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GRAPHS WITH A GIVEN GRAPHICAL PARAMETER

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ABSTRACT. In this paper the existence of a graph with a given covering number/neighbourhood number is established. A simple method (using very simple number theoretic concept) of constructing a Liouville's graph and a 1-graph with a given covering number/neighbourhood number is also given.

1. INTRODUCTION

The graphs considered here are finite, undirected, without loops or multiple edges. The notations and terminology used in this paper for Number Theory are the same as in Apostol [1] and for Graph theory are the same as in Harary [2]. Other references on this topic are [5] and [6].

Definition 1.1. *[1] The Liouville's function* λ *is defined as* $\lambda(1) = 1$ *and if* $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$*, where* p_i*'s are primes then* $\lambda(n) = (-1)^{a_1+a_2+\dots+a_k}$

Definition 1.2. *The Liouville's graph denoted by* L *is defined to be an arithmetic graph with its vertex set as* $V = \{1, 2, 3, \dots, m\}$*, the set of first m natural numbers where two distinct vertices* a, b *are adjacent if and only if* $\lambda(ab) = +1$.

The following result is an immediate consequence.

Theorem 1.1. *The Liouville's graph* L *is the union of two disjoint cliques with vertex sets* V_1 *and* V_2 *respectively, where* $V_1 = \{a \mid \lambda(a) = 1\}$ *and* $V_2 = \{b \mid \lambda(b) = -1\}$ *are the partitions of the vertex set* $V = \{1, 2, 3, \dots, m\}$ *of* L.

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Definition 1.3. *[2] A set of points which cover all the lines of a graph G is called a point cover for the graph G. The smallest number of points in any point cover for a graph G is called the point covering number (or simply the covering number) for the graph G and is denoted by $\alpha_0(G)$ or α_0.*

2. Main Results

We have the following result:

Theorem 2.1. *The Covering number of a Liouville's graph on m vertices $\{1, 2, 3, \ldots, m\}$ is $m - 2$ where $m \geq 4$.*

Proof. Let L be a Liouville's graph on m vertices $\{1, 2, 3, \ldots, m\}$. We can partition the vertex set $V = \{1, 2, 3, \ldots, m\}$ of L into two subsets V_1 and V_2 where $V_1 = \{a \mid \lambda(a) = 1\}$ and $V_2 = \{b \mid \lambda(b) = -1\}$ as in Theorem 1.1. Also we have, $L = L(V_1) \cup L(V_2)$ where $L(V_1)$ is the graph $G[V_1]$ and $L(V_2)$ is the graph $G[V_2]$. The Covering number of the required graph L is $\alpha_0(L) = \alpha_0[L(V_1)] + \alpha_0[L(V_2)] = m - 2$. □

Conversely, we have the following.

Theorem 2.2. *Given a positive integer m there exists a Liouville's graph for which m is the covering number.*

Proof. Take $n = m + 2$ where m is the given positive integer. Let L be a graph with vertex set as $V = \{1, 2, 3, \ldots, n\}$ where two distinct vertices a, b are adjacent if $\lambda(ab) = 1$. Let $V_1 = \{a \mid \lambda(a) = 1\}$ and $V_2 = \{b \mid \lambda(b) = -1\}$ be the partitions of V as in Theorem 1.1. Let $|V_1| = t_1,\ |V_2| = t_2$. Then $\alpha_0[L(V_1)] = t_1 - 1$ and $\alpha_0[L(V_2)] = t_2 - 1$. Therefore, $\alpha_0(L) = t_1 + t_2 - 2 = m$. Thus L is the graph for which m is the Covering number. □

Definition 2.1. *For a vertex $v \in V(G)$, the neighbourhood of a point is defined as $N(v) = \{u \in V(G) \mid uv \in V(G)\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of $V(G)$ is a neighbourhood set (written as n- set) of G if $G = \cup_{v \in V} \langle N[v] \rangle$ where $< N[v] >$ is the subgraph of G induced by $N[v]$. The neighbourhood number of G denoted by $n_0(G)$ is the minimum cardinality of an $n-$set of G.*

This number was extensively studied by Sampathkumar and Neeralagi [4]. Analogous to the arithmetic graph defined by Nathanson [3] we define a $1-$graph as follows.

Definition 2.2. *A $1-$graph is an arithmetic graph with its vertex set as the first m natural numbers and two vertices a, b are adjacent in this graph if $a+b \equiv 1(\mod 2)$.*

The following results are the direct consequences of definition.

Theorem 2.3. *A $1-$graph with vertex set $\{1, 2, 3, \ldots, m\}$ is a complete bipartite graph with partitions V_1 and V_2 of V where $V_1 = \{$ all even integers $\leq m\}$ and $V_2 = \{$ all odd integers $\leq m\}$.*

Theorem 2.4. *Let G be a $1-$graph with vertex set as the first m natural numbers then the neighbourhood number of G is $n_0(G) = \lceil \frac{m}{2} \rceil$ where $\lceil x \rceil$ denotes the smallest integer $\leq x$.*

Proof. Construct a $1-$graph G on m vertices $\{1, 2, 3, \ldots, m\}$. We can partition the vertex set $V(G)$ into two parts V_1 and V_2 where $V_1 = \{x \mid x \equiv 0(mod 2)\}$ and $V_2 = \{y \mid y \equiv 1(mod 2)\}$. The required $1-$graph G is a complete bipartite graph obtained by joining every vertex of V_1 to every vertex in V_2. To find the neighbourhood number of graph G, we consider the following cases.
Case1: Assume that m is even. Then $|V_1 = |V_2|$ and $V_1 = \{2, 4, 6, \ldots, m\}$. Then clearly $G = \bigcup_{v \in S} \langle N[v] \rangle$. Thus the set $V_1 = \{2, 4, 6, \ldots, m\}$ is an $n-$set of G. By the similar argument we can conclude that the set $V_2 = \{1, 3, 5, \ldots, m-1\}$ is also an $n-$set of the graph G. Since $|V_1 = |V_2|$ and there is no smaller neighbourhood set than these two $n-$sets, we conclude that the neighbourhood number is $\frac{m}{2}$. Therefore, $n_0(G) = \frac{m}{2} = \lceil \frac{m}{2} \rceil$ in this case.
Case2: Assume that m is odd. Let $V_1 = \{2, 4, 6, \ldots, m-1\}$ and $V_2 = \{1, 3, 5, \ldots, m\}$ be a partition of V. Then clearly $|V_1| < |V_2|$. We see that V_1, V_2 form $n-$sets by the same argument as in case-1. But we have $|V_1| < |V_2|$ and hence it can be easily seen that, the neighbourhood number of $G = |V_1| = \frac{(m-1)}{2}$. Thus in this case also $n_0(G) = \frac{(m-1)}{2} = \lceil \frac{m}{2} \rceil$. Hence the theorem follows. □

The following theorem establishes that given a positive integer m there exists a connected graph whose neighbourhood number is m.

Theorem 2.5. *Given a positive integer m there exists a $1-$graph for which m is a neighbourhood number.*

Proof. Let m be a given positive integer. Also let $n = 2m$ or $2m + 1$. By Theorem 1.5, the $1-$graph on n vertices will have the neighbourhood number as $\lceil \frac{n}{2} \rceil$ where $\lceil x \rceil$ denotes the smallest integer $\leq x$. Therefore, this $1-$graph on n vertices with $n = 2m$ or $2m + 1$ is the required graph with neighbourhood number m. $\square$

Illustrations: The graphs with a given covering number as 11 and a neighbourhood number as 7 are illustrated in figures 1 and 2 respectively.

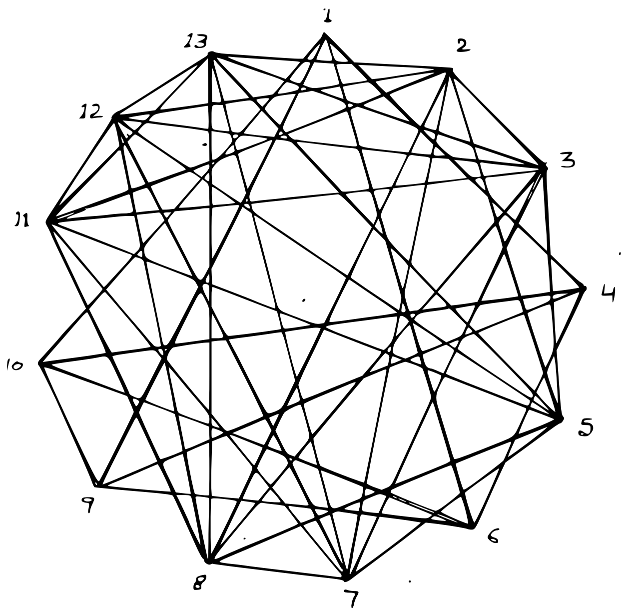

FIGURE 1. Graph with covering number 11

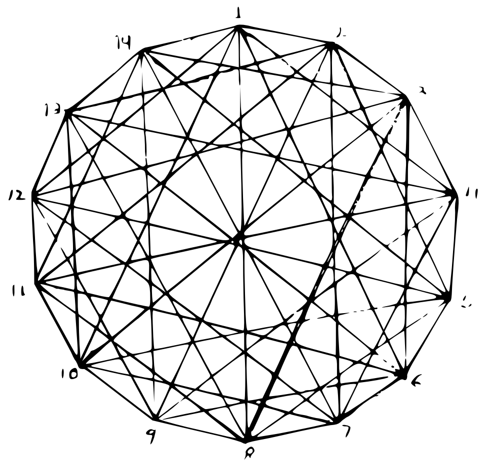

FIGURE 2. Graph with neighbourhood number 7

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