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4th HANKEL DETERMINANT FOR α BOUNDED TURNING FUNCTION

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ABSTRACT. In this present paper, Author have investigated the fourth Hankel determinant for a function of bounded turning in unit disk. Twofold and threefold symmetric functions also investigated for the same class.

1. INTRODUCTION

Let us define the most basic class $\mathcal{A}$ which represents the set of all analytic (holomorphic) functions f in region $\mathcal{U} = \{z; |z| < 1\}$ having form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

In addition, let subfamily $\mathcal{S}$ of $\mathcal{A}$ i.e. $\mathcal{S} \subset \mathcal{A}$ be the class of all functions which are univalent in $\mathcal{U}$. Let $\mathcal{P}$ represents the class of analytic functions p whose real parts are positive in $\mathcal{U}$ having the form $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$.

Let $\mathcal{R}(\alpha), \alpha \in [0,1)$ such that $Re(f'(z)) > \alpha, z \in \mathcal{U}$ and $\mathcal{R}(0) = \mathcal{R}$ class of bounded turning because $Re(f'(z)) > 0$ is equivalent to $|\arg f'(z)| < \frac{\pi}{2}$ and $\arg f(z)$ is the angle of rotation of the image of a line segment starting from z under the mapping f. [6] Hankel determinant for $f \in \mathcal{S}$ as $H_{q,n}(f)$ where $q, n \geq 1$ as

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$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

Computing the upper bound of $H_{q,n}$ over different subfamilies of $\mathcal{A}$ is an interesting problem to study. Firstly, Janteng [3], Babalola [2] have found the second and third order Hankel determinant for the class $\mathcal{R}$ respectively. In 2017, Zaprawa ([7]) improved the results obtained in [2] for the class of bounded turning function $\mathcal{R}$. Fourth Hankel determinant for function with bounded turning is studied in [1].

To find the upper bound of $H_{4,1}(f)$ we need the following results:

Lemma 1.1. *[5] If $p \in \mathcal{P}$ then $|c_n| \le 2$ for $n \in \mathbb{N}$,*

$$|c_{n+k} - \lambda c_n c_k| \le 2 \qquad (0 \le \lambda \le 1),$$

and

$$|c_m c_n - c_k c_l| \le 4 \qquad (m + n = k + l).$$

Theorem 1.1. *[1] Let $g \in \mathcal{S}^*$ where $g(z) = z + \sum_{n=1}^{\infty} b_n z^n$ then for any real λ*

$$|b_2^2(b_3 - \lambda b_2^2)| = \begin{cases} 4(3-4\lambda) & for \quad \lambda \le \frac{5}{8}; \\ \frac{1}{2(2\lambda-1)} & for \quad \lambda \in [\frac{5}{8}, \frac{3}{4}]; \\ \frac{1}{4(1-\lambda)} & for \quad \lambda \in [\frac{3}{4}, \frac{7}{8}]; \\ 4(4\lambda - 3) & for \quad \lambda \ge \frac{7}{8}. \end{cases}$$

Theorem 1.2. *[4] If $f \in \mathcal{R}(\alpha)$, $0 \le \alpha < 1$, then $|a_n| \le \frac{2(1-\alpha)}{n}$, $n \ge 2$, and*

$$|H_{3,1}(f)| \le \frac{A^2}{3}\left[\frac{40A + 36}{45} + \frac{(1+4A)^{\frac{3}{2}}}{12\sqrt{3}}\right], \tag{1.3}$$

where $A = 1 - \alpha$.

2. Bounds of fourth Hankel determinant

First, $H_{4,1}(f)$ where $f \in \mathcal{A}$ is of the form (1.1) can be written as

$$H_{4,1}(f) = a_7 H_{3,1}(f) - a_6\Delta_1 + a_5\Delta_2 - a_4\Delta_3 \tag{2.1}$$

where $\Delta_1 = (a_3a_6 - a_4a_5) - a_2(a_2a_6 - a_3a_5) + a_4(a_2a_4 - a_3^2)$, $\Delta_2 = (a_4a_6 - a_5^2) - a_2(a_3a_6 - a_4a_5) + a_3(a_3a_5 - a_4^2)$, and $\Delta_3 = a_2(a_4a_6 - a_5^2) - a_3(a_3a_6 - a_4a_5) + a_4(a_3a_5 - a_4^2)$.

Theorem 2.1. *If* $f \in \mathcal{R}(\alpha)$ *then* $|H_{4,1}(f)| \leq (1-\alpha)^3 \left[\frac{1}{42}\left(\frac{5-4\alpha}{3}\right)^{\frac{3}{2}} + \frac{1139(1-\alpha)}{3760} + \frac{10508}{23625}\right]$.

Proof. Let $f \in \mathcal{R}(\alpha), \alpha \in [0,1)$. Then $\frac{f'(z)-\alpha}{1-\alpha} = p(z)$, where $p \in \mathcal{P}$, and

$$\left(1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right) - \alpha = (1-\alpha)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right).$$

By identifying the coefficients, $a_n = \frac{A}{n} c_{n-1}$, $A = (1-\alpha)$ and substitute in Δ_i's:

$$\begin{aligned}
\Delta_1 &= \frac{A^2 c_5}{24}(c_2 - Ac_1^2) + \frac{A^2 c_3}{36}(c_4 - Ac_2^2) - \frac{A^2 c_3}{32}(c_4 - Ac_1c_3)\\
&\quad - \frac{67A^2 c_4}{1440}(c_3 - Ac_1c_2) + \frac{19A^2 c_2}{1440}(c_5 - Ac_1c_4) + \frac{A^2 c_2 c_5}{1440};\\
\Delta_2 &= \frac{A^2 c_5}{36}(c_3 - Ac_1c_2) - \frac{A^2 c_4}{45}(c_4 - Ac_2^2) + \frac{A^2 c_3}{48}(c_5 - Ac_2c_3)\\
&\quad - \frac{13A^2 c_3}{1800}(c_5 - Ac_1c_4) - \frac{4A^2}{225}(c_4 - Ac_1c_3) + \frac{A^2 c_3 c_5}{3600};\\
\Delta_3 &= \frac{A^3 c_5}{54}(c_4 - c_2^2) - \frac{A^3 c_5}{48}(c_4 - c_1c_3) + \frac{A^3 c_3}{64}c_3(c_6 - c_3^2) - \frac{A^3 c_3}{64}(c_6 - c_2c_4)\\
&\quad + \frac{A^3 c_4}{50}(c_5 - c_1c_4) - \frac{17A^3 c_4}{960}(c_5 - c_2c_3) + \frac{A^3 c_4 c_5}{43200}.
\end{aligned}$$

Using the triangular inequality along with lemma (1.1), we obtain

$$|\Delta_1| \leq \frac{29}{45}(1-\alpha)^2, \quad |\Delta_2| \leq \frac{173}{450}(1-\alpha)^2, \quad |\Delta_3| \leq \frac{13}{30}(1-\alpha)^3. \tag{2.2}$$

Now using (2.2),(1.3) and (1.2) in (2.1), we obtain our required result. □

Remark 2.1. *Choosing* $\alpha = 0$ *in (2.1), we get [1]* $H_{4,1}(f) \leq \frac{73757}{94500} \simeq 0.78050$.

A function f is said to be n-fold symmetric if $f(\varepsilon z) = \varepsilon f(z)$ holds for all $z \in \mathcal{U}$, where $\varepsilon = \exp(\frac{2\Pi\iota}{n})$. The set of all n-fold symmetric functions belonging to $\mathcal{S}$ is denoted by $\mathcal{S}^{(\backslash)} = f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1}$, $z \in \mathcal{U}$.

An analytic function $f \in \mathcal{S}^{(n)}$ belongs to the family $(\mathcal{R}(\alpha))^{(n)}$ if and only if $\frac{f'(z)-\alpha}{1-\alpha} = p(z)$ with $p \in \mathcal{P}^{(n)} = \{p(z) : p(z) = 1 + \sum_{k=1}^{\infty} c_{nk} z^{nk}\}$.

Theorem 2.2. *If* $f \in (\mathcal{R}(\alpha))^{(3)}$, $\alpha \in [0,1)$ *then* $|H_{4,1}(f)| \leq \frac{(1-\alpha)^2}{49}$.

Proof. Let $f \in (\mathcal{R}(\alpha))^{(3)}$, $\exists \tilde{g} \in \mathcal{S}^{*(3)}$ of the form $z + d_4 z^4 + d_7 z^7 + ...$ such that $\frac{z\tilde{g}'(z)}{\tilde{g}(z)}$ Since $f \in (\mathcal{R}(\alpha))^{(3)} \subset \mathcal{S}^{(n)}$ for $n = 3$, we have $1 + 3d_4 z^3 + (6d_7 - 3d_4^2)z^6 + \cdots = 1 + \frac{4}{1-\alpha} a_4 z^3 + \frac{7}{1-\alpha} a_7 z^6 + \cdots$, after equating $3d_4 = \frac{4}{1-\alpha} a_4, \quad 6d_7 - 3d_4^2 = \frac{7}{1-\alpha} a_7$, Since $\tilde{g} \in \mathcal{S}^{*(3)}$, $\exists g$ in $\mathcal{S}^*$ of the form (1.1) such that $\tilde{g}(z) = \sqrt[3]{g(z^3)}$. Thus $z + d_4 z^4 + d_7 z^7 + ... = z + \frac{1}{3} b_2 z^4 + (\frac{1}{3} b_3 - \frac{1}{9} b_2^2) z^7 + \cdots$.

$$d_4 = \frac{1}{3} b_2, \ \ d_7 = (\frac{1}{3} b_3 - \frac{1}{9} b_2^2).$$

Now by rearranging the coefficients $a_4 = \frac{1-\alpha}{4} b_2$, $a_7 = \frac{1-\alpha}{7}(2b_3 - b_2^2)$. We observe that $a_2 = a_3 = a_5 = a_6 = 0$. It is also clear that $H_{4,1}(f) = a_4^2(a_4^2 - a_7)$. This implies $|H_{4,1}(f)| = \frac{(1-\alpha)^3}{56} |b_2^2(b_3 - \frac{(23-7\alpha)}{32} b_2^2)|$.

Using Theorem 1.1 where $\lambda = \frac{(23-7\alpha)}{32} \in [\frac{5}{8}, \frac{3}{4}]$, we get our desired result. □

Theorem 2.3. *If* $f \in (\mathcal{R}(\alpha))^{(2)}$, $\alpha \in [0,1)$ *then* $|H_{4,1}(f)| \leq \frac{368(1-\alpha)^3}{2625}$.

Proof. Since $f \in \mathcal{S}^{*(2)}$, $\exists p \in \mathcal{P}^{(2)}$ such that $\frac{f(z)-\alpha}{1-\alpha} = p(z)$, after equating $a_3 = \frac{(1-\alpha)}{3} c_2, \quad a_5 = \frac{(1-\alpha)}{5} c_4, \quad a_7 = \frac{(1-\alpha)}{7} p_6$ and $H_{4,1}(f) = a_3 a_5 a_7 - a_3^3 a_7 + a_3^2 a_5^2 - a_5^3$.

$$|H_{4,1}(f)| \leq \frac{(1-\alpha)^3}{105} \cdot \left| \left(c_2 c_6 - \frac{21}{25} c_4^2 \right) \right| \cdot \left| \left(c_4 - \frac{5(1-\alpha)}{9} c_2^2 \right) \right| = \frac{368(1-\alpha)^3}{2625}.$$

□

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