_{n-1})\} < d(x_{n-1}, x_n) \; \forall n = 1, 2, 3, \cdots .$$

Thus it follows from (2.1) that $\forall n = 1, 2, 3, \cdots$

$$\begin{aligned} 0 &\leq \zeta(d(Tx_{n-1}, Tx_n), \max\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]\} \\ &\quad + n(x_{n-1}, x_n)) \\ &= \zeta(d(x_n, x_{n+1}), \max\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\}) \\ &< \max\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\} - d(x_n, x_{n+1}), \end{aligned} \tag{2.3}$$

which implies that for all $n = 1, 2, 3, \cdots$,

$$d(x_n, x_{n+1}) < \max\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\}. \tag{2.4}$$

If

$$\max\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\} = d(x_{n-1}, x_n)$$

then

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \; \forall n = 1, 2, 3, \cdots .$$

Let

$$d(x_{n-1}, x_n) \leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})].$$

Then from (2.4) we have

$$d(x_n, x_{n+1}) < \frac{d(x_{n-1}, x_n)}{2} + \frac{d(x_n, x_{n+1})}{2},$$

which implies

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \; \forall n = 1, 2, 3, \cdots .$$

Hence $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence, and so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = r.$$

We now show that $r = 0$. Assume that $r \neq 0$. Let $t_n = d(x_n, x_{n+1})$ and $s_n = \max\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\}$. Then $t_n \leq s_n$ and $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = r > 0$, and so we have

$$0 \leq \lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0,$$

which is a contradiction. Thus we have

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{2.5}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence.

On the contrary, assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which

$$m(k) > n(k) > k,\; d(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } d(x_{m(k)-1}, x_{n(k)}) < \epsilon. \tag{2.6}$$

From (2.5) we have

$$\begin{aligned} \epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq d(x_{n(k)}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< \epsilon + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}). \end{aligned} \tag{2.7}$$

Letting $k \to \infty$ in (2.6), we obtain

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon.$$

On the other hand, we obtain

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \le d(x_{n(k)+1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1}).$$

Thus

$$\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon.$$

We infer that

$$\begin{aligned}
&d(x_{m(k)}, x_{n(k)+1})\\
&\le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})\\
&< d(x_{m(k)}, x_{m(k)-1}) + \epsilon + d(x_{n(k)}, x_{n(k)+1}),
\end{aligned}$$

which yields

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \le \epsilon.$$

We have

$$d(x_{n(k)}, x_{m(k)+1}) \le d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1}),$$

which yields

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) \le \epsilon.$$

We infer that

$$\begin{aligned}
&\lim_{k\to\infty} n(x_{n(k)}, x_{m(k)})\\
&= \lim_{k\to\infty} \min\{d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1}), d(x_{m(k)}, x_{n(k)+1}), d(x_{n(k)}, x_{m(k)+1})\}\\
&\le \min\{0, 0, \epsilon, \epsilon\}\\
&= 0,
\end{aligned}$$

which yields

$$\lim_{k\to\infty} n(x_{n(k)}, x_{m(k)}) = 0.$$

It follows from (2.5) that there exists $N \in \mathbb{N}$ such that

$$d(x_{n(k)}, x_{n(k)+1}) < \epsilon, \ \forall k > N.$$

Thus

$$\frac{1}{2}d(x_{n(k)}, Tx_{n(k)}) = \frac{1}{2}d(x_{n(k)}, x_{n(k)+1}) < \epsilon \le d(x_{n(k)}, x_{m(k)}), \ \forall k > N,$$

which yields

$$\frac{1}{2}\min\{d(x_{n(k)}, Tx_{n(k)}), d(x_{m(k)}, Tx_{n(k)})\} < d(x_{n(k)}, x_{m(k)}), \ \forall k > N.$$

It follows from (2.1) that

$$
\begin{aligned}
0 &\leq \zeta(d(Tx_{n(k)}, Tx_{m(k)}), \max\{d(x_{n(k)}, x_{m(k)}), \frac{1}{2}[d(x_{n(k)}, Tx_{n(k)}) \\
&\quad + d(x_{m(k)}, Tx_{m(k)})]\} + Ln(x_{n(k)}, x_{m(k)})) \\
&= \zeta(d(x_{n(k)+1}, x_{m(k)+1}), \max\{d(x_{n(k)}, x_{m(k)}), \frac{1}{2}[d(x_{n(k)}, x_{n(k)+1}) \\
&\quad + d(x_{m(k)}, x_{m(k)+1})]\} + Ln(x_{n(k)}, x_{m(k)})) \\
&< \max\{d(x_{n(k)}, x_{m(k)}), \frac{1}{2}[d(x_{n(k)}, x_{n(k)+1}) + d(x_{m(k)}, x_{m(k)+1})]\} \\
&\quad + Ln(x_{n(k)}, x_{m(k)}) - d(x_{n(k)+1}, x_{m(k)+1})
\end{aligned} \tag{2.8}
$$

which implies

$$
\begin{aligned}
d(x_{n(k)+1}, x_{m(k)+1}) &< \max\{d(x_{n(k)}, x_{m(k)}), \frac{1}{2}[d(x_{n(k)}, x_{n(k)+1}) + d(x_{m(k)}, x_{m(k)+1})]\} \\
&\quad + Ln(x_{n(k)}, x_{m(k)}).
\end{aligned}
$$

Let

$$
\begin{aligned}
t_k &= d(x_{n(k)+1}, x_{m(k)+1}), \\
s_k &= \max\{d(x_{n(k)}, x_{m(k)}), \frac{1}{2}[d(x_{n(k)}, x_{n(k)+1}) + d(x_{m(k)}, x_{m(k)+1})]\} \\
&\quad + Ln(x_{n(k)}, x_{m(k)}).
\end{aligned} \tag{2.9}
$$

Then

$$
t_k < s_k \ \forall k = 1, 2, 3, \cdots, \quad \text{and} \quad \lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = \epsilon > 0,
$$

and so we have

$$
0 \leq \lim_{k\to\infty} \sup \zeta(t_k, s_k) < 0,
$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $z \in X$ such that

$$
\lim_{n\to\infty} d(x_n, z) = 0.
$$

Hence we may assume that

$$
d(x_{n+1}, z) \leq d(x_n, z), \ \forall n = 1, 2, 3, \cdots.
$$

We have

$$\begin{aligned}&\frac{1}{2}\min\{d(x_n,Tx_n),d(z,Tx_n)\}\\ =&\ \frac{1}{2}\min\{d(x_n,x_{n+1}),d(z,x_{n+1})\}\\ <&\ d(x_n,z),\ \forall n=1,2,3,\cdots.\end{aligned}$$

It follows from (2.1) that

$$\begin{aligned}0 &\leq \zeta(d(Tx_n,Tz),\max\{d(x_n,z),\frac{1}{2}[d(x_n,Tx_n)+d(z,Tz)]\}+Ln(x_n,z))\\ &= \zeta(d(x_{n+1},Tz),\max\{d(x_n,z),\frac{1}{2}[d(x_n,x_{n+1})+d(z,Tz)]\}+Ln(x_n,z))\\ &< \max\{d(x_n,z),\frac{1}{2}[d(x_n,x_{n+1})+d(z,Tz)]\}+Ln(x_n,z)-d(x_{n+1},Tz),\end{aligned}$$

where $n(x_n,z)=\min\{d(x_n,x_{n+1}),d(z,Tz),d(z,x_{n+1}),d(x_n,Tz)\}$. Thus we obtain

$$d(x_{n+1},Tz)<\max\{d(x_n,z),\frac{1}{2}[d(x_n,x_{n+1})+d(z,Tz)]\}+Ln(x_n,z). \tag{2.10}$$

Since

$$d(z,Tz)\leq d(z,x_n)+d(x_n,x_{n+1})+d(x_{n+1},Tz),$$

it follows from (2.7) that

$$\begin{aligned}&d(x_{n+1},Tz)\\ &<\max\{d(x_n,z),d(x_n,x_{n+1})+\frac{1}{2}[d(x_n,z)+d(x_{n+1},Tz)]\}+Ln(x_n,z),\end{aligned}$$

which implies

$$\lim_{n\to\infty}d(x_{n+1},Tz)=0.$$

By Lemma 1.1, $z=Tz$. □

We give an example to illustrate Theorem 2.1.

Example 2. *Let* $X=\{1,2,3,4\}$ *and define* $d:X\times X\to[0,\infty)$ *as follows:*

$$\begin{aligned}&d(1,2)=d(2,1)=3,\\ &d(2,3)=d(3,2)=d(1,3)=d(3,1)=1,\\ &d(1,4)=d(4,1)=d(2,4)=d(4,2)=d(3,4)=d(4,3)=4,\\ &d(x,x)=0\ \forall x\in X.\end{aligned}$$

Then (X, d) *is a complete generalized metric space, but not a metric space (see [3]). Define a map* $T : X \to X$ *by*

$$Tx = \begin{cases} 3 & (x \neq 4), \\ 1 & (x = 4). \end{cases}$$

We now show that T *is a* $\mathcal{Z}$*-contraction with respect to* ζ_b*, where* $\zeta_b(t, s) = ks - t \ \ \forall t, s \geq 0$*, where* $k = \frac{2}{3}$*. Let* $\lambda = \frac{1}{3}$ *and* $L = 1$*.*

Note that if $x = y = 4$*, then*

$$\frac{1}{2}\min\{d(x, Tx), d(x, Ty)\} = \frac{1}{2}\min\{d(4, 1), d(4, 1)\} = 2 > 0 = d(x, y).$$

Thus we consider the following two cases.

1° : *Let* $x = 4$ *and* $y \neq 4$*.*
Then $d(x, Tx) = 4, d(y, Ty) = 0$ *or* 1*, and* $d(Tx, Ty) = 1$*. We infer that*

$$n(4, 1) = 0, n(4, 2) = 1, n(4, 3) = 0.$$

Hence $n(x, y) = 0$ *or* 1*. We have*

$$\frac{1}{2}\min\{d(x, Tx), d(x, Ty)\} = \frac{1}{2}\min\{d(4, 1), d(4, 3)\} = 2 < 4 = d(x, y)$$

and

$$k\max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)]\} + Ln(x, y) - d(Tx, Ty) \geq \frac{5}{3}.$$

Thus we have

$$\begin{aligned} \frac{5}{3} &\leq k\max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)]\} + Ln(x, y) - d(Tx, Ty) \\ &= \zeta_b(d(Tx, Ty), \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)]\} + Ln(x, y)). \end{aligned}$$

2° : *Let* $x \neq 4$ *and* $y \neq 4$*.*
Then

$$d(x, Tx) = d(y, Ty) = 0 \text{ or } 1, d(Tx, Ty) = 0, \text{ and } n(x, y) = 0 \text{ or } 1.$$

Hence we have

$$k\max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)]\} + Ln(x, y) - d(Tx, Ty) \geq \frac{2}{3}.$$

Hence

$$\begin{aligned}\frac{2}{3} &\le k\max\{d(x,y), \frac{1}{2}[d(x,Tx)+d(y,Ty)]\} + Ln(x,y) - d(Tx,Ty)\\ &= \zeta_b(d(Tx,Ty), \max\{d(x,y), \frac{1}{2}[d(x,Tx)+d(y,Ty)]\} + Ln(x,y)).\end{aligned}$$

Thus all conditions of Theorem 2.1 are satisfied. By Theorem 2.1, T has a fixed point $z = 3$.

Corollary 2.1. *Let (X, d) be a complete generalized metric space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $\frac{1}{2}\min\{d(x,Tx), d(y,Tx)\} < d(x,y)$,*

$$\zeta(d(Tx,Ty), d(x,y) + Ln(x,y)) \ge 0$$

where $L \ge 0$. Then T has a unique fixed point.

Proof. From Theorem 2.1 T has a fixed point. If u and v are fixed point of T such that $u \neq v$, then $\frac{1}{2}\min\{d(u,Tu), d(v,Tu)\} = 0 < d(u,v)$, and so

$$0 \le \zeta(d(Tu,Tv), d(u,v) + Ln(u,v)) = \zeta(d(u,v), d(u,v)) < 0,$$

which is contradiction. Hence $u = v$, and hence T has a unique fixed point. □

Remark 2.1. *Corollary 2.1 is a generalization of Theorem 2 of [5] to generalized metric space with Suzuki type condition.*

Corollary 2.2. *Let (X, d) be a complete generalized metric space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $\frac{1}{2}\min\{d(x,Tx), d(y,Tx)\} < d(x,y)$,*

$$\zeta(d(Tx,Ty), \frac{1}{2}[d(x,Tx)+d(y,Ty)] + Ln(x,y)) \ge 0$$

where $L \ge 0$. Then T has a fixed point.

Corollary 2.3. *Let (X, d) be a complete generalized metric space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $\frac{1}{2}\min\{d(x,Tx), d(y,Tx)\} < d(x,y)$,*

$$\zeta(d(Tx,Ty), \frac{1}{2}[d(x,Tx)+d(y,Ty)]) \ge 0.$$

Then T has a fixed point.

From Theorem 2.1 we have the following corollary.

Corollary 2.4. *Let (X,d) be complete a generalized metric space, and let $T:X\to X$ be a mapping such that for all $x,y\in X$ with $\frac{1}{2}\min\{d(x,Tx),d(y,Tx)\}<d(x,y)$,*

$$d(Tx,Ty))\leq\lambda[d(x,Tx)+d(y,Ty)]+Ln(x,y), \tag{2.11}$$

where $\lambda\in(0,\frac{1}{2})$ and $L\geq 0$. Then T has a unique fixed point.

Proof. Let $\zeta_b(t,s)=ks-t, k\in(0,1)$, and let $\lambda=\frac{k}{2}$. It follows from (2.11) that for all $x,y\in X$ with $\frac{1}{2}\min\{d(x,Tx),d(y,Tx)\}<d(x,y)$,

$$\begin{aligned}0&\leq\lambda[d(x,Tx)+d(y,Ty)]+Ln(x,y)-d(Tx,Ty)\\&=\frac{k}{2}[d(x,Tx)+d(y,Ty)]+Ln(x,y)-d(Tx,Ty)\\&=\zeta_b(d(Tx,Ty),\frac{1}{2}[d(x,Tx)+d(y,Ty)]+\frac{L}{k}n(x,y)).\end{aligned}$$

By Theorem 2.1, T has a fixed point. Let u and v be fixed points of T such that $u\neq v$. Then from (2.11) we have

$$\begin{aligned}0<d(u,v)&=d(Tu,Tv)\\&\leq\lambda[d(u,Tu)+d(v,Tv)]+Ln(u,v)\\&\leq\lambda[d(u,u)+d(v,v)]+Ln(u,v)\\&=0,\end{aligned}$$

which is a contradiction. Thus T has a unique fixed point. $\square$

Corollary 2.5. *Let (X,d) be a complete generalized metric space, and let $T:X\to X$ be a mapping such that for all $x,y\in X$ with $\frac{1}{2}\min\{d(x,Tx),d(y,Tx)\}<d(x,y)$,*

$$d(Tx,Ty))\leq\eta(\frac{1}{2}[d(x,Tx)+d(y,Ty)]+Ln(x,y)), \tag{2.12}$$

where $L\geq 0$ and $\eta:[0,\infty)\to[0,\infty)$ is an upper semicontinuous function such that $\eta(t)<t\ \forall t>0$ and $\eta^{-1}(\{0\})=0$. Then T has a unique fixed point.

Proof. Let $\zeta_\eta(t,s)=\eta(s)-t$, where $\eta:[0,\infty)\to[0,\infty)$ is an upper semicontinuous function such that $\eta(t)<t\ \forall t>0$ and $\eta^{-1}(\{0\})=0$. It follows from (2.12) that for all $x,y\in X$ with $\frac{1}{2}\min\{d(x,Tx),d(y,Tx)\}<d(x,y)$,

$$\begin{aligned}0&\leq\eta(\frac{1}{2}[d(x,Tx)+d(y,Ty)]+Ln(x,y))-d(Tx,Ty))\\&=\zeta_\eta(d(Tx,Ty),\frac{1}{2}[d(x,Tx)+d(y,Ty)]+Ln(x,y)).\end{aligned}$$

By Theorem 2.1, T has a fixed point.

Let u and v be fixed points of T such that $u \neq v$. Then it follows from (2.12) that

$$\begin{aligned}0 < d(u,v) &= d(Tu,Tv)\\ &\leq \eta(\frac{1}{2}[d(u,Tu)+d(v,Tv)]+Ln(u,v))\\ &= \eta(\frac{1}{2}[d(u,u)+d(v,v)]+Ln(u,v))\\ &=0,\end{aligned}$$

which is a contradiction. Thus T has a unique fixed point. □

Corollary 2.6. *Let (X,d) be a complete generalized metric space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $\frac{1}{2}\min\{d(x,Tx), d(y,Tx)\} < d(x,y)$,*

$$d(Tx,Ty) \leq \frac{1}{2}[d(x,Tx)+d(y,Ty)] - \varphi(d(x,Tx), d(y,Ty)), \tag{2.13}$$

where $\varphi : [0,\infty)\times[0,\infty) \to [0,\infty)$ is continuous such that $\varphi(t,s) = 0 \Leftrightarrow s = t = 0$. Then T has a unique fixed point.

Proof. Let $\zeta(t,s) = s - \varphi(s_1,s_2) - t$, where $s = \frac{s_1+s_2}{2}$. Then $\zeta \in \mathcal{Z}$. From (2.13) we have that, for all $x, y \in X$ with $\frac{1}{2}\min\{d(x,Tx), d(y,Tx)\} < d(x,y)$,

$$\begin{aligned}0 &\leq \frac{1}{2}[d(x,Tx)+d(y,Ty)] - \varphi(d(x,Tx), d(y,Ty)) - d(Tx,Ty)\\ &= \zeta(d(Tx,Ty), \frac{1}{2}[d(x,Tx)+d(y,Ty)]).\end{aligned}$$

By Corollary 2.3, T has a fixed point.

Let u and v be fixed points of T such that $u \neq v$. Then we have

$$\begin{aligned}0 < d(u,v) &= d(Tu,Tv)\\ &\leq \frac{1}{2}[d(u,Tu)+d(v,Tv)] - \varphi(d(u,Tu), d(v,Tv))\\ &= \frac{1}{2}[d(u,u)+d(v,v)] - \varphi(d(u,u), d(v,v))\\ &= 0,\end{aligned}$$

which is a contradiction. Hence T has a unique fixed point. □

Corollary 2.7. *Let (X, d) be a complete generalized metric space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $\frac{1}{2}\min\{d(x, Tx), d(y, Tx)\} < d(x, y)$,*

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Tx) + d(y, Ty)] - \psi(\frac{1}{2}[d(x, Tx) + d(y, Ty)]),$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous and $\psi^{-1}(\{0\}) = 0$. Then T has a unique fixed point.

Corollary 2.8. *Let (X, d) be a complete generalized metric space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$,*

$$\zeta(d(Tx, Ty), \frac{1}{2}[d(x, Tx) + d(y, Ty)] + Ln(x, y)) \geq 0,$$

where $L \geq 0$. Then T has a fixed point.

Corollary 2.9. *Let (X, d) be a complete generalized metric space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$,*

$$\zeta(d(Tx, Ty), \frac{1}{2}[d(x, Tx) + d(y, Ty)]) \geq 0.$$

Then T has a fixed point.

Remark 2.2. *By taking $\zeta(t, s) = \zeta_b(t, s) = ks - t$, where $k \in (0, 1)$, Corollary 2.9 reduces to Theorem 2.1 of [3].*

3. Application

Let $q : \mathbb{I} = [a, b] \to \mathbb{R}$, $H : \mathbb{I} \times \mathbb{I} \to \mathbb{R}_+$ and $f : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Consider the following integral equation:

$$p(r) = q(r) + \int_a^b H(r, z)f(z, p(z))dz, r \in \mathbb{I}. \tag{3.1}$$

We give an application of our result to prove existence of solution of above integral equation.

Let $X = C(\mathbb{I}, \mathbb{R})$ and $d(p, q) = \sup_{r\in\mathbb{I}} \mid p(r) - q(r) \mid$, $\forall p, q \in X$. Then (X, d) is a complete metric space, and hence it is a complete generalized metric space. Define a map $T : X \to X$ by

$$Tp(r) = q(r) + \int_a^b H(r, z)f(z, p(z))dz.$$

Then the existence of unique solution of (3.1) is equivalent to the existence of unique fixed pont of T.

Theorem 3.1. *Assume that the followings are satisfied.*

(1) $\sup_{r\in\mathbb{I}}\int_a^b H(r,z)dz \leq \frac{1}{b-a}$;

(2) $\forall p,q\in\mathbb{R}$,

$$| f(z,p)-f(z,q) | \leq \eta(\frac{1}{2}[| p-Tp | + | q-Tq |]+Ln(p,q)),$$

where $L\geq 0$ *and* $\eta:[0,\infty)\to[0,\infty)$ *is upper semicontinuous with* $\eta(t)<t\ \forall t>0$ *and* $\eta^{-1}(\{0\})=0$.

Then the integral equation (3.1) has a unique solution.

Proof. We deduce that

$$\begin{aligned}
& d(Tp_1,Tp_2)\\
&=\sup_{r\in\mathbb{I}} | Tp_1(r)-Tp_2(r) |\\
&=\sup_{r\in\mathbb{I}} | q(r)+\int_a^b H(r,z)f(z,p_1(z))dz-q(r)-\int_a^b H(r,z)f(z,p_2(z))dz |\\
&=\sup_{r\in\mathbb{I}} | \int_a^b H(r,z)[f(z,p_1(z))-f(z,p_2(z))]dz |\\
&\leq \sup_{r\in\mathbb{I}}\{\int_a^b H(r,z)dz\}\{\int_a^b | f(z,p_1(z))-f(z,p_2(z)) | dz\}\\
&\leq \frac{1}{b-a}\int_a^b \eta(\frac{1}{2}[d(p_1,Tp_1)+d(p_2,Tp_2)]+Ln(p_1,p_2))dz\\
&=\eta(\frac{1}{2}[d(p_1,Tp_1)+d(p_2,Tp_2)]+Ln(p_1,p_2)).
\end{aligned}$$

Thus all conditions of Corollary 2.8, and T has a unique fixed point. Hence equation (3.1) has a unique solution. □

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