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# NUMERICAL RESOLUTION OF A DEGENERATE ELLIPTIC-PARABOLIC SEAWATER INTRUSION PROBLEM USING FINITE VOLUME SUSHI METHOD

MOHAMED MANDARI AND MOHAMED RHOUDAF

ABSTRACT. In this paper, we propose an approximation for a seawater intrusion problem in a confined aquifer, This model consists in a coupled system of an elliptic and a de-generate parabolic equation, using finite volume SUSHI (Schema Using Stabilisation and Hybrid Interfaces) method,we demonstrate the convergence of Schema using stabilisation and hybrid interface by numerical simulations tests proposed.

## 1. INTRODUCTION

Saltwater intrusion is the movement of saline water into freshwater, as result of higher seawater density than freshwater, ground-water pumping from coastal wells, or from construction of navigation channels or oils field canals. This phenomenon could affect human activity and wild life that depend on freshwater in the area, see [10], [11], [12].

A lot of research has been done on this problem. Jacob Bear in [3], H. I. Essaid [6], O.Kolditz et al [7], S. Sorek et al [9], H.-J.G. Diersch et O. Koldizt [8]. All these works can be classified in the following way. First, the physically approach-Hidden diffuse interfaces (sharp-diffuse) see [21, 22]. Second approach is the sharp interface where the two fluids are immiscible and the domains occupied by each fluids are assumed to be separated by an interface called sharp interface, we

[1]*corresponding author*

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refer to [2–5,13–16] for more details about sea intrusion problem with sharp interface approach. Third approach consists in considering the existence of a transition zone with variable concentrations of salt (see [17–19]). The fourth approach is to assume that no interface between the fluids, and this fluids are miscible see [20].

We are interested in the numerical resolution of the problem of seawater intrusion in the case of the first approach this work is the objective of some works even [2–5, 13–16].

In the literature there exists many methods to discretise the diffusion term, let us cite the discrete duality finite volume (DDFV) schemes by Domelevo and Omnes [25, 26]. Among other approaches let us mention the multipoints flux approximation schemes by Aavatsmark et al. [23,24], the scheme using stabilization and hybrid interfaces by Eymard, Gallouet, and Herbin [27, 28]. and the mixed finite volume (MFV) schemes by Droniou and Eymard [29, 30]. This is also the case for the mimetic finite difference (MFD) schemes [31,32], the mixed finite element methods on distorted meshes by Y. Kuznetzov and S. Repin [33]. All these methods have been compared in 2008 on a benchmark organized by Herbin and Hubert [34]

In this work, we will discretise the seawater intrusion problem by the finite volume method SUSHI, the advantage of this method is that we can use this method for any type of polyhedral mesh, moreover the hydraulic conductivity matrix-valued function is variable and defined positive, see [28].

1.1. **Seawater intrusion problem.** Let $\Omega$ an open, bounded connected subset of $\mathbb{R}^d$ ($d = 2$ or $d = 3$), which supported tube polygonal $(d = 2)$ or polyhedral $(d = 3)$, and $\partial\Omega$ stands for its boundary. The confined aquifer bounded by two horizontal and impermeable layers. The upper to $z = 0$ and the lower surface corresponds to $z = -H_2$, $H_2$ is the thickness of the aquifer assumed to be such that $H_2 > \delta > 0$ and $t_s(h) = H_2 - h$ is the thickness of saltwater zone.

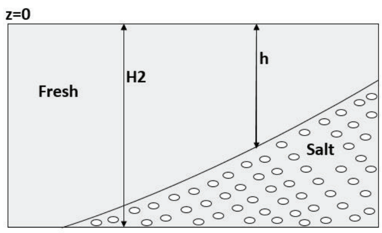


FIGURE 1. Saltwater intrusion phenomena.

Here $h$ is the depth of the interface and $f$ is the freshwater hydraulic head, then $(h, f)$ satisfy the following system (see [2]):

$$\begin{cases} \frac{\partial h}{\partial t} - div(t_s(h)K(x)\nabla h) + div(t_s(h)K(x)\nabla f) = -I_s & \text{in } \Omega \times [0,T], \\ -div(K(x)\nabla f) + div(t_s(h)K(x)\nabla h) = I_f + I_s & \text{in } \Omega \times [0,T]. \end{cases} \tag{1.1}$$

With the following hypothesis

- The initial condition

$$h(x,0) = h_0(x) \text{ in } \Omega, \tag{1.2}$$

such that

$$h^0 \in L^2(\Omega) \quad satisfies \quad \delta \leq h^0(x) \leq H_2 \;\; for\; a.e. \quad x \in \Omega. \tag{1.3}$$

- The boundary conditions

$$\begin{cases} h = 0 & \text{on } \partial\Omega \times [0,T], \\ f = 0 & \text{on } \partial\Omega \times [0,T]. \end{cases} \tag{1.4}$$

- Let $T_s$ be the function defined by, $T_s(x) = \int_0^x t_s(r)dr$
- $K$ is the hydraulic conductivity matrix-valued function satisfying,

$$\begin{cases} K : \Omega \times \mathbb{R} \to M_2(\mathbb{R}) \\ for\; all\; \xi \in \mathbb{R}^2, \exists (K^-, K^+) \in \mathbb{R}^2 \\ such\; that\; 0 < K^-|\xi|^2 \leq \sum\limits_{i,j=1,2} K_{i,j}(x)\xi_i.\xi_j \leq K^+|\xi|^2 < \infty \\ for\; a.e. \;\; x \in \Omega \;\; and \;\; \xi_i \neq 0 \;\; for \;\; i = 1 \; or \; i = 2. \end{cases} \tag{1.5}$$

- $I_s$ and $I_f$ are the supply functions represent the distributed supply surface of fresh and salt water into the aquifer such that:

$$(I_s, I_f) \in L^2(0, T, L^2(\Omega)). \tag{1.6}$$

The outline of this paper is as follows. In section 2 we presents the mesh notations, the discrete operators ( gradient and convection ) and some mathematical properties. Section 2 is devoted to the main results, it is divided to two subsections, the discretization of the problem is given in the first subsection and then the numerical validation in the second subsection.

## 2. A FINITE VOLUME METHOD SUSHI FOR THIS SEA INTRUSION PROBLEM

Now, we construct an approximate solution of Problem (1.1)-(1.6) corresponding to a time implicit discretization and a finite volume scheme SUSHI. The flux is constructed using the idea of the stabilised discrete gradient proposed by Eymard et. al. [19]. This scheme is inspired by hybrid finite volume (HFV) and the cell centred (SUCCES) schemes. They are based on two fundamental ideas: one where the unknown on the edges are introduced only where they are needed , and second where the unknown on the edges introduced on all edges of the mesh.

### 2.1. **Space and time discretization.**

**Definition 2.1.** *Now let's define some notations of the discretization of* $\Omega$.

- *A discretization of* $\Omega$*, denoted* $\mathcal{D}$ *is defined by a triplet* $\mathcal{D} = (\mathcal{M}, \mathcal{E}, P)$.
- $\mathcal{M}$ *is a family of connected non-empty open subspaces included in* $\Omega$ *(set of control volumes* $\mathcal{K}$*).*
- $\sigma$ *is a non-empty open of* $\mathbb{R}$.
- *The set of interfaces of the mesh* $\mathcal{D}$ *is denoted* $\mathcal{E}$.
- *This set is decomposed into two subdomains* $\mathcal{E}_{int}$ *and* $\mathcal{E}_{ext}$ *which respectively represent the set of internal faces and faces located on the edge* $\partial\Omega$ *of the domain.*
- *For all* $\mathcal{K} \in \mathcal{M}$, $M_\sigma = \{\mathcal{K}, \sigma \in \mathcal{E}_\mathcal{K}\}$. *If* $M_\sigma$ *content one element then* $\sigma \in \mathcal{E}_{ext}$*, else* $\sigma \in \mathcal{E}_{int}$.
- $x_\sigma$ *and* $x_\mathcal{K}$ *are respectively the center and the barycentre of* $\sigma$ *and* $\mathcal{K}$.
- $m_\mathcal{K}$ *and* $m_\sigma$ *are respectively the measure of control volume* $\mathcal{K}$ *and of interface* $\sigma$.
- $n_{\mathcal{K},\sigma}$ *is the unit vector normal to* $\sigma$ *outward to* $\mathcal{K}$.
- $P$ *is the set of points of* $\Omega$.
- $C_{\mathcal{K},\sigma}$ *is the cone with vertex* $x_\mathcal{K}$ *and basis* $\sigma$.

**Definition 2.2.** *We consider* $X_\mathcal{D}$, $X_{\mathcal{D},0}$ *and* $X_{\mathcal{D},0,\mathcal{B}}$ *three spaces defined as follow:*

$$
\begin{aligned}
X_\mathcal{D} &= \{v = ((v_\mathcal{K})_{\mathcal{K}\in\mathcal{M}}, (v_\sigma)_{\sigma\in\mathcal{E}}); v_\mathcal{K} \in \mathbb{R}, v_\sigma \in \mathbb{R}\},\\
X_{\mathcal{D},0} &= \{v \in X_\mathcal{D} \text{ such that } \Lambda_\mathcal{K}\nabla^n_{\mathcal{K},\sigma} v.n_{\mathcal{K},\sigma} = 0, \forall\sigma \in \mathcal{E}_{ext}\},\\
X_{\mathcal{D},0,\mathcal{B}} &= \{v \in X_{\mathcal{D},0} / \exists \beta^\mathcal{K}_\sigma \in \mathbb{R},\\
v_\sigma &= \sum_{\mathcal{K}\in\mathcal{M}} \beta^\mathcal{K}_\sigma v_\mathcal{K}\}.
\end{aligned}
$$

$\mathcal{B}$ is defined in the next definition.

**Definition 2.3.** *Let:*

$$u_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta_\sigma^{\mathcal{K}} u_{\mathcal{K}}, \tag{2.1}$$

*where $(\beta_\sigma^{\mathcal{K}})_{\mathcal{K}\in\mathcal{M},\sigma\in\mathcal{E}_{int}}$ is a family of real numbers, with $\beta_\sigma^{\mathcal{K}} \neq 0$ only for some control volumes $\mathcal{K}$ close to $\sigma$, and such that*

$$\sum_{\mathcal{K}\in\mathcal{M}} \beta_\sigma^{\mathcal{K}} = 1 \textit{ and } x_\sigma = \sum_{\mathcal{K}\in\mathcal{M}} \beta_\sigma^{\mathcal{K}} x_{\mathcal{K}}.$$

*$\mathcal{B}$ is the set of the eliminated unknowns using* (2.1)*, and $\mathcal{H} = \mathcal{E}_{int}/\mathcal{B}$.*

The space $X_{\mathcal{D}}$ is equipped with the semi-norm $|.|_{X_{\mathcal{D}}}$ defined by

$$|v|_{X_{\mathcal{D}}}^2 = \sum_{\mathcal{K}\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \frac{m_\sigma}{d_{\mathcal{K}\sigma}} (v_\sigma - v_{\mathcal{K}})^2, \text{ for all } v \in X_{\mathcal{D}}.$$

Note that $|.|_{X_{\mathcal{D}}}$ is a norm on the spaces $X_{\mathcal{D},0}$ and $X_{\mathcal{D},0,\mathcal{B}}$.

2.2. **The discrete gradient.** It is always possible to deduce an expression for $\nabla_{\mathcal{D}} u(x)$ as a linear combination of $(u_\sigma - u_{\mathcal{K}})_{\sigma\in\mathcal{E}_{\mathcal{K}}}$.

Let us first define

$$\begin{aligned} \nabla_{\mathcal{K}} : & X_{\mathcal{D}} \to (X_{\mathcal{D}})^2 \\ & u^{n+1} \mapsto \nabla_{\mathcal{K}} u^{n+1}, \end{aligned}$$

such that

$$u^{n+1} \in X_D, \nabla_{\mathcal{K}} u^{n+1} = \frac{1}{|\mathcal{K}|} \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} |\sigma| [u_\sigma^{n+1} - u_{\mathcal{K}}^{n+1}] \nu_{\mathcal{K},\sigma}.$$

However, we find that this discrete gradient is zero for any $u_{\mathcal{K}}^{n+1} \in \mathcal{K}$, if $u_\sigma^{n+1}$ are zero, so it is not coercive. We thus seek a new coherent discrete gradient with the previous and coercive in the $C_{\mathcal{K},\sigma}$ (cone the vertex $x_{\mathcal{K}}$ and basis $\sigma$). This corresponds to the previous step gradient to which we add a correction term. We define the discrete gradient as follows

$$\nabla_{\mathcal{K},\sigma} u^{n+1} = \nabla_{\mathcal{K}} u^{n+1} + \mathcal{R}_{\mathcal{K},\sigma}(u^{n+1}) \nu_{\mathcal{K},\sigma},$$

with

$$\mathcal{R}_{\mathcal{K},\sigma}(u^{n+1}) = \frac{\sqrt{d}}{d_{\mathcal{K},\sigma}} (u_\sigma^{n+1} - u_{\mathcal{K}}^{n+1} - \nabla_{\mathcal{K}} u^{n+1}.[x_\sigma - x_{\mathcal{K}}]).$$

(Recall that $d$ is the space dimension and $d_{\mathcal{K},\sigma}$ is the Euclidean distance between $x_{\mathcal{K}}$ and $x_\sigma$.) We obtain the following stable discrete gradient

$$\nabla_{\mathcal{K},\sigma} u^{n+1} = \nabla_{\mathcal{K}} u^{n+1} + \mathcal{R}_{\mathcal{K},\sigma} u^{n+1}.\nu_{\mathcal{K},\sigma}. \tag{2.2}$$

We may then define $\nabla_{\mathcal{D}}$ as the piece-wise constant function equal to $\nabla_{\mathcal{K},\sigma}$ a.e. in the cone $C_{\mathcal{K},\sigma}$ with vertex $x_{\mathcal{K}}$ and basis $\sigma$

$$\nabla_{\mathcal{D}} u^{n+1} = \nabla_{\mathcal{K},\sigma} u^{n+1} \text{ for a.e } x \in C_{\mathcal{K},\sigma}.$$

Then we have

$$\nabla_{\mathcal{K},\sigma} u^{n+1} = \sum_{\sigma' \mathcal{E}_{\mathcal{K}}} Y^{\sigma,\sigma'}(u_{\sigma'}^{n+1} - u_{\mathcal{K}}^{n+1}),$$

with $Y^{\sigma,\sigma'}$ giving by

$$Y^{\sigma,\sigma'} = \begin{cases} \frac{m_\sigma}{m_{\mathcal{K}}}\nu_{\mathcal{K}\sigma} + \frac{\sqrt{d}}{d_{\mathcal{K},\sigma}}(1 - \frac{m_\sigma}{m_{\mathcal{K}}}\nu_{\mathcal{K}\sigma}.[x_\sigma - x_{\mathcal{K}}])\nu_{\mathcal{K}\sigma} & \text{if } \sigma = \sigma', \\ \frac{m_{\sigma'}}{m_{\mathcal{K}}}\nu_{\mathcal{K}\sigma'} - \frac{\sqrt{d}}{d_{\mathcal{K},\sigma} m_{\mathcal{K}}}\nu_{\mathcal{K},\sigma'}.[x_\sigma - x_{\mathcal{K}}]\nu_{\mathcal{K},\sigma} & \text{otherwise.} \end{cases} \tag{2.3}$$

2.3. **The discrete convection term.** To treat the convection term in the concentration equation, we define the following convection operator discrete:

$$\begin{aligned} divc_\sigma \ &: \ X_{\mathcal{D}} \times X_{\mathcal{D}} \to X_{\mathcal{D}} \\ &(\xi_{\mathcal{D}}, v_{\mathcal{D}}) \mapsto div(\xi_{\mathcal{D}}, v_{\mathcal{D}}), \end{aligned}$$

with

$$divc_\sigma(\xi_{\mathcal{D}}, v_{\mathcal{D}}) = \begin{cases} v_{\mathcal{K}}\nu_{\sigma,\mathcal{K}}\xi_{\mathcal{K}} & \text{if } v_{\mathcal{K}}\nu_{\sigma,\mathcal{K}} \geq 0, \\ -v_{\mathcal{K}}\nu_{\sigma,\mathcal{K}}\xi_{\mathcal{L}} & \text{if } v_{\mathcal{K}}\nu_{\sigma,\mathcal{K}} < 0. \end{cases}$$

## 3. Main results

### 3.1. **Discrete weak formulation.**

3.1.1. *Elliptic equation.* We begin with the discretisation of this equation

$$-div(K(x)\nabla f) + div(K(x)t_s(h)\nabla h) = I_s - I_f.$$

We integer over $\mathcal{K}$ for any $\mathcal{K} \in \mathcal{M}$ an in the interval $]t^n, t^{n+1}[\subset ]0, T[$

$$\int_{t_n}^{t_{n+1}} \Delta t \int_{\mathcal{K}} -div(K_2(x)\nabla f^{n+1}) + div(K(x)T_s(h^n)\nabla h^{n+1} = \int_{t_n}^{t_{n+1}} \Delta t \int_{\mathcal{K}} (I_f + I_s),$$

then we get

$$\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\int_{\sigma}K(x)\nabla f^{n+1}.\nu_{\mathcal{K},\sigma}-\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\int_{\sigma}K(x)t_s(h^n)\nabla h^n.\nu_{\mathcal{K},\sigma}=m_{\mathcal{K}}(I^{n+1}_{f,\mathcal{K}}+I^{n+1}_{s,\mathcal{K}}),$$

finally

$$\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}^1_{\mathcal{K},\sigma}(f^{n+1})-\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}^2_{\mathcal{K},\sigma}(h^n)=m_{\mathcal{K}}(I^{n+1}_{f,\mathcal{K}}+I^{n+1}_{s,\mathcal{K}}). \tag{3.1}$$

Withe the following discretisation of boundary condition

$$f^{n+1}_{\sigma}=0;\quad \text{for all }\ \sigma\in\mathcal{E}_{ext}.$$

The fact that the flow is continuous at the interface of the two elements, we have

$$\mathcal{F}^1_{\mathcal{K},\sigma}(f^{n+1})+\mathcal{F}^1_{\mathcal{L},\sigma}(f^{n+1})=0\ \text{ for all }\sigma\in\mathcal{E}_{int}. \tag{3.2}$$

Let, multiplying the equation (3.1) by $v^{n+1}_{\mathcal{K}}$ for all $\mathcal{K}\in\mathcal{M}$ and all $n=0,\dots,N-1$, then sum over $\mathcal{K}$ and over $n=0,\dots,N-1$, and using the equation (3.2) we get

$$\left\{\begin{array}{l}\displaystyle\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}^1_{\mathcal{K}}(f^{n+1})[v^{n+1}_{\mathcal{K}}-v^{n+1}_{\sigma}]-\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}}\mathcal{F}^2_{\mathcal{K}}(h^{n+1})v^{n+1}_{\mathcal{K}}=\\ \displaystyle\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}v^{n+1}_{\mathcal{K}}m_{\mathcal{K}}(I^{n+1}_{s,\mathcal{K}}-I^{n+1}_{f,\mathcal{K}}),\end{array}\right.$$

3.1.2. *Parabolic equation.* Let discrete the following equation

$$\frac{\partial h}{\partial t}-div(K(x)t_s(h)\nabla h)+div(K(x)t_s(h)\nabla f)=-I_s.$$

Integering over $\mathcal{K}$ for any $\mathcal{K}\in\mathcal{M}$ an in the interval $]t^n,t^{n+1}[\subset]0,T[$, we get

$$\left\{\begin{array}{l}\int_{t_n}^{t_{n+1}}\Delta t\int_{\mathcal{K}}\frac{\partial h}{\partial t}+\int_{t_n}^{t_{n+1}}\Delta t\int_{\mathcal{K}}-div(K(x)t_s(h)\nabla h)+\\ div(K(x)T_s(h)\nabla f=\int_{t_n}^{t_{n+1}}\Delta t\int_{\mathcal{K}}(-I_s),\end{array}\right.$$

then we get

$$\left\{\begin{array}{l}\displaystyle m_{\mathcal{K}}(h^{n+1}_{\mathcal{K}}-h^n_{\mathcal{K}})+\Delta t\sum_{\sigma\in\mathcal{K}}\mathcal{F}^2_{\mathcal{K},\sigma}(h^{n+1})\\ \displaystyle+\Delta t\sum_{\sigma\in\mathcal{K}}m_{\sigma}divc_{\sigma}(t_s(h^{n+1}_{\mathcal{D}},K_{\mathcal{D}}\nabla_{\mathcal{D}}f^{n+1})=\Delta t m_{\mathcal{K}}(-I^{n+1}_{s,\mathcal{K}}),\end{array}\right. \tag{3.3}$$

with the following discretization of boundary condition

$$h^{n+1}_{\sigma}=0;\quad \text{for all }\ \sigma\in\mathcal{E}_{ext}.$$

The fact that the flow is continuous at the interface of the two elements, we have

$$\mathcal{F}^2_{\mathcal{K},\sigma}(h^{n+1}) + \mathcal{F}^2_{\mathcal{L},\sigma}(h^{n+1}) = 0 \text{ for all } \sigma \in \mathcal{E}_{int}. \tag{3.4}$$

We multiplying (3.3) by $w^{n+1}_{\mathcal{K}}$ for all $\mathcal{K} \in \mathcal{M}$ and all $n = 0, \dots, N-1$, then sum over $\mathcal{K}$ and over $n = 0, \dots, N-1$, then we get

$$\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w^{n+1}_{\mathcal{K}}(h^{n+1}_{\mathcal{K}} - h^n_{\mathcal{K}}) + \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^2_{\mathcal{K},\sigma}(h^{n+1})w^{n+1}_{\mathcal{K}} +$$
$$\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w^{n+1}_{\mathcal{K}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma div c_\sigma(t_s(h^{n+1}_{\mathcal{D}}), K_{\mathcal{D}}\nabla_{\mathcal{D}} f^{n+1}) = \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w^{n+1}_{\mathcal{K}} I^{+,n+1}_{s,\mathcal{K}}.$$

Bearing in mind (3.4), from above, we get

$$\begin{cases} \displaystyle\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w^{n+1}_{\mathcal{K}}(h^{n+1}_{\mathcal{K}} - h^n_{\mathcal{K}}) + \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^2_{\mathcal{K},\sigma}(h^{n+1})[w^{n+1}_{\mathcal{K}} - w^{n+1}_{\sigma}] + \\ \displaystyle\delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w^{n+1}_{\mathcal{K}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} m_\sigma div c_\sigma(t_s(h^{n+1}_{\mathcal{D}}), K_{\mathcal{D}}\nabla_{\mathcal{D}} f^{n+1}) = \delta t\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}} w^{n+1}_{\mathcal{K}} I^{+,n+1}_{s,\mathcal{K}}. \end{cases}$$

3.1.3. *The discrete flux.* The discrete flux $\mathcal{F}^1_{\mathcal{K},\sigma}$ and $\mathcal{F}^2_{\mathcal{K},\sigma}$ are expressed in terms of the discrete unknowns. For this purpose we apply the SUSHI scheme proposed in [27]. The idea is based upon the identification of the numerical flux through the mesh dependent bilinear form, using the expression of the discrete gradient

$$\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^1_{\mathcal{K},\sigma}(f^{n+1})(v_{\mathcal{K}} - v_\sigma) \approx \int_0^T\int_\Omega K(x)\nabla_{\mathcal{D}} f^{n+1} K(x,s)\nabla_{\mathcal{D}} v, \forall f^{n+1}, \tag{3.5}$$

$v \in X_{0,\mathcal{D}}$, and

$$\sum_{n=0}^{N-1}\sum_{\mathcal{K}\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \mathcal{F}^2_{\mathcal{K},\sigma}(h^{n+1})(w_{\mathcal{K}} - w_\sigma) \approx \int_0^T\int_\Omega \nabla_{\mathcal{D}} T_s(h^{n+1}) K(x)\nabla_{\mathcal{D}} w, \forall h^{n+1}, \tag{3.6}$$

$w \in X_{0,\mathcal{D}}$. The identification of the numerical fluxes using relation (3.5) and (3.6) leads to the expression

$$F^1_{\mathcal{K},\sigma}(f^n) = \sum_{\sigma'\in\mathcal{E}_{\mathcal{K}}} K^{\sigma,\sigma'}_{\mathcal{K}}(f^{n+1}_{\mathcal{K}} - f^{n+1}_{\sigma}),$$

$$F^2_{\mathcal{K},\sigma}(h^{n+1}) = \sum_{\sigma'\in\mathcal{E}_{\mathcal{K}}} K^{\sigma,\sigma'}_{\mathcal{K}}(T_s(h^{n+1}_{\mathcal{K}}) - T_s(h^{n+1}_{\sigma})).$$

Thus

$$\int_{\mathcal{K}} \nabla_{\mathcal{D}} f^{n+1} K(x) \nabla_{\mathcal{D}} v = \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \sum_{\sigma'\in\mathcal{E}_{\mathcal{K}}} K_{\mathcal{K}}^{\sigma,\sigma'} (f_{\mathcal{K}}^{n+1} - f_{\sigma'}^{n+1})(v_{\sigma'} - v_{\mathcal{K}}).$$

$$\int_{\mathcal{K}} \nabla_{\mathcal{D}} T_s(h^{n+1}) K(x) \nabla_{\mathcal{D}} w = \sum_{\sigma\in\mathcal{E}_{\mathcal{K}}} \sum_{\sigma'\in\mathcal{E}_{\mathcal{K}}} K_{\mathcal{K}}^{\sigma,\sigma'} (T_s(h_{\mathcal{K}}^{n+1}) - T_s(h_{\sigma'}^{n+1}))(w_{\sigma'} - w_{\mathcal{K}}).$$

With $\sigma, \sigma' \in \mathcal{E}_{\mathcal{K}}$ and

$$\begin{cases} K_{\mathcal{K}}^{\sigma,\sigma'} = \sum_{\sigma''\in\mathcal{E}_{\mathcal{K}}} Y^{\sigma'',\sigma} \Gamma_{\mathcal{K}}^{\sigma''} Y^{\sigma'',\sigma'} \quad \text{with } \Gamma_{\mathcal{K}}^{\sigma''} = \int_{\mathcal{K},C_{\sigma''}} K(x)dx, \end{cases}$$

The local matrices $K_{\mathcal{K}}^{\sigma,\sigma'}$ is symmetric positive and $Y^{\sigma'',\sigma'}$ is defined in (2.3).

3.2. **The numerical scheme.** Using (2.2) we have

$$\begin{cases} \nabla_{\mathcal{K},\sigma} f^{n+1} = \nabla_{\mathcal{K}} f^{n+1} + \mathcal{R}_{\mathcal{K},\sigma}(f^{n+1}).\nu_{\mathcal{K},\sigma}, \\ \nabla_{\mathcal{K},\sigma} h^{n+1} = \nabla_{\mathcal{K}} h^{n+1} + \mathcal{R}_{\mathcal{K},\sigma}(h^{n+1}).\nu_{\mathcal{K},\sigma}, \end{cases}$$

and

$$divc_{\sigma}(\xi_{\mathcal{D}}, v_{\mathcal{D}}) = \begin{cases} v_{\mathcal{K}} \nu_{\sigma,\mathcal{K}} \xi_{\mathcal{K}} & \text{if } v_{\mathcal{K}} \nu_{\sigma,\mathcal{K}} \geq 0, \\ -v_{\mathcal{K}} \nu_{\sigma,\mathcal{K}} \xi_{\mathcal{L}} & \text{if } v_{\mathcal{K}} \nu_{\sigma,\mathcal{K}} < 0. \end{cases}$$

The discretisation of the problem (1.1), (1.2) and (1.4) is defined as following

$$h(x,0) = \frac{1}{m_{\mathcal{K}}} \int_{\mathcal{K}\in\mathcal{M}} h_0(x)dx.$$

$$\begin{cases} K_{\mathcal{K},\sigma} \nabla_{\mathcal{D}} f.\nu = 0 \\ K_{\mathcal{K},\sigma} \nabla_{\mathcal{D}} h.\nu = 0 = 0. \end{cases}$$

At each time step $n$, the numerical solution will be given by $(h_{\mathcal{D}}^{n+1}, f_{\mathcal{D}}^{n+1}) \in \mathbb{R}_{\mathcal{D}}^2$. Then, the scheme for (1.1) writes for all $0 \leq n \leq N-1$,

$$I_{s,\mathcal{K}}^{n+1} = \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \frac{1}{m_{\mathcal{K}}} \int_{\mathcal{K}} I_s(t,x)dt,$$

$$I_{f,\mathcal{K}}^{n+1} = \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} \frac{1}{m_{\mathcal{K}}} \int_{\mathcal{K}} I_f(t,x)dt,$$

$$\text{(3.7)} \quad \begin{cases} -\sum_{\mathcal{K}\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_\mathcal{K}} \mathcal{F}^1_{\mathcal{K},\sigma}(f^{n+1})(v_\sigma - v_\mathcal{K}) + \sum_{\mathcal{K}\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_\mathcal{K}} \mathcal{F}^2_{\mathcal{K},\sigma}(h^n)(v_\sigma - v_\mathcal{K}) = \\ \sum_{\mathcal{K}\in\mathcal{T}} v_\mathcal{K}\left(I^{n+1}_{s,\mathcal{K}} + I^{n+1}_{f,\mathcal{K}}\right), \\ \sum_{\mathcal{K}\in\mathcal{T}} m_\mathcal{K} v_\mathcal{K} \dfrac{h^{n+1}_\mathcal{K} - h^n_\mathcal{K}}{\Delta t} - \sum_{\mathcal{K}\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_\mathcal{K}} \mathcal{F}^2_{\mathcal{K},\sigma}(h^{n+1})(v_\sigma - v_\mathcal{K}) + \\ \sum_{\mathcal{K}\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_\mathcal{K}} m_\sigma divc(t_s(h^{n+1}_\mathcal{D}), K_\mathcal{D}\nabla_{\mathcal{K},\sigma} f^{n+1}) v_\mathcal{K} = -\sum_{\mathcal{K}\in\mathcal{T}} v_\mathcal{K} m_\mathcal{K} I^{n+1}_{s,\mathcal{K}}, \\ \mathcal{F}^1_{\mathcal{K},\sigma}(f^{n+1}) = \sum_{\sigma'\in\mathcal{E}_\mathcal{K}} A^{\sigma,\sigma'}_\mathcal{K}(f^{n+1}_{\sigma'} - f^{n+1}_\mathcal{K}), \\ \mathcal{F}^2_{\mathcal{K},\sigma}(h^{n+1}) = \sum_{\sigma'\in\mathcal{E}_\mathcal{K}} A^{\sigma,\sigma'}_\mathcal{K}(T_s(h^{n+1}_{\sigma'}) - T_s(h^{n+1}_\mathcal{K})), \end{cases}$$

such that,

$$A^{\sigma,\sigma'}_\mathcal{K} = \sum_{\sigma''\in\mathcal{E}_\mathcal{K}} Y^{\sigma,\sigma''} K Y^{\sigma',\sigma''},$$

with $Y^{\sigma,\sigma'}$ is given by (2.3).

### 3.3. **Numerical convergence of the SUSHI scheme.**

3.3.1. *Convergence of the diffusion equation.* In this numerical test, we are interested to demonstrate the convergence of the following diffusion equation with Dirichlet boundary

$$\begin{cases} -div(K(x)\nabla S) = f(x)\ in \Omega, \\ S = 0\ \ on \partial\Omega. \end{cases}$$

We take the exact solution $S_1(x, y) = sin(\pi x)^2 sin(\pi y)^2$ in a first case and in a second case $S_2(x, y) = x^2y^2(x-1)^2(y-1)^2$ the permeability $K(x, y)$ is given by

$$K_1(x, y) = 80 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

or by

$$K_2(x, y) = \frac{1}{x^2 + y^2} \begin{bmatrix} 10^{-3}x^2 + y^2 & (10^{-3} - 1)xy \\ (10^{-3} - 1)xy & 10^{-3}y^2 + x^2 \end{bmatrix},$$

in both cases $\Omega = (0, 1)^2$. Then we get the convergence tables in norm $L^2$, $L^1$ and $L^\infty$ following,

TABLE 1. Convergence results of the SUSHI schemes, with $S_{ext} = S_1$ and $K = K_1$ .

| **Refinement level** | $\|S_{\mathcal{T}} - S_{ext}\|_{L^2(\Omega)}$ | $\|S_{\mathcal{T}} - S_{ext}\|_{L^1(\Omega)}$ | $\|S_{\mathcal{T}} - S_{ext}\|_{L^\infty(\Omega)}$ |
|---|---|---|---|
| 1 | 0.0149 | 0.0045 | 0.1920 |
| 2 | 0.0033 | 0.0010 | 0.0848 |
| 3 | 8.0426e-04 | 2.4747e-04 | 0.0414 |
| 4 | 2.0011e-04 | 6.1305e-05 | 0.0206 |

TABLE 2. Convergence results of the SUSHI schemes, with $S_{ext} = S_2$ and $K = K_2$.

| **Refinement level** | $\|S_{\mathcal{T}} - S_{ext}\|_{L^2(\Omega)}$ | $\|S_{\mathcal{T}} - S_{ext}\|_{L^1(\Omega)}$ | $\|S_{\mathcal{T}} - S_{ext}\|_{L^\infty(\Omega)}$ |
|---|---|---|---|
| 1 | 0.0159 | 0.0017 | 0.3076 |
| 2 | 0.0050 | 4.7231e-04 | 0.1941 |
| 3 | 0.0015 | 1.2424e-04 | 0.1124 |
| 4 | 4.2783e-04 | 3.1108e-05 | 0.0623 |

3.3.2. *Convergence of a nonlinear elliptic-parabolic equations:* In this subsection, we present the schema 2D-SUSHI apply to a simple test case with analytical solution in order to study the convergence properties. For this test case we give the formulas for the diffusion tensor $K$ and the exacts solutions $h_e$ and $f_e$ from which we deduce the source terms $g_1$ and $g_2$, of the system (3.8), to be used in the numerical computations.

$$
\left\{
\begin{array}{l}
\frac{\partial h}{\partial t} - div(K(H_2 - h)\nabla h) + div(K(H_2 - h)\nabla f) = g_1, \text{ in } \Omega \times [0, T], \\
-div(K\nabla f) + div(K(H_2 - h)\nabla h) = g_2, \text{ in } \Omega \times [0, T], \\
f = 0 \text{ and } h = 0, \text{ on } \partial\Omega \times [0, T], \\
h(x, 0) = h_0, \text{ in } \Omega.
\end{array}
\right.
\tag{3.8}
$$

The algorithm used to compute numerical solution of the system (3.7) is the following: at each time step, we first calculate $f^n$ solution of the linear system given by the first equation of (3.7) and next we compute $h^{n+1}$ as the solution of the nonlinear system defined by the second equation of (3.7).

Consider the following data: $Lx = 1, \quad Ly = 1$ (the length and the width of the domain $\Omega = ]0, 1[\times]0, 1[$), the rectangular domain is covered by triangles. Further,

$\delta t = 10^{-4}$, the diffusion tensor $K = 1$ and the second members $g_1$ and $g_2$ of the bouts equations are defined such that the exacts solutions are,

$$\begin{cases} h(x,t) = tsin(\pi x)^2 sin(\pi y)^2, \\ f(x,t) = x(x-1)y(y-1). \end{cases}$$

We introduce the relative error in $L^1(\Omega)$, $L^2(\Omega)$, $L^\infty(\Omega)$, between the exact and the numerical solution by,

$$\begin{cases} err_1(h) = \frac{\|h_e - h_{\mathcal{D}}\|_{L^1(\Omega)}}{\|h_e\|_{L^1(\Omega)}}, \\ err_2(h) = \frac{\|h_e - h_{\mathcal{D}}\|_{L^2(\Omega)}}{\|h_e\|_{L^2(\Omega)}}, \\ err_\infty(h) = \frac{\|h_e - h_{\mathcal{D}}\|_{L^\infty(\Omega)}}{\|h_e\|_{L^\infty(\Omega)}}. \end{cases}$$

In the Table bellow, we calculate the norms $L^1(\Omega)$, $L^2(\Omega)$, and $L^\infty(\Omega)$ of the difference between the exact solution $h(x,t)$ and the numerical solution $h_{\mathcal{T}}$.

TABLE 3. $L^\infty - norm$ Convergence results of the SUSHI method on the deft $h$ at $t = 10^{-3}$ with $\delta t = 10^{-4}$.

| **N.U** | $err_\infty$ | Order | $err_1$ | Order | $err_2$ | Order |
|---|---|---|---|---|---|---|
| 44 | 9.7600e-03 | - | 3.6692e-07 | - | 3.8518e-07 | - |
| 168 | 1.1289e-04 | 1.9633 | 2.2946e-07 | 1.0317 | 3.2735e-07 | 1.0110 |
| 656 | 1.1056e-04 | 1.0023 | 1.3789e-07 | 1.0333 | 1.6480e-07 | 1.0460 |
| 2592 | 1.0944e-04 | 1.0011 | 6.8453e-08 | 1.0443 | 8.0808e-08 | 1.0456 |
| 10304 | 1.0862e-04 | 1.0008 | 3.5054e-08 | 1.0406 | 4.4356e-08 | 1.0367 |
| 12560 | 1.079e-04 | 1.0007 | 2.0286e-08 | 1.0319 | 2.8983e-08 | 1.0251 |

3.3.3. *Example describes the interface evolution from a salt intrusion problem in a confined aquifer:* In this subsection, we illustrate the behavior of the SUSHI scheme by applying it to the system (1.1), which describes the seawater intrusion. In this simulations, we solve numerically the full bi-dimensional problem and we plot the transversal section average (with respect to $y$) of the depth $h_{\mathcal{D}}(x)$ at different times.

Let's use the data from the original Keulegan article [1] for our numerical simulations. In this example the evolution of the interface is described from a salt intrusion problem in a confined aquifer. The fresh and salty waters are separated by interface described by a linear profile pivoting around a fixed point $(0,(D/2))$ . More precisely, Keulegan has proposed the following formula:

$$Z(x,t)=\frac{D}{2}\left(1+\frac{x}{L(t)}\right),\quad with\quad L(t)=\left(\frac{KD\alpha t}{n_e}\right)^2.$$

With $D=10m$ stands for the depth of the aquifer, $\alpha=0.1$, the effective porosity $n_e=0$ and the hydraulic conductivity $K=39,024m/j$. We consider homogeneous Dirichlet conditions. Let $L(t_0)=50m$ for the initial data.

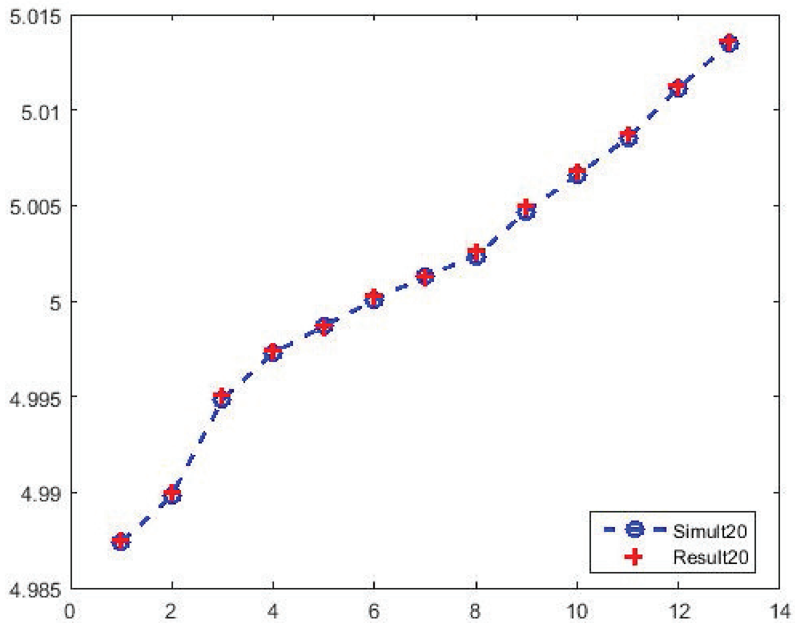


FIGURE 2. Comparison between the approximate and exact solution at time $t=20$

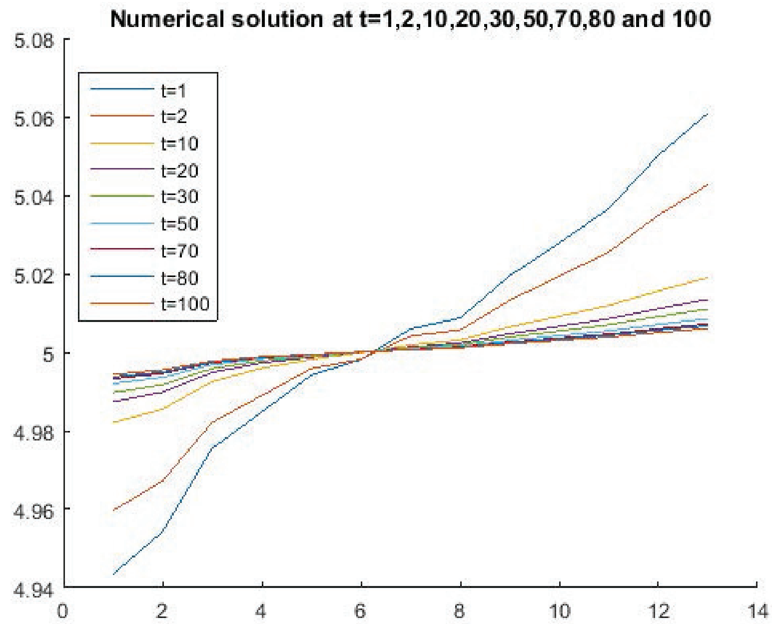


FIGURE 3. Approximate solution at time $t = 1, 2, 10, 20, 30, 50, 70, 80, 100$

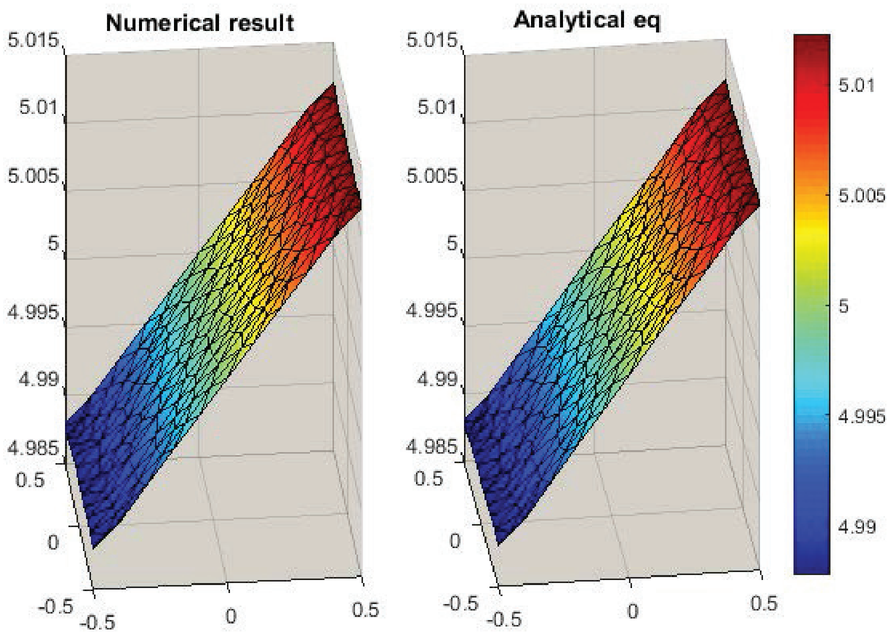


FIGURE 4. Comparison between the approximate and exact solution at time $t = 30$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY MOULAY ISMAIL, FACULTY OF SCIENCES
MEKNES, MOROCCO
*Email address*: mandariart@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY MOULAY ISMAIL, FACULTY OF SCIENCES
MEKNES, MOROCCO
*Email address*: rhoudafmohamed@gmail.com