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# 17-DECOMPOSITION MATRICES FOR THE SPIN CHARACTERS OF SYMMETRIC GROUP $S_N$, $17 \leq N \leq 22$

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ABSTRACT. In this paper, we compute the Brauer trees of the symmetric group $S_n$, $17 \leq n \leq 22$, which can give the decomposition matrices of spin characters of $S_n$, $17 \leq n \leq 22$, modulo $p = 17$. The method $(r,\ r')$-inducing (restricting) is used.

## 1. INTRODUCTION

The representation group $\bar{S}_n$ of the symmetric group $S_n$ has order $2(n!)$ and it has a central subgroup $Z = \{-1,\ 1\}$ such that $\bar{S}_n/Z \approx S_n$, see [1]. Then, the irreducible representations or characters of $\bar{S}_n$ fall into two classes [1,2]:

(1) Those, which have $Z$ in their kernel, will be referred to as ordinary representations or characters. The irreducible representations and characters are indexed by partitions $\lambda$ of $n$ and the character is denoted by $[\lambda]$.
(2) The representations which do not have $Z$ in their kernel are called the spin representation of $S_n$. The irreducible spin representations are indexed by partitions of $n$ with distinct parts which are called bar partitions of $n$ and denoted by $\langle\lambda\rangle$, see [2,3].

In fact, if $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda \mapsto n$ and if $n - m$ is even, then there is one irreducible spin character denoted by $\langle\lambda\rangle$ which is self-associate, and if $n - m$ is

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odd, then there are two associate spin characters denoted by $\langle\lambda\rangle$ and $\langle\lambda\rangle'$. The decomposition matrix gives the relationship between the irreducible spin characters and projective indecomposable spin characters of $S_n$.

In this paper, we determined the decomposition matrices of spin characters of $S_n$, $17 \leq n \leq 22$, modulo $p = 17$. The method $(r,\ r')$-inducing (restricting) is used [3], to distribute the spin characters into $p$-blocks [4, 5]. The Brauer trees for spin characters of $S_n$, $13 \leq n \leq 20$ modulo $p = 13$ are found by Taban and Jawad [6], for $n = 21$ are found by Yaseen [7] and for $n = 22$ are found by Yaseen and Tahir [8].

## 2. Preliminaries

The following results are very useful to find the modular characters:

(1) The degree of spin characters $\langle\lambda\rangle$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is [1, 9]:

$$deg\langle\lambda\rangle = 2^{\left[\frac{(n-m)}{2}\right]} \frac{n!}{\prod_{i=1}^{m} \lambda!} \prod_{1\leq i<j\leq m} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

(2) Every spin (modular, projective) character of $S_n$ can be written as a linear combination with non-negative integer coefficients of the irreducible spin (irreducible modular, projective indecomposable)characters respectively [10].

(3) Let $H$ be a subgroup of the group $G$ [11], then:

   a) If $\varphi$ is a modular (principal) character of a subgroup $H$ of $G$, then $\varphi \uparrow G$ is a modular (principal) character of $G$, (where $\uparrow$ denotes inducing).

   b) If $\psi$ is a modular (principal) character of group $G$, then $\psi \downarrow H$ is a modular (principal) character of a subgroup $H$, (where $\downarrow$ denotes the restricting).

(4) Let $B$ be a $p$-block $G$ of defect one and let $b$ be the number of $p$-conjugate characters to the irreducible ordinary character $\chi$ of $G$ [12], then:

   a) There exists a positive integer number $N$ such that the irreducible ordinary characters lying in the block $B$ can be partitioned into two disjoint classes: $B_1 = \{x \in B \mid b\ deg\ x \equiv N\ mod\ p^a\}$, $B_2 = \{x \in B \mid b\ deg\ x \equiv -N\ mod\ p^a\}$.

b) Each coefficient of the decomposition matrix of the block $B$ is 1 or 0.

c) If $\alpha_1$ and $\alpha_2$ are not $p$-conjugate characters and belong to the same partition class $B_1$ or $B_2$ above, then they have no irreducible modular character in common.

d) For every irreducible ordinary character $\chi$ in $B_1$, there exists irreducible ordinary character $\varphi$ in $B_2$ such that they have one irreducible modular character in common with multiplicity one.

(5) Let $\lambda$ and $\mu$ be bar partitions such that $\lambda \neq \mu$, then $\langle\lambda\rangle$ and $\langle\mu\rangle$ are in the same $p$-block if and only if $\lambda(\bar{p}) = \mu(\bar{p})$, where $p$ is an odd prime. The associative irreducible spin characters $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ are in the same $p$-block if $\lambda(\bar{p}) \neq \lambda$, see [3].

(6) Let $G$ be a group of order $m = m_0 p^a$, where $(p, m_0) = 0$. If $c$ is a principal character of $H$, then $deg\ c \equiv 0\ mod\ p^a$, see [13].

(7) If $c$ is a principal character of $G$ for an odd prime $p$ and all entries in $c$ are divisible by positive integer $q$, then $c/q$ is a principal character of $G$, see [11].

(8) Let $p$ be odd and $n$ be even, then from [14]:

a If $p \nmid n$, then $\langle n\rangle = \varphi\langle n\rangle$ and $\langle n\rangle' = \varphi\langle n\rangle'$ are distinct irreducible modular spin characters of degree $2^{(n-2)/2}$.

b If $p \nmid n$ and $p \nmid (n-1)$, then $\langle n-1, 1\rangle = \varphi\langle n-1, 1\rangle^*$ is an irreducible modular spin characters of degree $2^{(n-2)/2}(n-2)$.

(9) Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ be a bar partition of $n$, not a $p$-bar core, and $B$ be the block containing $\langle\alpha\rangle$, then:

a If $n - m - m_0$ is even, then all irreducible modular spin characters in $B$ are double.

b If $n - m - m_0$ is odd, then all irreducible modular spin characters in $B$ are associate, where $m_0$ is the number of parts of $\alpha$ divisible by $p$ [13]. For more details, see [16–18].

We shall use the following notations next: Irreducible modular spin characters (i.m.s), modular spin characters (m.s), principal indecomposable spin character (p.i.s), and principal spin character (p.s).

## 3. Decomposition Matrices for the Spin Characters of the symmetric groups $S_n$, $17 \leq n \leq 22$ for the prime $p = 17$

In the following sections, we calculate the decomposition matrices of the spin characters of the symmetric group $S_n$, when $17 \leq n \leq 22$ for the prime $p = 17$. In each we find the irreducible spin characters and $(17,\ \alpha)$–regular classes of $S_n$, $17 \leq n \leq 22$ when $p = 17$. All blocks in these sections are 17-blocks.

3.1. **Decomposition Matrix for the Spin Characters of $S_{17}$.** The symmetric group $S_{17}$ has 57 irreducible spin characters and $S_{17}$ has 56 $(17,\ \alpha)$-regular classes, then the decomposition matrix of the spin characters for $S_{17}$, $p = 17$ has 57 rows and 56 columns. There are forty-one 17-block, (Preliminary 5). The principal block $B_1$ (the block which contains the spin character $\langle n\rangle$ or $\langle n\rangle'$), where $B_1$ of defect one contains the characters $\langle 17\rangle^*$, $\langle 16,\ 1\rangle$, $\langle 16,\ 1\rangle'$, $\langle 15,\ 2\rangle$, $\langle 15,\ 2\rangle'$, $\langle 14,\ 3\rangle$, $\langle 14,\ 3\rangle'$, $\langle 13,\ 4\rangle$, $\langle 13,\ 4\rangle'$, $\langle 12,\ 5\rangle$, $\langle 12,\ 5\rangle'$, $\langle 11,\ 6\rangle$, $\langle 11,\ 6\rangle'$, $\langle 10,\ 7\rangle$, $\langle 10,\ 7\rangle'$, $\langle 9,\ 8\rangle$, $\langle 9,\ 8\rangle'$ with 17-bar core $\varphi$. All the 40 remaining characters $B_2, B_3, \cdots, B_{41}$ form their own blocks of defect 0, see [10], which are irreducible modular spin characters.

3.2. **The Brauer tree for the spin characters of $S_p$.** The spin characters of $S_p$ are $\{\langle P-r,\ r\rangle | r = 0, 1, 2, \cdots, (p-1)/2\}$ belong to the same $p$-block since they have empty $\bar{p}$-core. These characters of defect 1 since $p$ does not divide their degrees. Then, we apply the following theorem to find the Brauer tree for spin characters of $S_p$ to determine the decomposition matrix for $S_p$.

**Theorem 3.1.** *The Brauer tree for $S_p$ is:*
$\langle (p+1)/2,\ (p-1)/2\rangle' \_ \ldots \_ \langle p-1,\ 1\rangle' \_ \langle p^*\rangle \_ \langle p-1,\ 1\rangle \_ \ldots \_ \langle (p+1)/2,\ (p-1)/2\rangle$.

*Proof.* See [3]. □

**Proposition 3.1.** *By using above Theorem 3.1, the Brauer tree for the principal block $B_1$ is:*

$$\begin{array}{ll} & \langle 16,1\rangle \_ \langle 15,2\rangle \_ \langle 14,3\rangle \_ \langle 13,4\rangle \_ \langle 12,5\rangle \_ \langle 11,6\rangle \_ \langle 10,7\rangle \_ \langle 9,8\rangle \\ & \diagup \\ \langle 17\rangle^* & \\ & \diagdown \\ & \langle 16,1\rangle' \_ \langle 15,2\rangle' \_ \langle 14,3\rangle' \_ \langle 13,4\rangle' \_ \langle 12,5\rangle' \_ \langle 11,6\rangle' \_ \langle 10,7\rangle' \_ \langle 9,8\rangle' \end{array}$$

*Hence the decomposition matrix for this block $D^{(1)}_{17,17}$ in Table 1.*

TABLE 1. $D^{(1)}_{17,17}$

| *The spin characters* | *The decomposition matrix for the block* $B_1$ | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 17\rangle^*$ | *1* | *1* | | | | | | | | | | | | | | |
| $\langle 16,1\rangle$ | *1* | | *1* | | | | | | | | | | | | | |
| $\langle 16,1\rangle'$ | | *1* | | *1* | | | | | | | | | | | | |
| $\langle 15,2\rangle$ | | | *1* | | *1* | | | | | | | | | | | |
| $\langle 15,2\rangle'$ | | | | *1* | | *1* | | | | | | | | | | |
| $\langle 14,3\rangle$ | | | | | *1* | | *1* | | | | | | | | | |
| $\langle 14,3\rangle'$ | | | | | | *1* | | *1* | | | | | | | | |
| $\langle 13,4\rangle$ | | | | | | | *1* | | *1* | | | | | | | |
| $\langle 13,4\rangle'$ | | | | | | | | *1* | | *1* | | | | | | |
| $\langle 12,5\rangle$ | | | | | | | | | *1* | | *1* | | | | | |
| $\langle 12,5\rangle'$ | | | | | | | | | | *1* | | *1* | | | | |
| $\langle 11,6\rangle$ | | | | | | | | | | | *1* | | *1* | | | |
| $\langle 11,6\rangle'$ | | | | | | | | | | | | *1* | | *1* | | |
| $\langle 10,7\rangle$ | | | | | | | | | | | | | *1* | | *1* | |
| $\langle 10,7\rangle'$ | | | | | | | | | | | | | | *1* | | *1* |
| $\langle 9,8\rangle$ | | | | | | | | | | | | | | | *1* | |
| $\langle 9,8\rangle'$ | | | | | | | | | | | | | | | | *1* |
| | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ | $d_7$ | $d_8$ | $d_9$ | $d_{10}$ | $d_{11}$ | $d_{12}$ | $d_{13}$ | $d_{14}$ | $d_{15}$ | $d_{16}$ |

3.3. **Decomposition Matrix for the Spin Characters of $S_{18}$.** The group $S_{18}$ has $69$ irreducible spin characters and $68$ of $(17,\alpha)$-regular classes, then the decomposition matrix for the spin characters of $S_{18}$, $p = 17$ has $69$ rows and $68$ columns. By using (Preliminary $5$), there are $51$ blocks of $S_{18}$. In the spin block $B_1$ of defect $1$, all i.m.s. of the decomposition matrix are associate (Preliminary $9$) and $\langle\beta\rangle \neq \langle\beta\rangle'$. The block $B_1$ contains the characters $\langle 18\rangle$, $\langle 18\rangle'$, $\langle 17,1\rangle^*$, $\langle 15,2,1\rangle$, $\langle 15,2,1\rangle'$, $\langle 14,3,1\rangle$, $\langle 14,3,1\rangle'$, $\langle 13,4,1\rangle$, $\langle 13,4,1\rangle'$, $\langle 12,5,1\rangle$, $\langle 12,5,1\rangle'$, $\langle 11,6,1\rangle$, $\langle 11,6,1\rangle'$, $\langle 10,7,1\rangle$, $\langle 10,7,1\rangle'$, $\langle 9,8,1\rangle$, $\langle 9,8,1\rangle'$ with $17$-bar core $\langle 1\rangle$. The other Blocks $B_2, B_3, \cdots, B_{51}$ of defect zero.

**Proposition 3.2.** *The Brauer tree for $B_1$ is:*

$\langle 18\rangle$ $\langle 15,2,1\rangle\_\langle 14,3,1\rangle\_\langle 13,4,1\rangle\_\langle 12,5,1\rangle\_\langle 11,6,1\rangle\_\langle 10,7,1\rangle\_\langle 9,8,1\rangle$

$\diagdown$ $\diagup$

$\langle 17,1\rangle^*$

$\diagup$ $\diagdown$

$\langle 18\rangle'$ $\langle 15,2,1\rangle'\_\langle 14,3,1\rangle'\_\langle 13,4,1\rangle'\_\langle 12,5,1\rangle'\_\langle 11,6,1\rangle'\_\langle 10,7,1\rangle'\_\langle 9,8,1\rangle'$

*Proof.*

(a) $deg\ \{\langle 18\rangle, \langle 18\rangle', \langle 15,2,1\rangle, \langle 15,2,1\rangle', \langle 13,4,1\rangle, \langle 13,4,1\rangle'\} \equiv 1\ mod\ 17$; $deg\ \{\langle 17,1\rangle^*, \langle 14,3,1\rangle, \langle 14,3,1\rangle', \langle 12,5,1\rangle, \langle 12,5,1\rangle', \langle 10,7,1\rangle, \langle 10,7,1\rangle'\} \equiv -1\ mod\ 17$ (Preliminary 4).

(b) By using $(r, r')$-inducing of p.i.s. of $S_{17}$ (see $D^{(1)}_{17,17}$) to $S_{18}$, we have:

$$d_1 \uparrow^{(1,0)} S_{18} = \langle 18\rangle + \langle 18\rangle' + 2\langle 17,1\rangle^* = K = D_1 + D_2$$
$$d_3 \uparrow^{(1,0)} S_{18} = \langle 17\rangle^* + \langle 15,2,1\rangle = D_3$$
$$d_4 \uparrow^{(1,0)} S_{18} = \langle 17\rangle^* + \langle 15,2,1\rangle' = D_4$$
$$d_5 \uparrow^{(1,0)} S_{18} = \langle 15,2,1\rangle + \langle 14,3,1\rangle = D_5$$
$$d_6 \uparrow^{(1,0)} S_{18} = \langle 15,2,1\rangle' + \langle 14,3,1\rangle' = D_6$$
$$d_7 \uparrow^{(1,0)} S_{18} = \langle 14,3,1\rangle + \langle 13,4,1\rangle = D_7$$
$$d_8 \uparrow^{(1,0)} S_{18} = \langle 14,3,1\rangle' + \langle 13,4,1\rangle' = D_8$$
$$d_9 \uparrow^{(1,0)} S_{18} = \langle 13,4,1\rangle + \langle 12,5,1\rangle = D_9$$
$$d_{10} \uparrow^{(1,0)} S_{18} = \langle 13,4,1\rangle' + \langle 12,5,1\rangle' = D_{10}$$
$$d_{11} \uparrow^{(1,0)} S_{18} = \langle 12,5,1\rangle + \langle 11,6,1\rangle = D_{11}$$
$$d_{12} \uparrow^{(1,0)} S_{18} = \langle 12,5,1\rangle' + \langle 11,6,1\rangle' = D_{12}$$
$$d_{13} \uparrow^{(1,0)} S_{18} = \langle 11,6,1\rangle + \langle 10,7,1\rangle = D_{13}$$
$$d_{14} \uparrow^{(1,0)} S_{18} = \langle 11,6,1\rangle' + \langle 10,7,1\rangle' = D_{14}$$
$$d_{15} \uparrow^{(1,0)} S_{18} = \langle 10,7,1\rangle + \langle 9,8,1\rangle = D_{15}$$
$$d_{16} \uparrow^{(1,0)} S_{18} = \langle 10,7,1\rangle' + \langle 9,8,1\rangle' = D_{16}$$

$\langle 18,1\rangle \downarrow_{(1,0)} S_{18} = D_1$ since $\langle 18,1\rangle$ i.m. in $S_{19}$, and $\langle 18,1\rangle' \downarrow_{(1,0)} S_{18} = D_2$ since $\langle 18,1\rangle'$ i.m. in $S_{19}$. So we have the Brauer tree for $B_1$ and the decomposition matrix for this block $D^{(2)}_{18,17}$ in Table 2. □

TABLE 2. $D^{(2)}_{18,17}$

| The spin characters | The decomposition matrix for the block $B_1$ | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 18\rangle$ | 1 | | | | | | | | | | | | | | | |
| $\langle 18\rangle'$ | | 1 | | | | | | | | | | | | | | |
| $\langle 17,1\rangle^*$ | 1 | 1 | 1 | 1 | | | | | | | | | | | | |
| $\langle 15,2,1\rangle$ | | | 1 | | 1 | | | | | | | | | | | |
| $\langle 15,2,1\rangle'$ | | | | 1 | | 1 | | | | | | | | | | |
| $\langle 14,3,1\rangle$ | | | | | 1 | | 1 | | | | | | | | | |
| $\langle 14,3,1\rangle'$ | | | | | | 1 | | 1 | | | | | | | | |
| $\langle 13,4,1\rangle$ | | | | | | | 1 | | 1 | | | | | | | |
| $\langle 13,4,1\rangle'$ | | | | | | | | 1 | | 1 | | | | | | |
| $\langle 12,5,1\rangle$ | | | | | | | | | 1 | | 1 | | | | | |
| $\langle 12,5,1\rangle'$ | | | | | | | | | | 1 | | 1 | | | | |
| $\langle 11,6,1\rangle$ | | | | | | | | | | | 1 | | 1 | | | |
| $\langle 11,6,1\rangle'$ | | | | | | | | | | | | 1 | | 1 | | |
| $\langle 10,7,1\rangle$ | | | | | | | | | | | | | 1 | | 1 | |
| $\langle 10,7,1\rangle'$ | | | | | | | | | | | | | | 1 | | 1 |
| $\langle 9,8,1\rangle$ | | | | | | | | | | | | | | | 1 | |
| $\langle 9,8,1\rangle'$ | | | | | | | | | | | | | | | | 1 |
| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $D_8$ | $D_9$ | $D_{10}$ | $D_{11}$ | $D_{12}$ | $D_{13}$ | $D_{14}$ | $D_{15}$ | $D_{16}$ |

3.4. **Decomposition Matrix for the Spin Characters of $S_{19}$.** The group $S_{19}$ has $74$ irreducible spin characters and $72$ of $(17,\alpha)$-regular classes, then the decomposition matrix for the spin characters of $S_{19}$, $p = 17$ has $74$ rows and $72$ columns. By using (Preliminary $5$), there are $65$ blocks of $S_{19}$. In the principal block $B_1$ of defect $1$, all i.m.s. of the decomposition matrix are double (Preliminary $9$) and $\langle\beta\rangle \neq \langle\beta\rangle'$. The block $B_1$ contains the characters $\langle 19\rangle^*$, $\langle 17,2\rangle$, $\langle 17,2\rangle'$, $\langle 16,2,1\rangle^*$, $\langle 14,3,2\rangle^*$, $\langle 13,4,2\rangle^*$, $\langle 12,5,2\rangle^*$, $\langle 11,6,2\rangle^*$, $\langle 10,7,2\rangle^*$, $\langle 9,8,2\rangle^*$ with 17-bar core $\langle 2\rangle$. The other Blocks $B_2, B_3, \cdots, B_{65}$ of defect zero.

**Proposition 3.3.** *The Brauer tree for* $B_1$ *is:* $\langle 19^*\rangle\_\langle 17,2\rangle = \langle 17,2\rangle'\_\langle 16,2,1\rangle^*\_\langle 14,3,2\rangle^*\_\langle 13,4,2\rangle^*\_\langle 12,5,2\rangle^*\_\langle 11,6,2\rangle^*\_\langle 10,7,2\rangle^*\_\langle 9,8,2\rangle^*$

*Proof.* $deg \{\langle 19^*\rangle, \langle 16,2,1\rangle^*, \langle 13,4,2\rangle^*, \langle 11,6,2\rangle^*, \langle 9,8,2\rangle^*\} \equiv 2 \ mod \ 17$; $deg \{\langle 17,2\rangle, \langle 17,2\rangle', \langle 14,3,2\rangle^*, \langle 12,5,2\rangle^*, \langle 10,7,2\rangle^*\} \equiv -2 \ mod \ 17$ (Preliminary 4). By using $(r, r')$-inducing of p.i.s. of $S_{18}$ (see $D_{18,17}$) to $S_{19}$, we have:

$$D_1 \uparrow^{(2,16)} S_{19} = \langle 19^*\rangle + \langle 17,2\rangle + \langle 17,2\rangle' = e_1$$
$$D_2 \uparrow^{(2,16)} S_{19} = \langle 17,2\rangle + \langle 17,2\rangle' + \langle 16,2,1\rangle^* = e_2$$
$$D_3 \uparrow^{(2,16)} S_{19} = \langle 16,2,1\rangle^* + \langle 14,3,2\rangle^* = e_3$$
$$D_4 \uparrow^{(2,16)} S_{19} = \langle 14,3,2\rangle^* + \langle 13,4,2\rangle^* = e_4$$
$$D_5 \uparrow^{(2,16)} S_{19} = \langle 13,4,2\rangle^* + \langle 12,5,2\rangle^* = e_5$$
$$D_6 \uparrow^{(2,16)} S_{19} = \langle 12,5,2\rangle^* + \langle 11,6,2\rangle^* = e_6$$
$$D_7 \uparrow^{(2,16)} S_{19} = \langle 11,6,2\rangle^* + \langle 10,7,2\rangle^* = e_7$$
$$D_8 \uparrow^{(2,16)} S_{19} = \langle 10,7,2\rangle^* + \langle 9,8,2\rangle^* = e_8$$

So we have the Brauer tree for $B_1$ and the decomposition matrix for this block $D^{(3)}_{19,17}$ in Table 3. □

TABLE 3. $D^{(3)}_{19,17}$

| The spin characters | The decomposition matrix for the block $B_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\langle 19^*\rangle$ | 1 | | | | | | | |
| $\langle 17,2\rangle$ | 1 | 1 | | | | | | |
| $\langle 17,2\rangle'$ | 1 | 1 | | | | | | |
| $\langle 16,2,1\rangle^*$ | | 1 | 1 | | | | | |
| $\langle 14,3,2\rangle^*$ | | | 1 | 1 | | | | |
| $\langle 13,4,2\rangle^*$ | | | | 1 | 1 | | | |
| $\langle 12,5,2\rangle^*$ | | | | | 1 | 1 | | |
| $\langle 11,6,2\rangle^*$ | | | | | | 1 | 1 | |
| $\langle 10,7,2\rangle^*$ | | | | | | | 1 | 1 |
| $\langle 9,8,2\rangle^*$ | | | | | | | | 1 |
| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ |

3.5. **Decomposition Matrix for the Spin Characters of $S_{20}$.** The group $S_{20}$ has $81$ irreducible spin characters and $78$ of $(17, \alpha)$-regular classes, then the decomposition matrix for the spin characters of $S_{20}$, $p = 17$ has $81$ rows and $78$ columns. By using (Preliminary $5$), there are $69$ blocks of $S_{20}$ two of them $B_1$, $B_2$ of defect $1$. All the $67$ remaining characters form their own blocks $B_3, B_4, \cdots, B_{69}$ of defect zero [11].

In the principal block $B_1$, all i.m.s. of the decomposition matrix are associate (Preliminary $9$) and $\langle\beta\rangle \neq \langle\beta\rangle'$. The block $B_1$ contains the characters $\langle 20\rangle$, $\langle 20\rangle'$, $\langle 17,3\rangle^*$, $\langle 16,3,1\rangle$, $\langle 16,3,1\rangle'$, $\langle 15,3,2\rangle$, $\langle 15,3,2\rangle'$, $\langle 13,4,3\rangle$, $\langle 13,4,3\rangle'$, $\langle 12,5,3\rangle$, $\langle 12,5,3\rangle'$, $\langle 11,6,3\rangle$, $\langle 11,6,3\rangle'$, $\langle 10,7,3\rangle$, $\langle 10,7,3\rangle'$, $\langle 9,8,3\rangle$, $\langle 9,8,3\rangle'$ with 17-bar core $\langle 3\rangle$.

In the spin block $B_2$, all i.m.s. of the decomposition matrix are double (Preliminary $9$) and $\langle\beta\rangle \neq \langle\beta\rangle'$. The block $B_2$ contains the irreducible spin characters $\langle 19,1\rangle^*$, $\langle 18,2\rangle^*$, $\langle 17,2,1\rangle$, $\langle 17,2,1\rangle'$, $\langle 14,3,2,1\rangle^*$, $\langle 13,4,2,1\rangle^*$, $\langle 12,5,2,1\rangle^*$, $\langle 10,7,2,1\rangle^*$ has 17-bar core $\langle 2,1\rangle$.

**Proposition 3.4.** *The Brauer tree for principal block $B_1$ is:*

$\langle 20\rangle$ $\qquad\qquad$ $\langle 16,3,1\rangle\_\langle 15,3,2\rangle\_\langle 13,4,3\rangle\_\langle 12,5,3\rangle\_\langle 11,6,3\rangle\_\langle 10,7,3\rangle\_\langle 9,8,3\rangle$

$\qquad\diagdown \qquad \diagup$

$\qquad\quad \langle 17,3\rangle^*$

$\qquad\diagup \qquad \diagdown$

$\langle 20\rangle'$ $\qquad\qquad$ $\langle 16,3,1\rangle'\_\langle 15,3,2\rangle'\_\langle 13,4,3\rangle'\_\langle 12,5,3\rangle'\_\langle 11,6,3\rangle'\_\langle 10,7,3\rangle'\_\langle 9,8,3\rangle'$

*Proof.* $deg\ \{\langle 20\rangle, \langle 20\rangle', \langle 16,3,1\rangle, \langle 16,3,1\rangle', \langle 13,4,3\rangle, \langle 13,4,3\rangle', \langle 11,6,3\rangle, \langle 11,6,3\rangle', \langle 9,8,3\rangle, \langle 9,8,3\rangle'\} \equiv 3\ mod\ 17$;
$deg\ \{\langle 17,3\rangle^*, \langle 15,3,2\rangle, \langle 15,3,2\rangle', \langle 12,5,3\rangle, \langle 12,5,3\rangle', \langle 10,7,3\rangle, \langle 10,7,3\rangle'\} \equiv - 3\ mod\ 17$ (Preliminary $4$).
By using $(3,15)$-inducing of p.i.s. of $S_{19}$ (see $D^{(3)}_{19,17}$) to $S_{20}$, we have:

$$e_1 \uparrow^{(3,15)} S_{20} = \langle 20\rangle + \langle 20\rangle' + 2\langle 17,3\rangle^* = K_1 = E_1 + E_2$$
$$e_2 \uparrow^{(3,15)} S_{20} = 2\langle 17,3\rangle^* + \langle 16,3,1\rangle + \langle 16,3,1\rangle' = K_2 = E_3 + E_4$$
$$e_3 \uparrow^{(3,15)} S_{20} = \langle 16,3,1\rangle + \langle 16,3,1\rangle' + \langle 15,3,2\rangle + \langle 15,3,2\rangle' = K_3 = E_5 + E_6$$
$$e_4 \uparrow^{(3,15)} S_{20} = \langle 15,3,2\rangle + \langle 15,3,2\rangle' + \langle 13,4,3\rangle + \langle 13,4,3\rangle' = K_4 = E_7 + E_8$$
$$e_5 \uparrow^{(3,15)} S_{20} = \langle 13,4,3\rangle + \langle 13,4,3\rangle' + \langle 12,5,3\rangle + \langle 12,5,3\rangle' = K_5 = E_9 + E_{10}$$

$$e_6 \uparrow^{(3,15)} S_{20} = \langle 12,5,3\rangle + \langle 12,5,3\rangle' + \langle 11,6,3\rangle + \langle 11,6,3\rangle' = K_6 = E_{11} + E_{12}$$
$$e_7 \uparrow^{(3,15)} S_{20} = \langle 11,6,3\rangle + \langle 11,6,3\rangle' + \langle 10,7,3\rangle + \langle 10,7,3\rangle' = K_7 = E_{13} + E_{14}$$
$$e_8 \uparrow^{(3,15)} S_{20} = \langle 10,7,3\rangle + \langle 10,7,3\rangle' + \langle 9,8,3\rangle + \langle 9,8,3\rangle' = K_8 = E_{15} + E_{16}$$

Since $\langle 20\rangle \neq \langle 20\rangle'$ are distinct irreducible modular spin characters Property (6) and $\langle 17,3\rangle^*$ contains $\langle 20\rangle$, $\langle 20\rangle'$ with the same multiplicity [12] then $K_1$ split to $E_1$, $E_2$.

$\langle 16,3\rangle \uparrow^{(1,0)} S_{20} = \langle 17,3\rangle^* + \langle 16,3,1\rangle = E_3$ since $\langle 16,3\rangle$ i.m. in $S_{19}$ and $\langle 16,3\rangle' \uparrow^{(1,0)} S_{20} = \langle 17,3\rangle^* + \langle 16,3,1\rangle' = E_4$ since $\langle 16,3\rangle'$ i.m. in $S_{19}$, also we have $\langle 15,3,2\rangle \neq \langle 15,3,2\rangle'$, $\langle 13,4,3\rangle \neq \langle 13,4,3\rangle'$, $\langle 12,5,3\rangle \neq \langle 12,5,3\rangle'$, $\langle 11,6,3\rangle \neq \langle 11,6,3\rangle'$, $\langle 10,7,3\rangle \neq \langle 10,7,3\rangle'$, $\langle 9,8,3\rangle \neq \langle 9,8,3\rangle'$ on $(17,\alpha)$-regular classes, then $K_3$, $K_4$, $K_5$, $K_6$, $K_7$, $K_8$ are split, respectively. So, we have the Brauer tree for $B_1$ and the decomposition matrix for this block $D^{(4)}_{20,17}$ in Table 4. □

**Proposition 3.5.** *The Brauer tree for the block $B_2$ is:*
$\langle 19,1\rangle^*\_\langle 18,2\rangle^*\_\langle 17,2,1\rangle = \langle 17,2,1\rangle'\_\langle 16,2,1\rangle^*\_\langle 14,3,2,1\rangle^*\_\langle 13,4,2,1\rangle^*\_$
$\langle 12,5,2,1\rangle^*\_\langle 11,6,2,1\rangle^*\_\langle 10,7,2,1\rangle^*\_\langle 9,8,2,1\rangle^*$

*Proof.* $deg\ \{\langle 19,1\rangle^*, (\langle 17,2,1\rangle + \langle 17,2,1\rangle')\langle 13,4,2,1\rangle^*, \langle 11,6,2,1\rangle^*, \langle 9,8,2,1\rangle^*\} \equiv 2\ mod\ 17$; $deg\ \{\langle 18,2\rangle^*, \langle 14,3,2,1\rangle^*, \langle 12,5,2,1\rangle^*, \langle 10,7,2,1\rangle^*\} \equiv -2\ mod\ 17$ (Preliminary 4). By using $(1,0)$-inducing of p.i.s. of $S_{19}$ (see $D^{(3)}_{19,17}$) to $S_{20}$, we have:

$$e_1 \uparrow^{(1,0)} S_{20} = \langle 19,1\rangle^* + 2\langle 18,2\rangle^* + \langle 17,2,1\rangle + \langle 17,2,1\rangle' = K_1$$
$$e_2 \uparrow^{(1,0)} S_{20} = 2\langle 18,2\rangle^* + 2\langle 17,2,1\rangle + 2\langle 17,2,1\rangle' = 2F_2$$
$$e_3 \uparrow^{(1,0)} S_{20} = \langle 17,2,1\rangle + \langle 17,2,1\rangle' + \langle 14,3,2,1\rangle^* = F_3$$
$$e_4 \uparrow^{(1,0)} S_{20} = \langle 14,3,2,1\rangle^* + \langle 13,4,2,1\rangle^* = F_4$$
$$e_5 \uparrow^{(1,0)} S_{20} = \langle 13,4,2,1\rangle^* + \langle 12,5,2,1\rangle^* = F_5$$
$$e_6 \uparrow^{(1,0)} S_{20} = \langle 12,5,2,1\rangle^* + \langle 11,6,2,1\rangle^* = F_6$$
$$e_7 \uparrow^{(1,0)} S_{20} = \langle 11,6,2,1\rangle^* + \langle 10,7,2,1\rangle^* = F_7$$
$$e_8 \uparrow^{(1,0)} S_{20} = \langle 10,7,2,1\rangle^* + \langle 9,8,2,1\rangle^* = F_8.$$

Since $\langle 19,2\rangle \downarrow_{(1,0)} S_{20} = \langle 19,1\rangle^* + \langle 18,2\rangle^* = K_1 - F_2 = F_1$ since $\langle 19,2\rangle$ i.m. in $S_{21}$. So, we have the Brauer tree for $B_2$ and the decomposition matrix for this block $D^{(5)}_{20,17}$ in Table 5. □

TABLE 4. $D^{(4)}_{20,17}$

| The spin characters | The decomposition matrix for the block $B_1$ | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 20\rangle$ | 1 | | | | | | | | | | | | | | | |
| $\langle 20\rangle'$ | | 1 | | | | | | | | | | | | | | |
| $\langle 17,3\rangle^*$ | 1 | 1 | 1 | 1 | | | | | | | | | | | | |
| $\langle 16,3,1\rangle$ | | | 1 | | 1 | | | | | | | | | | | |
| $\langle 16,3,1\rangle'$ | | | | 1 | | 1 | | | | | | | | | | |
| $\langle 15,3,2\rangle$ | | | | | 1 | | 1 | | | | | | | | | |
| $\langle 15,3,2\rangle'$ | | | | | | 1 | | 1 | | | | | | | | |
| $\langle 13,4,2\rangle$ | | | | | | | 1 | | 1 | | | | | | | |
| $\langle 13,4,2\rangle'$ | | | | | | | | 1 | | 1 | | | | | | |
| $\langle 12,5,3\rangle$ | | | | | | | | | 1 | | 1 | | | | | |
| $\langle 12,5,3\rangle'$ | | | | | | | | | | 1 | | 1 | | | | |
| $\langle 11,6,3\rangle$ | | | | | | | | | | | 1 | | 1 | | | |
| $\langle 11,6,3\rangle'$ | | | | | | | | | | | | 1 | | 1 | | |
| $\langle 10,7,3\rangle$ | | | | | | | | | | | | | 1 | | 1 | |
| $\langle 10,7,3\rangle'$ | | | | | | | | | | | | | | 1 | | 1 |
| $\langle 9,8,3\rangle$ | | | | | | | | | | | | | | | 1 | |
| $\langle 9,8,3\rangle'$ | | | | | | | | | | | | | | | | 1 |
| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ | $E_8$ | $E_9$ | $E_{10}$ | $E_{11}$ | $E_{12}$ | $E_{13}$ | $E_{14}$ | $E_{15}$ | $E_{16}$ |

3.6. **Decomposition Matrix for the Spin Characters of $S_{21}$.** The group $S_{21}$ has $114$ irreducible spin characters and $111$ of $(17,\alpha)$-regular classes, then the decomposition matrix of the spin characters for $S_{21}$, $p = 17$ has $114$ rows and $111$ columns. By using (Preliminary $5$), there are $89$ blocks of $S_{20}$ two of them $B_1$, $B_2$ of defect $1$. All the $87$ remaining characters form their own blocks $B_3, B_4, \cdots, B_{89}$ of defect zero [11].

In the principal block $B_1$, all i.m.s. of the decomposition matrix are double (Preliminary $9$) and $\langle\beta\rangle \neq \langle\beta\rangle'$. The block $B_1$ contains the irreducible spin characters $\langle 21\rangle^*$, $\langle 17,4\rangle$, $\langle 17,4\rangle'$, $\langle 16,4,1\rangle^*$, $\langle 15,4,2\rangle^*$, $\langle 14,4,3\rangle^*$, $\langle 12,5,4\rangle^*$, $\langle 11,6,4\rangle^*$, $\langle 10,7,4\rangle^*$, $\langle 9,8,4\rangle^*$ has 17-bar core $\langle 4\rangle$.

TABLE 5. $D^{(5)}_{20,17}$

| The spin characters | The decomposition matrix for the block $B_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\langle 19^*, 1\rangle$ | 1 | | | | | | | |
| $\langle 18^*, 2\rangle$ | 1 | 1 | | | | | | |
| $\langle 17, 2, 1\rangle$ | | 1 | 1 | | | | | |
| $\langle 17, 2, 1\rangle'$ | | 1 | 1 | | | | | |
| $\langle 14, 3, 2, 1\rangle^*$ | | | 1 | 1 | | | | |
| $\langle 13, 4, 2, 1\rangle^*$ | | | | 1 | 1 | | | |
| $\langle 12, 5, 2, 1\rangle^*$ | | | | | 1 | 1 | | |
| $\langle 11, 6, 2, 1\rangle^*$ | | | | | | 1 | 1 | |
| $\langle 10, 7, 2, 1\rangle^*$ | | | | | | | 1 | 1 |
| $\langle 9, 8, 2, 1\rangle^*$ | | | | | | | | 1 |
| | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ |

In the spin block $B_2$, all i.m.s. of the decomposition matrix are associate (Preliminary 9) and $\langle\beta\rangle \neq \langle\beta\rangle'$. The block $B_2$ contains the irreducible spin characters $\langle 20, 1\rangle$, $\langle 20, 1\rangle'$, $\langle 18, 3\rangle$, $\langle 18, 3\rangle'$, $\langle 17, 3, 1\rangle^*$, $\langle 15, 3, 2, 1\rangle$, $\langle 15, 3, 2, 1\rangle'$, $\langle 13, 4, 3, 1\rangle$, $\langle 13, 4, 3, 1\rangle'$, $\langle 12, 5, 3, 1\rangle$, $\langle 12, 5, 3, 1\rangle'$, $\langle 11, 6, 3, 1\rangle$, $\langle 11, 6, 3, 1\rangle'$, $\langle 10, 7, 3, 1\rangle$, $\langle 10, 7, 3, 1\rangle'$, $\langle 9, 8, 3, 1\rangle$, $\langle 9, 8, 3, 1\rangle'$ has 17-bar core $\langle 3, 1\rangle$.

**Proposition 3.6.** *The Brauer tree for principal block $B_1$ is:* $\langle 21\rangle^*\_\langle 17, 4\rangle =$ $\langle 17, 4\rangle'\_\langle 16, 4, 1\rangle^*\_\langle 15, 4, 2\rangle^*\_\langle 14, 4, 3\rangle^*\_\langle 12, 5, 4\rangle^*\_\langle 11, 6, 4\rangle^*\_\langle 10, 7, 4\rangle^*\_\langle 9, 8, 4\rangle^*$

*Proof.* $deg\ \{\langle 21\rangle^*, \langle 16, 4, 1\rangle^*, \langle 14, 4, 3\rangle^*, \langle 11, 6, 4\rangle^*, \langle 9, 8, 4\rangle^*\} \equiv 4\ mod\ 17$;
$deg\ \{(\langle 17, 4\rangle + \langle 17, 4\rangle'), \langle 15, 4, 2\rangle^*, \langle 12, 5, 4\rangle^*, \langle 10, 7, 4\rangle^*\} \equiv -4\ mod\ 17$ (Preliminary 4). We apply $(4, 14)$-inducing of p.i.s. of $S_{20}$ (see $D^{(4)}_{20,17}$) to $S_{21}$, we have:

$$E_1 \uparrow^{(4,14)} S_{21} = \langle 21\rangle^* + \langle 17, 4\rangle + \langle 17, 4\rangle' = f_1$$
$$E_3 \uparrow^{(4,14)} S_{21} = \langle 17, 4\rangle + \langle 17, 4\rangle' + \langle 16, 4, 1\rangle^* = f_2$$
$$E_5 \uparrow^{(4,14)} S_{21} = \langle 16, 4, 1\rangle^* + \langle 15, 4, 2\rangle^* = f_3$$
$$E_7 \uparrow^{(4,14)} S_{21} = \langle 15, 4, 2\rangle^* + \langle 14, 4, 3\rangle^* = f_4$$

$$E_9 \uparrow^{(4,14)} S_{21} = \langle 14,4,3\rangle^* + \langle 12,5,4\rangle^* = f_5$$
$$E_{11} \uparrow^{(4,14)} S_{21} = \langle 12,5,4\rangle^* + \langle 11,6,4\rangle^* = f_6$$
$$E_{13} \uparrow^{(4,14)} S_{21} = \langle 11,6,4\rangle^* + \langle 10,7,4\rangle^* = f_7$$
$$E_{15} \uparrow^{(4,14)} S_{21} = \langle 10,7,4\rangle^* + \langle 9,8,4\rangle^* = f_8.$$

So we have the Brauer tree for $B_1$ and the decomposition matrix for this block $D^{(6)}_{21,17}$ in Table 6. □

TABLE 6. $D^{(6)}_{21,17}$

| The spin characters | The decomposition matrix for the block $B_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\langle 21\rangle^*$ | 1 | | | | | | | |
| $\langle 17,4\rangle$ | 1 | 1 | | | | | | |
| $\langle 17,4\rangle'$ | 1 | 1 | | | | | | |
| $\langle 16,4,1\rangle^*$ | | 1 | 1 | | | | | |
| $\langle 15,4,2\rangle^*$ | | | 1 | 1 | | | | |
| $\langle 14,4,3\rangle^*$ | | | | 1 | 1 | | | |
| $\langle 12,5,4\rangle^*$ | | | | | 1 | 1 | | |
| $\langle 11,6,4\rangle^*$ | | | | | | 1 | 1 | |
| $\langle 10,7,4\rangle^*$ | | | | | | | 1 | 1 |
| $\langle 9,8,4\rangle^*$ | | | | | | | | 1 |
| | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ | $f_8$ |

**Proposition 3.7.** *The Brauer tree for the block $B_2$ is:*

$\langle 20,1\rangle$-$\langle 18,3\rangle$ $\quad$ $\langle 15,3,2,1\rangle$-$\langle 13,4,3,1\rangle$-$\langle 12,5,3,1\rangle$-$\langle 11,6,3,1\rangle$-$\langle 10,7,3,1\rangle$-$\langle 9,8,3,1\rangle$

$\diagdown \quad \diagup$

$\langle 17,1\rangle^*$

$\diagup \quad \diagdown$

$\langle 20,1\rangle'$-$\langle 18,3\rangle'$ $\quad$ $\langle 15,3,2,1\rangle'$-$\langle 13,4,3,1\rangle'$-$\langle 12,5,3,1\rangle'$-$\langle 11,6,3,1\rangle'$-$\langle 10,7,3,1\rangle'$-$\langle 9,8,3,1\rangle'$

*Proof.* $deg\{\langle 20,1\rangle, \langle 20,1\rangle', \langle 17,3,1\rangle^*, \langle 13,4,3,1\rangle, \langle 13,4,3,1\rangle', \langle 11,6,3,1\rangle, \langle 11,6,3,1\rangle',$ $\langle 9,8,3,1\rangle, \langle 9,8,3,1\rangle'\} \equiv 4 \ mod \ 17;$ $deg \ \{\langle 18,3\rangle, \langle 18,3\rangle', \langle 15,3,2,1\rangle,$

$\langle 15,3,2,1\rangle', \langle 12,5,3,1\rangle, \langle 12,5,3,1\rangle', \langle 10,7,3,1\rangle, \langle 10,7,3,1\rangle'\} \equiv -4 \ mod \ 17$ (Preliminary 4). By using $(1,0)$-inducing of p.i.s. of $S_{20}$ (see $D^{(4)}_{20,17}$) to $S_{21}$, we have:

$$E_1 \uparrow^{(1,0)} S_{21} = \langle 20,1\rangle + \langle 18,3\rangle + \langle 18,3\rangle' + \langle 17,3,1\rangle^* = K_1$$
$$E_2 \uparrow^{(1,0)} S_{21} = \langle 20,1\rangle' + \langle 18,3\rangle + \langle 18,3\rangle' + \langle 17,3,1\rangle^* = K_2$$
$$E_3 \uparrow^{(1,0)} S_{21} = \langle 18,3\rangle + \langle 18,3\rangle' + 2\langle 17,3,1\rangle^* = K_3$$
$$E_5 \uparrow^{(1,0)} S_{21} = \langle 17,3,1\rangle^* + \langle 15,3,2,1\rangle = g_5$$
$$E_6 \uparrow^{(1,0)} S_{21} = \langle 17,3,1\rangle^* + \langle 15,3,2,1\rangle' = g_6$$
$$E_7 \uparrow^{(1,0)} S_{21} = \langle 15,3,2,1\rangle + \langle 13,4,3,1\rangle = g_7$$
$$E_8 \uparrow^{(1,0)} S_{21} = \langle 15,3,2,1\rangle' + \langle 13,4,3,1\rangle' = g_8$$
$$E_9 \uparrow^{(1,0)} S_{21} = \langle 13,4,3,1\rangle + \langle 12,5,3,1\rangle = g_9$$
$$E_{10} \uparrow^{(1,0)} S_{21} = \langle 13,4,3,1\rangle' + \langle 12,5,3,1\rangle' = g_{10}$$
$$E_{11} \uparrow^{(1,0)} S_{21} = \langle 12,5,3,1\rangle + \langle 11,6,3,1\rangle = g_{11}$$
$$E_{12} \uparrow^{(1,0)} S_{21} = \langle 12,5,3,1\rangle' + \langle 11,6,3,1\rangle' = g_{12}$$
$$E_{13} \uparrow^{(1,0)} S_{21} = \langle 11,6,3,1\rangle + \langle 10,7,3,1\rangle = g_{13}$$
$$E_{14} \uparrow^{(1,0)} S_{21} = \langle 11,6,3,1\rangle' + \langle 10,7,3,1\rangle' = g_{14}$$
$$E_{15} \uparrow^{(1,0)} S_{21} = \langle 10,7,3,1\rangle + \langle 9,8,3,1\rangle = g_{15}$$
$$E_{16} \uparrow^{(1,0)} S_{21} = \langle 10,7,3,1\rangle' + \langle 9,8,3,1\rangle' = g_{16}$$

$\langle 18,3,1\rangle \downarrow_{(1,0)} S_{21} = \langle 18,3\rangle + \langle 17,3,1\rangle^* = g_1$ since $\langle 18,3,1\rangle$ i.m. in $S_{22}$, and $\langle 18,3,1\rangle' \downarrow_{(1,0)} S_{21} = \langle 18,3\rangle' + \langle 17,3,1\rangle^* = g_2$ since $\langle 18,3,1\rangle'$ i.m. in $S_{22}$. Then $K_3$ split to $g_3$ and $g_4$. Since $F_1 \uparrow^{(3,15)} S_{21} = \langle 19,1\rangle^* + \langle 18,2\rangle^* \uparrow^{(3,15)} S_{21} = \langle 20,1\rangle + \langle 20,1\rangle' + \langle 18,3\rangle + \langle 18,3\rangle' = K_4 = K_1 + K_2 - g_3 - g_4$ and $\langle 20,1\rangle \neq \langle 20,1\rangle'$, $\langle 18,3\rangle \neq \langle 18,3\rangle'$ on $(17,\alpha)$-regular classes, then $K_1 - g_4 = g_1$, $K_2 - g_3 = g_2$. So, we have the Brauer tree for $B_2$ and the decomposition matrix for this block $D^{(7)}_{21,17}$ in Table 7. □

3.7. **Decomposition Matrix for the Spin Characters of $S_{22}$.** The group $S_{22}$ has 133 irreducible spin characters and 121 of $(17,\alpha)$-regular classes, then the decomposition matrix of the spin characters for $S_{22}$, $p = 17$ has 133 rows and 121 columns. By using (Preliminary 5), there are 101 blocks of $S_{22}$ two of them $B_1$, $B_2$, $B_3$ of defect 1. All the 98 remaining characters form their own blocks $B_4, B_5, \cdots, B_{101}$ of defect zero [11].

TABLE 7. $D^{(7)}_{21,17}$

| The spin characters characters | The decomposition matrix for the block $B_2$ | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 20,1\rangle$ | 1 | | | | | | | | | | | | | | | |
| $\langle 20,1\rangle'$ | | 1 | | | | | | | | | | | | | | |
| $\langle 18,3\rangle$ | 1 | | 1 | | | | | | | | | | | | | |
| $\langle 18,3\rangle'$ | | 1 | | 1 | | | | | | | | | | | | |
| $\langle 17,3,1\rangle^*$ | | | 1 | 1 | 1 | 1 | | | | | | | | | | |
| $\langle 15,3,2,1\rangle$ | | | | | 1 | | 1 | | | | | | | | | |
| $\langle 15,3,2,1\rangle'$ | | | | | | 1 | | 1 | | | | | | | | |
| $\langle 13,4,3,1\rangle$ | | | | | | | 1 | | 1 | | | | | | | |
| $\langle 13,4,3,1\rangle'$ | | | | | | | | 1 | | 1 | | | | | | |
| $\langle 12,5,3,1\rangle$ | | | | | | | | | 1 | | 1 | | | | | |
| $\langle 12,5,3,1\rangle'$ | | | | | | | | | | 1 | | 1 | | | | |
| $\langle 11,6,3,1\rangle$ | | | | | | | | | | | 1 | | 1 | | | |
| $\langle 11,6,3,1\rangle'$ | | | | | | | | | | | | 1 | | 1 | | |
| $\langle 10,7,3,1\rangle$ | | | | | | | | | | | | | 1 | | 1 | |
| $\langle 10,7,3,1\rangle'$ | | | | | | | | | | | | | | 1 | | 1 |
| $\langle 9,8,3,1\rangle$ | | | | | | | | | | | | | | | 1 | |
| $\langle 9,8,3,1\rangle'$ | | | | | | | | | | | | | | | | 1 |
| | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ | $g_{11}$ | $g_{12}$ | $g_{13}$ | $g_{14}$ | $g_{15}$ | $g_{16}$ |

In the principal block $B_1$, all i.m.s. of the decomposition matrix are associate (Preliminary 9) and $\langle\beta\rangle \neq \langle\beta\rangle'$. The principal block $B_1$ contains the irreducible spin characters $\langle 22\rangle$, $\langle 22\rangle'$, $\langle 17,5\rangle^*$, $\langle 16,5,1\rangle$, $\langle 16,6,1\rangle'$, $\langle 15,5,2\rangle$, $\langle 15,5,2\rangle'$, $\langle 14,5,3\rangle$, $\langle 14,5,3\rangle'$, $\langle 13,5,4\rangle$, $\langle 13,5,4\rangle'$, $\langle 11,6,5\rangle$, $\langle 11,6,5\rangle'$, $\langle 10,7,5\rangle$, $\langle 10,7,5\rangle'$, $\langle 9,8,5\rangle$, $\langle 9,8,5\rangle'$ has 17-bar core $\langle 5\rangle$.

In the block $B_2$, all i.m.s. of the decomposition matrix are double (Preliminary 9) and $\langle\beta\rangle \neq \langle\beta\rangle'$. The block $B_2$ contains the irreducible spin characters $\langle 21,1\rangle^*$, $\langle 18,4\rangle^*$, $\langle 17,4,1\rangle$, $\langle 17,4,1\rangle'$, $\langle 15,4,2,1\rangle^*$, $\langle 14,4,3,1\rangle^*$, $\langle 12,5,4,1\rangle^*$, $\langle 11,6,4,1\rangle^*$, $\langle 10,7,4,1\rangle^*$, $\langle 9,8,4,1\rangle^*$ has 17-bar core $\langle 4,1\rangle$.

In the block $B_3$, all i.m.s. of the decomposition matrix are double (Preliminary 9) and $\langle\beta\rangle \neq \langle\beta\rangle'$. The block $B_3$ contains the irreducible spin characters $\langle 20,2\rangle^*$, $\langle 19,3\rangle^*$, $\langle 17,3,2\rangle$, $\langle 17,3,2\rangle'$, $\langle 16,3,2,1\rangle^*$, $\langle 13,4,2,1\rangle^*$, $\langle 12,5,3,2\rangle^*$, $\langle 11,6,3,2\rangle^*$, $\langle 10,7,3,2\rangle^*$, $\langle 9,8,3,2\rangle^*$ has 17-bar core $\langle 3,2\rangle$.

**Proposition 3.8.** *The Brauer tree for principal block $B_1$ is:*

$$\langle 22\rangle \qquad \langle 16,5,1\rangle\_\langle 15,5,2\rangle\_\langle 14,5,3\rangle\_\langle 13,5,4\rangle\_\langle 11,6,5\rangle\_\langle 10,7,5\rangle\_\langle 9,8,5\rangle$$

$$\diagdown \qquad \diagup$$

$$\langle 17,5\rangle^*$$

$$\diagup \qquad \diagdown$$

$$\langle 22\rangle' \qquad \langle 16,5,1\rangle'\_\langle 15,5,2\rangle'\_\langle 14,5,3\rangle'\_\langle 13,5,4\rangle'\_\langle 11,6,5\rangle'\_\langle 10,7,5\rangle'\_\langle 9,8,5\rangle'$$

*Proof.* $deg \quad \{\langle 22\rangle, \langle 22\rangle', \langle 16,5,1\rangle, \langle 16,5,1\rangle', \langle 14,5,3\rangle, \langle 14,5,3\rangle', \langle 11,6,5\rangle, \langle 11,6,5\rangle', \langle 9,8,4\rangle, \langle 9,8,4\rangle'\} \equiv 4 \ mod\ 17$; $deg\ \{\langle 17,5\rangle^*, \langle 15,5,2\rangle, \langle 15,5,2\rangle', \langle 13,5,4\rangle, \langle 13,5,4\rangle', \langle 10,7,5\rangle, \langle 10,7,5\rangle'\} \equiv -4\ mod\ 17$ (Preliminary 4). By using $(4,14)$-inducing of p.i.s. of $S_{21}$ (see $D^{(6)}_{21,17}$) to $S_{22}$, we have:

$$f_1 \uparrow^{(4,14)} S_{22} = \langle 22\rangle + \langle 22\rangle' + 2\langle 17,5\rangle^* = K_1 = G_1 + G_2$$

$$f_2 \uparrow^{(4,14)} S_{22} = 2\langle 17,5\rangle^* + \langle 16,5,1\rangle + \langle 16,5,1\rangle' = K_2 = G_3 + G_4$$

$$f_3 \uparrow^{(4,14)} S_{22} = \langle 16,5,1\rangle + \langle 16,5,1\rangle' + \langle 15,5,2\rangle + \langle 15,5,2\rangle' = K_3 = G_5 + G_6$$

$$f_4 \uparrow^{(4,14)} S_{22} = \langle 15,5,2\rangle + \langle 15,5,2\rangle' + \langle 14,5,3\rangle + \langle 14,5,3\rangle' = K_4 = G_7 + G_8$$

$$f_5 \uparrow^{(4,14)} S_{22} = \langle 14,5,3\rangle + \langle 14,5,3\rangle' + \langle 13,5,4\rangle + \langle 13,5,4\rangle' = K_5 = G_9 + G_{10}$$

$$f_6 \uparrow^{(4,14)} S_{22} = \langle 13,5,4\rangle + \langle 13,5,4\rangle' + \langle 11,6,5\rangle + \langle 11,6,5\rangle' = K_6 = G_{11} + G_{12}$$

$$f_7 \uparrow^{(4,14)} S_{22} = \langle 11,6,5\rangle + \langle 11,6,5\rangle' + \langle 10,7,5\rangle + \langle 10,7,5\rangle' = K_7 = G_{13} + G_{14}$$

$$f_8 \uparrow^{(4,14)} S_{22} = \langle 10,7,5\rangle + \langle 10,7,5\rangle' + \langle 9,8,5\rangle + \langle 9,8,5\rangle' = K_8 = G_{15} + G_{16}$$

Since $\langle 22\rangle \neq \langle 22\rangle'$, $\langle 16,5,1\rangle \neq \langle 16,5,1\rangle'$, $\langle 15,5,2\rangle \neq \langle 15,5,2\rangle'$, $\langle 14,5,3\rangle \neq \langle 14,5,3\rangle'$, $\langle 13,5,4\rangle \neq \langle 13,5,4\rangle'$, $\langle 11,6,5\rangle \neq \langle 11,6,5\rangle'$, $\langle 10,7,5\rangle \neq \langle 10,7,5\rangle'$, $\langle 9,8,5\rangle \neq \langle 9,8,5\rangle'$ on $(17,\alpha)$-regular classes, then $K_1$, $K_2$, $K_3$, $K_4$, $K_5$, $K_6$, $K_7$, $K_8$ are split, respectively. So we have the Brauer tree for $B_1$ and the decomposition matrix for this block $D^{(8)}_{22,17}$ in Table 8. □

**Proposition 3.9.** *The Brauer tree for spin block $B_2$ is:* $\langle 21,1\rangle^*\_\langle 18,4\rangle^*\_\langle 17,4,1\rangle = \langle 17,4,1\rangle'\_\langle 15,4,2,1\rangle^*\_\langle 14,4,3,1\rangle^*\_\langle 12,5,4,1\rangle^*\_\langle 11,6,4,1\rangle^*\_\langle 10,7,4,1\rangle^*\_\langle 9,8,4,1\rangle^*$

TABLE 8. $D^{(8)}_{22,17}$

| The spin characters | The decomposition matrix for the block $B_1$ | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 22\rangle$ | 1 | | | | | | | | | | | | | | | |
| $\langle 22\rangle'$ | | 1 | | | | | | | | | | | | | | |
| $\langle 17,5\rangle^*$ | 1 | 1 | 1 | 1 | | | | | | | | | | | | |
| $\langle 16,5,1\rangle$ | | | 1 | | 1 | | | | | | | | | | | |
| $\langle 16,5,1\rangle'$ | | | | 1 | | 1 | | | | | | | | | | |
| $\langle 15,5,2\rangle$ | | | | | 1 | | 1 | | | | | | | | | |
| $\langle 15,5,2\rangle'$ | | | | | | 1 | | 1 | | | | | | | | |
| $\langle 14,5,3\rangle$ | | | | | | | 1 | | 1 | | | | | | | |
| $\langle 14,5,3\rangle'$ | | | | | | | | 1 | | 1 | | | | | | |
| $\langle 13,5,4\rangle$ | | | | | | | | | 1 | | 1 | | | | | |
| $\langle 13,5,4\rangle'$ | | | | | | | | | | 1 | | 1 | | | | |
| $\langle 11,6,5\rangle$ | | | | | | | | | | | 1 | | 1 | | | |
| $\langle 11,6,5\rangle'$ | | | | | | | | | | | | 1 | | 1 | | |
| $\langle 10,7,5\rangle$ | | | | | | | | | | | | | 1 | | 1 | |
| $\langle 10,7,5\rangle'$ | | | | | | | | | | | | | | 1 | | 1 |
| $\langle 9,8,5\rangle$ | | | | | | | | | | | | | | | 1 | |
| $\langle 9,8,5\rangle'$ | | | | | | | | | | | | | | | | 1 |
| | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ | $G_6$ | $G_7$ | $G_8$ | $G_9$ | $G_{10}$ | $G_{11}$ | $G_{12}$ | $G_{13}$ | $G_{14}$ | $G_{15}$ | $G_{16}$ |

***Proof.*** $deg\ \{\langle 21,1\rangle^*, (\langle 17,4,1\rangle + \langle 17,4,1\rangle'), \langle 14,4,3,1\rangle^*, \langle 11,6,4,1\rangle^*, \langle 9,8,4,1\rangle^*\} \equiv 12\ mod\ 17$; $deg\ \{\langle 18,4\rangle^*, \langle 15,4,2,1\rangle^*, \langle 12,5,4,1\rangle^*, \langle 10,7,4,1\rangle^*\} \equiv -12\ mod\ 17$ (Preliminary 4). We apply $(0,1)$-inducing of p.i.s. of $S_{21}$ (see $D^{(6)}_{21,17}$) to $S_{22}$, we have:

$$f_1 \uparrow^{(0,1)} S_{22} = \langle 21,1\rangle^* + 2\langle 18,4\rangle^* + \langle 17,4,1\rangle + \langle 17,4,1\rangle' = K_1$$

$$f_2 \uparrow^{(0,1)} S_{22} = 2\langle 18,4\rangle^* + 2\langle 17,4,1\rangle + 2\langle 17,4,1\rangle' = 2h_2$$

$$f_3 \uparrow^{(0,1)} S_{22} = \langle 17,4,1\rangle + \langle 17,4,1\rangle' + \langle 15,4,2,1\rangle^* = h_3$$

$$f_4 \uparrow^{(0,1)} S_{22} = \langle 15,4,2,1\rangle^* + \langle 14,4,3,1\rangle^* = h_4$$

$$f_5 \uparrow^{(0,1)} S_{22} = \langle 14,4,3,1\rangle^* + \langle 12,5,4,1\rangle^* = h_5$$

$$f_6 \uparrow^{(0,1)} S_{22} = \langle 12,5,4,1\rangle^* + \langle 11,6,4,1\rangle^* = h_6$$

$$f_7 \uparrow^{(0,1)} S_{22} = \langle 11,6,4,1\rangle^* + \langle 10,7,4,1\rangle^* = h_7$$
$$f_8 \uparrow^{(0,1)} S_{22} = \langle 10,7,4,1\rangle^* + \langle 9,8,4,1\rangle^* = h_8.$$

Since $\langle 20,1\rangle + \langle 18,3\rangle \uparrow^{(4,14)} S_{22} = \langle 21,1\rangle^* + \langle 18,4\rangle^* = K_1 - h_2 = h_1$. So we have the Brauer tree for $B_2$ and the decomposition matrix for this block $D^{(9)}_{22,17}$ in Table 9. □

TABLE 9. $D^{(9)}_{22,17}$

| The spin characters | The decomposition matrix for the block $B_2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\langle 21,1\rangle^*$ | 1 | | | | | | | |
| $\langle 18,4\rangle^*$ | 1 | 1 | | | | | | |
| $\langle 17,4,1\rangle$ | | 1 | 1 | | | | | |
| $\langle 17,4,1\rangle'$ | | 1 | 1 | | | | | |
| $\langle 15,4,2,1\rangle^*$ | | | 1 | 1 | | | | |
| $\langle 14,4,3,1\rangle^*$ | | | | 1 | 1 | | | |
| $\langle 12,5,4,1\rangle^*$ | | | | | 1 | 1 | | |
| $\langle 11,6,4,1\rangle^*$ | | | | | | 1 | 1 | |
| $\langle 10,7,4,1\rangle^*$ | | | | | | | 1 | 1 |
| $\langle 9,8,4,1\rangle^*$ | | | | | | | | 1 |
| | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ | $h_6$ | $h_7$ | $h_8$ |

**Proposition 3.10.** *The Brauer tree for spin block $B_3$ is:* $\langle 20,2\rangle^*$-$\langle 19,3\rangle^*$-$\langle 17,3,2\rangle = \langle 17,3,2\rangle'$-$\langle 16,3,2,1\rangle^*$-$\langle 13,4,3,2\rangle^*$-$\langle 12,5,3,2\rangle^*$-$\langle 11,6,3,2\rangle^*$-$\langle 10,7,3,2\rangle^*$-$\langle 9,8,3,2\rangle^*$

*Proof.* $deg$ $\{\langle 20,2\rangle^*, (\langle 17,3,2\rangle + \langle 17,3,2\rangle'), \langle 13,4,3,2\rangle^*, \langle 11,6,3,2\rangle^*, \langle 9,8,4,1\rangle^*\} \equiv 8$ $mod$ $17$; $deg$ $\{\langle 19,3\rangle^*, \langle 16,3,2,1\rangle^*, \langle 12,5,3,2\rangle^*, \langle 10,7,3,2\rangle^*\} \equiv -8$ $mod$ $17$ (Preliminary 4). We apply $(r,r')$-inducing of p.i.s. of $S_{21}$ (see $D_{20,17}$) to $S_{22}$, we have:

$$g_1 \uparrow^{(2,16)} S_{22} = \langle 20,2\rangle^* + 2\langle 19,3\rangle^* = H_1$$
$$g_3 \uparrow^{(2,16)} S_{22} = 2\langle 19,3\rangle^* + \langle 17,3,2\rangle + 2\langle 17,3,2\rangle' = H_2$$
$$g_5 \uparrow^{(2,16)} S_{22} = \langle 17,3,2\rangle + \langle 17,3,2\rangle' + \langle 16,3,2,1\rangle^* = H_3$$

$$g_7 \uparrow^{(2,16)} S_{22} = \langle 16,3,2,1\rangle^* + \langle 13,4,3,2\rangle^* = H_4$$
$$g_9 \uparrow^{(2,16)} S_{22} = \langle 13,4,3,2\rangle^* + \langle 12,5,3,2\rangle^* = H_5$$
$$g_{11} \uparrow^{(2,16)} S_{22} = \langle 12,5,3,2\rangle^* + \langle 11,6,3,2\rangle^* = H_6$$
$$g_{13} \uparrow^{(2,16)} S_{22} = \langle 11,6,3,2\rangle^* + \langle 10,7,3,2\rangle^* = H_7$$
$$g_{15} \uparrow^{(2,16)} S_{22} = \langle 10,7,3,2\rangle^* + \langle 9,8,3,2\rangle^* = H_8.$$

So we have the Brauer tree for $B_3$ and the decomposition matrix for this block $D^{(10)}_{22,17}$ in Table 10. □

TABLE 10. $D^{(10)}_{22,17}$

| The spin characters | The decomposition matrix for the block $B_3$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\langle 20,2\rangle^*$ | 1 | | | | | | | |
| $\langle 19,3\rangle^*$ | 1 | 1 | | | | | | |
| $\langle 17,3,2\rangle$ | | 1 | 1 | | | | | |
| $\langle 17,3,2\rangle'$ | | 1 | 1 | | | | | |
| $\langle 16,3,2,1\rangle^*$ | | | 1 | 1 | | | | |
| $\langle 13,4,3,2\rangle^*$ | | | | 1 | 1 | | | |
| $\langle 12,5,3,2\rangle^*$ | | | | | 1 | 1 | | |
| $\langle 11,6,3,2\rangle^*$ | | | | | | 1 | 1 | |
| $\langle 10,7,3,2\rangle^*$ | | | | | | | 1 | 1 |
| $\langle 9,8,3,2\rangle^*$ | | | | | | | | 1 |
| | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | $H_6$ | $H_7$ | $H_8$ |

## 4. CONCLUSION

In this work, motivated by previous results given in the papers [3, 10, 12, 13], we conclude that all blocks of defect one or zero and the decomposition numbers are one or zero. Also we compute the Brauer trees of the symmetric group $S_n$, $17 \leq n \leq 22$ modulo $P = 17$. Finally, all the 17-decomposition matrices of spin characters of $S_n$, $17 \leq n \leq 22$ are found.

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