

## MOMENTS OF PROGRESSIVE TYPE-II RIGHT CENSORED ORDER STATISTICS FROM POWER HAZARD RATE DISTRIBUTION

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**ABSTRACT.** In this paper, some new recurrence relations of the single and product moments of progressively Type-II censored order statistics from power hazard rate distribution are obtained. Further this distribution is characterized by several techniques. Also, the estimation of the parameters under progressively Type- II right censored order statistics are obtained by maximum likelihood method. Finally, a real data set has been analyzed for illustrative purpose.

### 1. INTRODUCTION

Progressively censoring is very important in life testing experiments. It allows for the removal of live- units from the experiment at various stages as it is an attractive feature as it will potentially save a lot for the experimenter in terms of cost and time. There are several types of censoring schemes used in lifetime analysis. In general scheme of progressively Type- II right censoring scheme,  $n$  units are placed on a life- testing experiment and only  $m(< n)$  are completely observed until failure. The censoring occurs progressively in  $m$  stages. Cohen [1], Thomas and Wilson [2] works are notable in the progressive censoring schemes. We refer readers to [3] for details.

Consider an experiment in which  $n$  units are placed on life test. In progressive scheme, the experimenter decides beforehand the quantity  $m$ , the number

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of failures to be observed. At the time of first failure is observed  $R_1$  of  $n - 1$  surviving units are randomly withdrawn from the experiment,  $R_2$  of  $n - 2 - R_1$  surviving units are withdrawn at the time of the second failure. The experiment finally terminates at the time of  $m^{th}$  failure when all remaining  $R_m = n - m - R_1 - R_2 - \dots - R_{m-1}$  surviving units are withdrawn. The censoring numbers  $(R_i, i = 1, \dots, m - 1)$  are prefixed. We will denote the  $m$  ordered failure times thus observed by  $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$ . It is evident that  $n = m + \sum_{k=1}^m R_k$ . The resulting  $m$  ordered values which are obtained from this type of censoring are referred to as progressively Type-II right-censored order statistics. If the failure times of the  $n$  items originally on test are from a continuous population with cumulative distribution function (cdf),  $F(\cdot)$  and probability density function (pdf),  $f(\cdot)$ . In progressive censoring, the following notations are used

- (i)  $n, m, R_1, R_2, \dots, R_m$  all are integers.
- (ii)  $m$  is the sample size (which may be random in some models).
- (iii)  $n$  is the total number of units in the experiment.
- (iv)  $R_j$  is the number of removals at the  $j^{th}$  censoring time.
- (v)  $(R_1, R_2, \dots, R_m)$  denote the censoring scheme.

Then the joint pdf of the progressively Type-II censored samples  $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$  is given by [4], as follows

$$f_{X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}}(x_1, x_2, \dots, x_m) = C(n, m - 1) \prod_{i=1}^m f(x_i) [1 - F(x_i)]^{R_i},$$

where,  $-\infty < x_1 < x_2 < \dots < x_m < \infty$ ,  $f(\cdot)$  and  $F(\cdot)$  are respectively, the pdf and cdf of the random sample and

$$C(n, m - 1) = n(n - R_1 - 1) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1).$$

Let the progressively Type- II right-censored sample  $X_{1:m:n}^{(R_1, R_2, \dots, R_m)}, X_{2:m:n}^{(R_1, R_2, \dots, R_m)}, \dots, X_{m:m:n}^{(R_1, R_2, \dots, R_m)}$ , with censoring scheme  $(R_1, R_2, \dots, R_m)$ ,  $m \leq n$  arise from power hazard rate distribution. The power hazard function has been defined in [5],

$$h(x) = \alpha x^\beta, \quad x > 0, \alpha > 0, \beta > -1.$$

Corresponding to this hazard function, its probability density function (pdf) is given by,

$$(1.1) \quad f(x) = \alpha x^\beta \exp \left\{ -\frac{\alpha}{\beta + 1} x^{\beta+1} \right\}, \quad x > 0, \alpha > 0, \beta > -1,$$

with the corresponding cdf

$$(1.2) \quad F(x) = 1 - \exp \left\{ -\frac{\alpha}{\beta + 1} x^{\beta+1} \right\}, \quad x > 0, \alpha > 0, \beta > -1.$$

The distribution with density function defined in (1.1) is known as power hazard rate (PHR) distribution. For  $-1 < \beta < 0$ , PHR distribution has a decreasing hazard function and  $\beta > 0$ , PHR distribution has an increasing hazard function and  $\beta = 0$ , PHR distribution has constant failure rate. For more details and properties of PHR distribution, see [6].

It is clear that some well-known life time distributions as Weibull, Rayleigh, exponential and linear failure rate distribution are special cases of PHR distribution, such that,

- (i) If  $\beta = \alpha - 1$ , then PHR distribution reduces to Weibull( $\alpha - 1$ ).
- (ii) If  $\alpha = \frac{1}{\theta}$ ,  $\beta = 1$ , then PHR distribution reduces to Rayleigh( $\theta$ ).
- (iii) If  $\beta = 0$ , then PHR distribution reduces to exponential distribution with mean( $\frac{1}{\alpha}$ ).
- (iv) If  $\beta = 1$ , then PHR distribution reduces to linear failure rate distribution ( $\alpha, 0$ ).

It may be noted that from (1.1) and (1.2), the characterizing differential equation is given by

$$(1.3) \quad f(x) = \alpha x^{\beta} [1 - F(x)].$$

The relation given in (1.3) will be used to derive some simple recurrence relations for the single and the product moments.

Means and variances of a distribution can be computed by using recurrence relations for the single and product moments for any continuous distribution. Recent works in this area of those of [7–16] and the references cited therein. To our best knowledge, statistical inference for unknown parameters of PHR distribution has not been studied under the progressive Type-II right censored order statistics.

The outline of this paper is as follows. Recurrence relations for single moments of progressive Type-II right censored order statistics from power hazard rate distribution are given in Section 2. Section 3, contains the recurrence relations for product moments of progressive Type-II right censored order statistics from power hazard rate distribution. Characterization results are also presented in Section 4. The

parameters estimation under proposed scheme for PHR distribution are obtained and illustrated by a real data set in Section 5.

## 2. RECURRENCE RELATIONS FOR SINGLE MOMENTS

In this section, we derive several new recurrence relations for the single moments of progressively Type- II censored order statistics for all sample size  $n$  and all censoring schemes  $(R_1, R_2, \dots, R_m)$ ,  $m \leq n$  from power hazard rate distribution. The single moments of the progressive type- II censored ordered statistics for PHR distribution can be written as,

$$(2.1) \quad \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} = E \left[ X_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} \right] \\ = C(n, m-1) \int \cdots \int_{0 < x_1 < \cdots < x_m < \infty} x_i^k f(x_1) [1 - F(x_1)]^{R_1} f(x_2) \times \\ [1 - F(x_2)]^{R_2} f(x_3) [1 - F(x_3)]^{R_3} \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_1 \cdots dx_m.$$

**Theorem 2.1.** For  $2 \leq m \leq n$  and  $k \geq 0$ ,

$$(2.2) \quad \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} = \frac{\alpha}{k + \beta + 1} \left[ (n - R_1 - 1) \mu_{1:m-1:n}^{(R_1+1+R_2, \dots, R_m)^{(k+\beta+1)}} + \right. \\ \left. (1 + R_1) \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k+\beta+1)}} \right]$$

*Proof.* From equations (2.1), we have

$$(2.3) \quad \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} = C(n, m-1) \int \cdots \int_{0 < x_1 < x_2 < \cdots < x_m < \infty} L(x_2) f(x_2) \times \\ [1 - F(x_2)]^{R_2} f(x_3) [1 - F(x_3)]^{R_3} \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_2 dx_3 \cdots dx_m$$

where

$$(2.4) \quad L(x_2) = \int_0^{x_2} x_1^k f(x_1) [1 - F(x_1)]^{R_1} dx_1.$$

On using (1.3) in (2.4), we have

$$(2.5) \quad L(x_2) = \alpha \int_0^{x_2} x_1^{k+\beta} [1 - F(x_1)]^{R_1+1} dx_1.$$

Integrating (2.5) by parts, we have.

$$(2.6) \quad L(x_2) = \frac{\alpha}{k + \beta + 1} \left[ [1 - F(x_2)]^{R_1+1} x_2^{k+\beta+1} + (R_1 + 1) \int_0^{x_2} x_1^{k+\beta+1} \times \right. \\ \left. [1 - F(x_1)]^{R_1} f(x_1) dx_1 \right].$$

Substituting the value of  $L(x_2)$  from (2.6) into (2.3), we have

$$\begin{aligned} \mu_{1:m:n}^{(R_1, \dots, R_m)^{(k)}} &= \left( \frac{\alpha}{k + \beta + 1} \right) \left[ C(n, m-1) \int \cdots \int_{0 < x_1 < \cdots < x_m < \infty} x_2^{k+\beta+1} \times \right. \\ &\quad \left. [1 - F(x_2)]^{R_1+R_2+1} f(x_2) \cdots f(x_m) [1 - F(x_m)]_m^R dx_2 dx_3 \cdots dx_m \right] \\ &\quad + (1 + R_1) C(n, m-1) \int \cdots \int_{0 < x_1 < x_2 < \cdots < x_m < \infty} x_1^{k+\beta+1} f(x_1) [1 - F(x_1)]^{R_1} \times \\ &\quad f(x_2) [1 - F(x_2)]^{R_2} \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_1 dx_2 dx_3 \cdots dx_m, \end{aligned}$$

which upon rearrangement, gives (2.2).  $\square$

**Corollary 2.1.** For  $m = 1$  and  $n = 1, 2, \dots, k \geq 0$ ,

$$\mu_{1:1:n}^{(n-1)^{(k)}} = \frac{n\alpha}{k + \beta + 1} \mu_{1:1:n}^{(n-1)^{(k+\beta+1)}}.$$

*Proof.* Corollary 2.1 can be proved on the same line of Theorem 2.1.  $\square$

**Theorem 2.2.** For  $2 \leq i \leq m-1$ ,  $m \leq n$  and  $k \geq 0$ ,

$$\begin{aligned} (2.7) \quad \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} &= \left( \frac{\alpha}{k + \beta + 1} \right) \times \\ &\quad \left[ (n - R_1 - R_2 - \dots - R_i - i) \mu_{i:m-1:n}^{(R_1, R_2, \dots, R_{i-1}, R_i + R_{i+1} + 1, R_{i+2}, \dots, R_m)^{(k+\beta+1)}} - \right. \\ &\quad (n - R_1 - R_2 - \dots - R_{i-1} - i + 1) \mu_{i-1:m-1:n}^{(R_1, R_2, \dots, R_{i-2}, R_{i-1} + R_i + 1, R_{i+1}, \dots, R_m)^{(k+\beta+1)}} \\ &\quad \left. + (1 + R_i) \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k+\beta+1)}} \right]. \end{aligned}$$

*Proof.* From (2.1), we have

$$\begin{aligned} (2.8) \quad \mu_{i:m:n}^{(R_1, \dots, R_m)^{(k)}} &= C(n, m-1) \int \cdots \int_{0 < x_1 < \cdots < x_m < \infty} I(x_{i-1}, x_{i+1}) \times \\ &\quad f(x_1) [1 - F(x_1)]^{R_1} \cdots f(x_{i-1}) [1 - F(x_{i-1})]^{R_{i-1}} f(x_{i+1}) [1 - F(x_{i+1})]^{R_{i+1}} \\ &\quad \times \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_1 \cdots dx_m \end{aligned}$$

where

$$(2.9) \quad I(x_{i-1}, x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^k f(x_i) [1 - F(x_i)]^{R_i} dx_i.$$

On using (1.3) in (2.9), we have

$$I(x_{i-1}, x_{i+1}) = \alpha \int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta} [1 - F(x_1)]^{R_1+1} dx_i.$$

Integrating the above equation by parts, we have

$$\begin{aligned} I(x_{i-1}, x_{i+1}) &= \left( \frac{\alpha}{k + \beta + 1} \right) \left[ x_{i+1}^{k+\beta+1} [1 - F(x_i)]^{R_i+1} - x_{i-1}^{k+\beta+1} [1 - F(x_i)]^{R_i+1} \right. \\ &\quad \left. + (1 + R_i) \int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta+1} [1 - F(x_1)]^{R_1} dx_i \right] \end{aligned}$$

Now, substituting the above resulting expression of  $I(x_{i-1}, x_{i+1})$  in (2.8),

$$\begin{aligned} \mu_{i:m:n}^{(R_1, \dots, R_m)^{(k)}} &= C(n, m-1) \int \cdots \int_{0 < x_1 < \cdots < x_m < \infty} \left( \frac{\alpha}{k + \beta + 1} \right) \times \\ &\quad \left[ x_{i+1}^{k+\beta+1} [1 - F(x_i)]^{R_i+1} - x_{i-1}^{k+\beta+1} [1 - F(x_i)]^{R_i+1} + (1 + R_i) \times \right. \\ &\quad \left. \int_{x_{i-1}}^{x_{i+1}} x_i^{k+\beta+1} [1 - F(x_1)]^{R_1} dx_i \right] f(x_1) [1 - F(x_1)]^{R_1} \cdots f(x_{i-1}) [1 - F(x_{i-1})]^{R_{i-1}} \\ &\quad f(x_{i+1}) [1 - F(x_{i+1})]^{R_{i+1}} \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_1 \cdots dx_m. \end{aligned}$$

after simplification it leads to (2.7).  $\square$

**Theorem 2.3.** For  $2 \leq m \leq n$  and  $k \geq 0$ ,

$$\begin{aligned} \mu_{m:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} &= \frac{\alpha}{k + \beta + 1} \left[ (1 + R_m) \mu_{m:m:n}^{(R_1, R_2, \dots, R_m)^{(k+\beta+1)}} \right. \\ &\quad \left. - (n - R_1 - R_2 - \cdots - R_{i-1} - i + 1) \mu_{m-1:m-1:n}^{(R_1, R_2, \dots, R_{m-2}, R_{m-1}+R_m+1, R_{i+1}, \dots, R_m)^{(k+\beta+1)}} \right]. \end{aligned}$$

*Proof.* Similar to the proof of Theorem 2.1.  $\square$

**Remark 2.1.** For  $\beta = 0$  and  $\alpha = 1$  in Theorem 2.1, we get the recurrence relations for single moments of progressively Type-II censored order statistics from the standard exponential distribution,

$$\mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} = \frac{1}{k + 1} \left[ (n - R_1 - 1) \mu_{1:m-1:n}^{(R_1+1+R_2, \dots, R_m)^{(k+1)}} + (1 + R_1) \mu_{1:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} \right],$$

as obtained in [17].

**Remark 2.2.** For  $\beta = 0$  and  $\alpha = 1$  in corollary 2.1

$$\mu_{1:1:n}^{(n-1)^{(k)}} = \frac{n}{k + 1} \mu_{1:1:n}^{(n-1)^{(k+1)}},$$

as obtained in [17].

**Remark 2.3.** For  $\beta = 0$  and  $\alpha = 1$  in Theorem 2.2, we get the recurrence relations for single moments of progressively Type-II censored order statistics from the standard exponential distribution,

$$\begin{aligned} \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} &= \left( \frac{1}{k+1} \right) \times \\ &\left[ (n - R_1 - R_2 - \dots - R_i - i) \mu_{i:m-1:n}^{(R_1, R_2, \dots, R_{i-1}, R_i + R_{i+1} + 1, R_{i+2}, \dots, R_m)^{(k+1)}} - \right. \\ &(n - R_1 - R_2 - \dots - R_{i-1} - i + 1) \mu_{i-1:m-1:n}^{(R_1, R_2, \dots, R_{i-2}, R_{i-1} + R_i + 1, R_{i+1}, \dots, R_m)^{(k+1)}} \\ &\left. + (1 + R_i) \mu_{i:m:n}^{(R_1, R_2, \dots, R_m)^{(k+1)}} \right], \end{aligned}$$

as obtained in [17].

**Remark 2.4.** For  $\beta = 0$  and  $\alpha = 1$  in Theorem 2.3, we get the recurrence relations for single moments of progressively Type-II censored order statistics from the standard exponential distribution,

$$\begin{aligned} \mu_{m:m:n}^{(R_1, R_2, \dots, R_m)^{(k)}} &= \frac{1}{k+1} \left[ (1 + R_m) \mu_{m:m:n}^{(R_1, R_2, \dots, R_m)^{(k+1)}} \right. \\ &\left. - (n - R_1 - R_2 - \dots - R_{i-1} - i + 1) \mu_{m-1:m-1:n}^{(R_1, R_2, \dots, R_{m-2}, R_{m-1} + R_m + 1, R_{i+1}, \dots, R_m)^{(k+1)}} \right], \end{aligned}$$

as obtained in [17].

**Deductions:** For special case  $R_1 = R_2 = \dots = R_m = 0$ , so that  $m = n$  in which the progressive censored order statistics become the usual order statistics  $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ , then

(i) From (2.2): For  $k \geq 0$ , we get

$$\mu_{1:n}^k = \frac{\alpha}{k + \beta + 1} \left[ \mu_{1:n}^{(k+\beta+1)} + (n-1) \mu_{1:n-1:n}^{(1,0,0,\dots,0)^{(k+\beta+1)}} \right].$$

(ii) From (2.7): For  $k \geq 0$ , we get

$$\mu_{i:n}^k = \frac{\alpha}{k + \beta + 1} \left[ (n-i) \mu_{i:n-1}^{(k+\beta+1)} - (n-i+1) \mu_{i-1:n-1}^{(k+\beta+1)} + (1 + R_i) \mu_{i:n}^{(k+\beta+1)} \right].$$

## 3. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

In this section, we derive some recurrence relations for product moments of the progressively Type-II right censored order statistics from the PHR distribution. The  $(i, j)^{th}$  product moments of the progressive Type- II right censored order statistics can be written as,

$$(3.1) \quad \mu_{i:j:m:n}^{(R_1, R_2, \dots, R_m)^{(r,s)}} = E \left[ X_{i:m:n}^{(R_1, R_2, \dots, R_m)^r} X_{j:m:n}^{(R_1, R_2, \dots, R_m)^s} \right] \\ = C(n, m-1) \int \cdots \int_{0 < x_1 < \cdots < x_m < \infty} x_i^r x_j^s f(x_1) [1 - F(x_1)]^{R_1} f(x_2) \\ \times [1 - F(x_2)]^{R_2} f(x_3) [1 - F(x_3)]^{R_3} \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_1 \cdots dx_m.$$

**Theorem 3.1.** For  $1 \leq i < j \leq m-1$  and  $m \leq n$ ,

$$(3.2) \quad \mu_{i:j:m:n}^{(R_1, R_2, \dots, R_m)^{(r,s)}} = \left( \frac{\alpha}{s + \beta + 1} \right) [(n - R_1 - 1 - \cdots - R_j - j) \\ \times \mu_{i:j:m-1:n}^{(R_1, R_2, \dots, R_{j-1}, R_j + R_{j+1} + 1, \dots, R_m)^{(r, s + \beta + 1)}} - (n - R_1 - 1 - \cdots - R_{j-1} - j + 1) \\ \times \mu_{i:j-1:m-1:n}^{(R_1, R_2, \dots, R_{j-1} + R_j + 1, \dots, R_m)^{(r, s + \beta + 1)}} + (1 + R_j) \mu_{i:j:m:n}^{(R_1, R_2, \dots, R_m)^{(r, s + \beta + 1)}}].$$

*Proof.* From (3.1), we have,

$$(3.3) \quad \mu_{i:j:m:n}^{(R_1, \dots, R_m)^{(r,s)}} = C(n, m-1) \int \cdots \int_{0 < x_1 < \cdots < x_m < \infty} x_i^r I(x_{j-1}, x_{j+1}) f(x_1) \\ \times [1 - F(x_1)]^{R_1} \cdots f(x_{j-1}) [1 - F(x_{j-1})]^{R_{j-1}} f(x_{j+1}) [1 - F(x_{j+1})]^{R_{j+1}} \\ \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_m$$

where

$$(3.4) \quad I(x_{j-1}, x_{j+1}) = \int_{x_{j-1}}^{x_{j+1}} x_j^s f(x_j) [1 - F(x_j)]^{R_j} dx_j.$$

On using (1.3) in (3.4), we have

$$(3.5) \quad I(x_{j-1}, x_{j+1}) = \alpha \int_{x_{j-1}}^{x_{j+1}} x_j^{s+\beta} f(x_j) [1 - F(x_j)]^{R_j+1} dx_j.$$

Integrating (3.5) by parts, we have

$$I(x_{j-1}, x_{j+1}) = \frac{\alpha}{\beta + s + 1} \left[ x_{j+1}^{\beta+s} [1 - F(x_{j+1})]^{R_j+1} - x_{j-1}^{\beta+s} [1 - F(x_{j-1})]^{R_j+1} \right] \\ + (1 + R_j) \int_{x_{j-1}}^{x_{j+1}} [1 - F(x_j)]^{R_j+1} f(x_j) x_j^{\beta+s} dx_j.$$



Substituting the expression of  $I(x_{j-1}, x_{j+1})$  in (3.3) and using (3.1), we have,

$$\begin{aligned} \mu_{i:j:m:n}^{(R_1, \dots, R_m)^{(r,s)}} &= \left( \frac{\alpha}{\beta + s + 1} \right) C(n, m-1) \left[ \int \cdots \int_{0 < x_1 < \cdots < x_m < \infty} x_i^r x_{j+1}^{\beta+s} f(x_1) \right. \\ &\quad [1 - F(x_1)]^{R_1} \cdots f(x_{j-1}) [1 - F(x_{j-1})]^{R_{j-1}} f(x_{j+1}) [1 - F(x_{j+1})]^{R_{j+1}} \cdots \\ &\quad \times \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_1 dx_2 \cdots dx_{j-1} dx_{j+1} \cdots dx_m \\ &\quad - \int \cdots \int_{0 < x_1 < \cdots < x_m < \infty} x_i^r x_{j-1}^{\beta+s} f(x_1) [1 - F(x_1)]^{R_1} \cdots f(x_{j-1}) [1 - F(x_{j-1})]^{R_{j-1}} \times \\ &\quad f(x_{j+1}) [1 - F(x_{j+1})]^{R_{j+1}} \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_m \\ &\quad + (1 + R_j) \int \cdots \int_{0 < x_1 < \cdots < x_m < \infty} x_i^r x_j^{\beta+s} f(x_1) [1 - F(x_1)]^{R_1} \cdots \\ &\quad \times \cdots f(x_m) [1 - F(x_m)]^{R_m} dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_m \Big]. \end{aligned}$$

After simplification, we get the required result. This completes the proof of Theorem 3.1.  $\square$

Theorem 3.1 can be reduced to Theorem 2.2 by putting  $i = 0$ .

**Corollary 3.1.** For  $1 \leq i \leq m-1$  and  $m \leq n$ ,

$$\begin{aligned} \mu_{i:j:m:n}^{(R_1, \dots, R_m)^{(r,s+\beta+1)}} &= \left( \frac{\alpha}{s + \beta + 1} \right) \left[ (1 + R_j) \mu_{i:j:m:n}^{(R_1, \dots, R_m)^{(r,s+\beta+1)}} \right. \\ &\quad \left. - (n - R_1 - 1 - \cdots - R_{j-1} - j + 1) \mu_{i:j-1:m-1:n}^{(R_1, \dots, R_{j-1}+R_j+1, \dots, R_m)^{(r,s+\beta+1)}} \right]. \end{aligned}$$

*Proof.* Similar to the proof of Theorem 3.1.  $\square$

**Remark 3.1.** For  $\beta = 0$  and  $\alpha = 1$  in Theorem 3.1, we get the recurrence relations for product moments of progressively Type-II censored order statistics from the standard exponential distribution,

$$\begin{aligned} \mu_{i:j:m:n}^{(R_1, \dots, R_m)^{(r,s)}} &= \left( \frac{1}{s + 1} \right) \times \\ &\quad \left[ (n - R_1 - 1 - \cdots - R_j - j) \mu_{i:j:m-1:n}^{(R_1, R_2, \dots, R_{j-1}, R_j+R_{j+1}+1, \dots, R_m)^{(r,s+1)}} - \right. \\ &\quad (n - R_1 - 1 - \cdots - R_{j-1} - j + 1) \mu_{i:j-1:m-1:n}^{(R_1, R_2, \dots, R_{j-1}+R_j+1, \dots, R_m)^{(1,s+1)}} + \\ &\quad \left. (1 + R_j) \mu_{i:j:m:n}^{(R_1, \dots, R_m)^{(r,s+1)}} \right] \end{aligned}$$

as obtained in [17].

**Remark 3.2.** For  $\beta = 0$  and  $\alpha = 1$  in corollary 3.1, we get the recurrence relations for product moments of progressively Type-II censored order statistics from the standard exponential distribution,

$$\mu_{i:j:m:n}^{(R_1, \dots, R_m)^{(r, s+1)}} = \left( \frac{1}{s+1} \right) \left[ (1 + R_j) \mu_{i:j:m:n}^{(R_1, R_2, \dots, R_m)} - (n - R_1 - 1 - \dots - R_{j-1} - j + 1) \mu_{i:j-1:m-1:n}^{(R_1, R_2, \dots, R_{j-1} + R_j + 1, \dots, R_m)^{(r, s+1)}} \right],$$

as obtained in [17].

**Remark 3.3.** When  $R_1 = R_2 = \dots = R_m = 0$  in (3.2) the recurrence relation for product moments of order statistics from PHR distribution as follows,

$$\mu_{i:j:n}^{(r, s)} = \left( \frac{\alpha}{s + \beta + 1} \right) \times \left[ (n - j) \mu_{i:j:n-1}^{(0, 0, \dots, 1, \dots, 0)^{(r, s+\beta+1)}} - (n - j + 1) \mu_{i:j-1:n-1}^{(0, 0, \dots, 1, \dots, 0)^{(r, s+\beta+1)}} + \mu_{i:j:n}^{(r, s+\beta+1)} \right]$$

#### 4. CHARACTERIZATIONS

An important area of statistical theory is characterization of probability distributions. Different methods are used for the characterization. In this paper, power hazard rate distribution is characterized by hazard function, recurrence relations for single moments and truncated moment respectively.

**Theorem 4.1.** Let  $X$  be a continuous random variable with pdf  $f(x)$  and cdf  $F(x)$  and survival function  $[1 - F(x)]$ . Then  $X$  has a power hazard rate distribution if

$$(4.1) \quad f(x) = \alpha x^\beta [1 - F(x)].$$

*Proof.* Necessary part from equation (1.1) and (1.2), we can easily obtain. For sufficiency part if (4.1) is true, then (4.1) can be written as

$$\frac{-d[1 - F(x)]}{1 - F(x)} = \alpha x^\beta dx.$$

On integrating the above equation, we have

$$(4.2) \quad -\ln [1 - F(x)] = \frac{\alpha x^{\beta+1}}{\beta + 1} + C,$$

where  $C$  is an arbitrary constant. Now since,  $1 - F(0) = 1$ , then putting  $x = 0$ , in (4.2), we have  $C = 0$ ,

$$-\ln [1 - F(x)] = \frac{\alpha x^{\beta+1}}{\beta + 1},$$

$$F(x) = 1 - \exp \left\{ -\frac{\alpha}{\beta + 1} x^{\beta+1} \right\}, \quad x > 0, \alpha > 0, \beta > -1.$$

This is the distribution function of power hazard rate distribution. This completes the proof.  $\square$

**Theorem 4.2.** For  $2 \leq i \leq m - 1$ ,  $m \leq n$  and  $k \geq 0$ , a necessary and sufficient condition for a random variable  $X$  to be distributed with pdf given in (1.1) is that

$$(4.3) \quad \mu_{i:m:n}^{(R_1, \dots, R_m)^{(k)}} = \left( \frac{\alpha}{k + \beta + 1} \right) \times$$

$$\left[ (n - R_1 - \dots - R_i - i) \mu_{i:m-1:n}^{(R_1, \dots, R_{i-1}, R_i + R_{i+1} + 1, R_{i+2}, \dots, R_m)^{(k+\beta+1)}} - \right.$$

$$(n - R_1 - \dots - R_{i-1} - i + 1) \mu_{i-1:m-1:n}^{(R_1, \dots, R_{i-2}, R_{i-1} + R_i + 1, R_{i+1}, \dots, R_m)^{(k+\beta+1)}} +$$

$$\left. (1 + R_i) \mu_{i:m:n}^{(R_1, \dots, R_m)^{(k+\beta+1)}} \right].$$

*Proof.* The necessary part follows from (2.7). On the other hand if the recurrence relation (4.3) is satisfied, then using (2.1), we have

$$(4.4) \quad \mu_{i:m:n}^{(R_1, \dots, R_m)^{(k+a)}} = C(n, m - 1) \int \dots \int_{0 < x_1 < x_{i-1} < x_{i+1} < \dots < x_m < \infty} I(x_{i-1}, x_{i+1})$$

$$\times f(x_1) [1 - F(x_1)]^{R_1} \dots f(x_{i-1}) [1 - F(x_{i-1})]^{R_{i-1}} f(x_{i+1}) [1 - F(x_{i+1})]^{R_{i+1}}$$

$$\times \dots f(x_m) [1 - F(x_m)]^{R_m} dx_1 \dots dx_m,$$

where,

$$(4.5) \quad I(x_{i-1}, x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^{k+a} f(x_i) [1 - F(x_i)]^{R_i} dx_i.$$

Integrating (4.5) by parts, we get

$$(4.6) \quad I(x_{i-1}, x_{i+1}) = \frac{1}{(R_i + 1)} [-x_{i+1}^{k+a} [1 - F(x_{i+1})]^{R_i+1} - x_{i-1}^{k+a} [1 - F(x_{i-1})]^{R_i+1}$$

$$+ (k + a) \int_{x_{i-1}}^{x_{i+1}} x_i^{k+a-1} [1 - F(x_i)]^{R_i+1} dx_i].$$

Upon substituting (4.6) in (4.4), and simplifying the resulting expression, we get

$$(4.7) \quad \mu_{i:m:n}^{(R_1, \dots, R_m)^{(k+a)}} = \frac{1}{(R_i + 1)} \times \\ \left[ -(n - R_1 - \dots - R_i - i) \mu_{i:m-1:n}^{(R_1, R_2, \dots, R_{i-1}, R_i + R_{i+1} + 1, R_{i+2}, \dots, R_m)^{(k+a+1)}} + \right. \\ \left. (n - R_1 - R_2 - \dots - R_{i-1} - i + 1) \mu_{i-1:m-1:n}^{(R_1, R_2, \dots, R_{i-2}, R_{i-1} + R_i + 1, R_{i+1}, \dots, R_m)^{(k+a+1)}} + \right. \\ \left. (k + a) C(n, m - 1) \int \dots \int_{0 < x_1 < x_{i-1} < x_{i+1} < \dots < x_m < \infty} x_i^{k+a-1} f(x_1) [1 - F(x_1)]^{R_1} \times \right. \\ \left. f(x_i) [1 - F(x_i)]^{R_i+1} \dots f(x_m) [1 - F(x_m)]^{R_m} dx_1 \dots dx_m \right].$$

Now substituting for  $\mu_{i:m:n}^{(R_1, \dots, R_m)^{(k+\beta+1)}}$  in (4.7) and simplifying the resulting expression, we get

$$(4.8) \quad C(n, m - 1) \int \dots \int_{0 < x_1 < x_{i-1} < x_{i+1} < \dots < x_m < \infty} x_i^k \left[ f(x_i) - \alpha x_i^\beta [1 - F(x_i)] \right] \\ \times f(x_1) [1 - F(x_1)]^{R_1} f(x_{i-1}) [1 - F(x_{i-1})]^{R_{i-1}} f(x_{i+1}) [1 - F(x_{i+1})]^{R_{i+1}} \\ \times f(x_m) [1 - F(x_m)]^{R_m} dx_1 \dots dx_m = 0.$$

Now applying a generalization of the Muntz- Szasz Theorem (see [18]) to (4.8), we obtain

$$f(x_i) = \alpha x_i^\beta [1 - F(x_i)],$$

which proves the theorem.  $\square$

The following theorem contains characterization result for power hazard rate distribution based on truncated moment.

**Theorem 4.3.** Suppose an absolutely continuous (with respect to Lebesgue measure) random variable  $X$  has the cdf  $F(x)$  and pdf  $f(x)$  for  $x \geq 0$  such that  $f'(x)$  and  $E(X|X \leq x)$  exist, then

$$E(X|X \leq x) = g(x)\eta(x)$$

where

$$g(x) = \left[ -\frac{x}{\alpha x^\beta} + \frac{\exp(\frac{\alpha}{\beta+1}x^{\beta+1})}{\alpha x^\beta} \int_0^x \exp(-\frac{\alpha}{\beta+1}u^{\beta+1}) du \right] \quad \text{and} \quad \eta(x) = \frac{f(x)}{F(x)}$$

if and only if

$$f(x) = \alpha x^\beta \exp(-\frac{\alpha}{\beta+1}x^{\beta+1}), \quad x > 0, \alpha > 0, \beta > -1.$$

*Proof.* If  $X$  has PHR distribution, we have

$$(4.9) \quad E(X|X \leq x) = \frac{1}{F(x)} \int_0^x u f(u) du = \frac{1}{F(x)} \int_0^x u \alpha u^\beta \exp\left(-\frac{\alpha}{\beta+1} u^{\beta+1}\right) du.$$

Integrating (4.9) by parts treating  $\alpha u^\beta \exp(-\frac{\alpha}{\beta+1} u^{\beta+1})$  for integration and rest for the integrand for differentiation, we get,

$$(4.10) \quad E(X|X \leq x) = \frac{1}{F(x)} \left[ -x \exp\left(-\frac{\alpha}{\beta+1} x^{\beta+1}\right) + \int_0^x \exp\left(-\frac{\alpha}{\beta+1} u^{\beta+1}\right) du \right].$$

After multiplying and dividing by  $f(x)$  in (4.10), we have

$$E(X|X \leq x) = \left[ -\frac{x}{\alpha x^\beta} + \frac{\exp(-\frac{\alpha}{\beta+1} x^{\beta+1})}{\alpha x^\beta} \int_0^x \exp\left(-\frac{\alpha}{\beta+1} u^{\beta+1}\right) du \right] \frac{f(x)}{F(x)}$$

therefore

$$E(X|X \leq x) = g(x)\eta(x).$$

This proves the necessary part.

To prove the sufficiency part. From [19], we have the following,

$$(4.11) \quad \frac{1}{F(x)} \int_0^x u f(u) du = \frac{g(x) f(x)}{F(x)} \quad \text{or} \quad \int_0^x u f(u) du = g(x) f(x) \\ x f(x) = g'(x) f(x) + g(x) f'(x).$$

Therefore,

$$(4.12) \quad \frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)},$$

where,

$$g'(x) = x - g(x) \left[ \frac{\beta}{x} - \alpha x^\beta \right].$$

Integrating both sides in (4.12) with respect to  $x$ , we get,

$$f(x) = c x^\beta \exp\left(-\frac{\alpha}{\beta+1} x^{\beta+1}\right).$$

Now, using the condition,  $\int_{-\infty}^{\infty} f(x) dx = 1$ , then

$$\int_0^{\infty} c x^\beta \exp\left(-\frac{\alpha}{\beta+1} x^{\beta+1}\right) dx = 1$$

$$\frac{1}{c} = \int_0^\infty x^\beta \exp\left(-\frac{\alpha}{\beta+1}x^{\beta+1}\right)dx \implies \frac{1}{c} = \frac{1}{\alpha}$$

This completes the proof.  $\square$

## 5. PARAMETER ESTIMATION UNDER PROGRESSIVE TYPE-II CENSORED ORDER STATISTICS

Let  $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}$  be the ordered  $m$  observed failures under Type-II progressively censored sample from PHR distribution  $(\alpha, \beta)$  with censoring scheme  $(R_1, R_2, \dots, R_m)$ . For notational convenience, we will use  $X_i$  in place of  $X_{i:m:n}$ . Thus the likelihood function is given by

$$\begin{aligned} L(x|\alpha, \beta) &= C(n, m-1) \prod_{i=1}^m f(x_i)[1 - F(x_i)]^{R_i} \\ &= C(n, m-1) \prod_{i=1}^m \left[ \alpha x_i^\beta \exp\left\{-\frac{\alpha(1+R_i)x_i^{\beta+1}}{\beta+1}\right\} \right]. \end{aligned}$$

The corresponding log-likelihood function is given by,

$$(5.1) \quad \ln L(x|\alpha, \beta) = D + m \ln \alpha + \beta \sum_{i=1}^m \ln(x_i) - \frac{\alpha}{\beta+1} \sum_{i=1}^m (1+R_i)x_i^{\beta+1}.$$

where  $D = \ln[C(n, m-1)]$ .

By differentiating the log-likelihood function (5.1). The MLEs of  $\alpha$  and  $\beta$  can be obtained by equating the first derivatives by zero and solving with respect to  $\alpha$  and  $\beta$ ,

$$\begin{aligned} \frac{\partial \ln L(x|\alpha, \beta)}{\partial \alpha} &= \frac{m}{\alpha} - \frac{1}{\beta+1} \sum_{i=1}^m (1+R_i)x_i^{\beta+1} = 0, \\ \frac{\partial \ln L(x|\alpha, \beta)}{\partial \beta} &= \sum_{i=1}^m \ln(x_i) + \frac{\alpha}{(\beta+1)^2} \sum_{i=1}^m (R_i+1)x_i^{\beta+1} \\ &\quad - \frac{\alpha}{\beta+1} \sum_{i=1}^m (R_i+1)x_i^{\beta+1} \ln(x_i) = 0. \end{aligned}$$

Once the ML estimates of  $\alpha$  and  $\beta$  are obtained, we can apply the asymptotic normality of the MLEs to compute the approximate CIs for the parameters. The

observed variance and covariance matrix for the MLEs of the unknown parameters  $\Theta = (\alpha, \beta)$  is

$$I^{-1}(\Theta) = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix}^{-1} = \begin{bmatrix} -I_{11} & -I_{12} \\ -I_{21} & -I_{22} \end{bmatrix}^{-1}.$$

The derivative in  $I(\Theta)$  are given as follows

$$\begin{aligned} I_{11} &= -\frac{m}{\alpha^2} \\ I_{22} &= -\frac{2\alpha}{(\beta+1)^3} \sum_{i=1}^m (R_i+1)x_i^{\beta+1} + \frac{2\alpha}{(\beta+1)^2} \sum_{i=1}^m (R_i+1)x_i^{\beta+1} \ln(x_i) \\ &\quad - \frac{\alpha}{\beta+1} \sum_{i=1}^m (R_i+1)x_i^{\beta+1} (\ln(x_i))^2 \\ I_{21} = I_{12} &= \frac{1}{(\beta+1)^2} \sum_{i=1}^m (R_i+1)x_i^{\beta+1} - \frac{1}{\beta+1} \sum_{i=1}^m (R_i+1)x_i^{\beta+1} \ln(x_i). \end{aligned}$$

Therefore, the above approach is used to derive the approximate 100 (1 -  $\delta$ )% confidence interval CIs of the parameters  $\Theta = (\alpha, \beta)$  as in the following form

$$\hat{\alpha} \pm -z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\alpha})}, \hat{\beta} \pm -z_{\frac{\delta}{2}} \sqrt{\text{var}(\hat{\beta})}.$$

Here  $z_{\frac{\delta}{2}}$  is the upper ( $\frac{\delta}{2}$ ) th percentile of the standard normal distribution.

**5.1. Data Analysis.** Now we use a real data set to show that the PHR distribution can be a better model, comparing with many known distributions such as the exponential ([20]) and Rayleigh. We consider the data set from [20].

$n = 15$  and  $m = 12$

(0.278,1), (2.009,1), (6.352,1), (8.286, 0), (18.325, 0), (19.332,0), (20.333, 0), (24.727, 0), (25.717, 0), (25.877,0), (41.47, 0), (84.676, 0).

By using this data set, the likelihood estimators, asymptotic confidence interval, log likelihood function, AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) are calculated in the following table.

The variance-covariance matrix can be obtained as follows.

$$V = \begin{bmatrix} 6.334 \times 10^{-4} & -5.102 \times 10^{-3} \\ -5.102 \times 10^{-3} & 0.052 \end{bmatrix}$$

TABLE 1. The Likelihood Estimation, LogL, AIC and BIC

Models	$\hat{\alpha}$	$\hat{\beta}$	LogL	AIC	BIC
PHRD	0.03936	0.02561	-7.8024	19.6048	21.020
Exp( $\theta$ )	<b>0.042</b>	–	-50.054	102.108	102.816
Rayleigh ( $\alpha = \frac{1}{\theta}$ )	$1.981 \times 10^{-3}$	1	-14.329	30.658	34.074

The asymptotic confidence interval for  $\alpha$  and  $\beta$  are (0,0.089) and (0,0.471), respectively.

Based on Table 1, it is shown that PHRD ( $\alpha, \beta$ ) model provides better fit to the data rather than other distributions which we compared with because it has the smallest value of AIC and BIC test.

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